Math 163, Due Date: ____________ **Name(s): ________________________**

Background Project 4: Factoring Polynomials

Objective

To quickly review some practical tools for factoring polynomials of the form (1) below by hand.

Narrative

A polynomial $p(x)$ of degree n is an algebraic expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \tag{1}$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants and $n \neq 0$. In this project we review some basic tools for factoring polynomials by hand: if we can factor a polynomial $p(x)$ then we can solve an equation of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \tag{2}$$

by finding the solutions to the equations we obtain by setting each factor of $p(x)$ equal to 0.

One of the most important results regarding the factoring of polynomials is the *Fundamental Theorem of Algebra*. This result states that any polynomial of the form (1) for which $a_n = 1$ and $a_0, a_1, \ldots, a_{n-1}$ are complex numbers can (up to a rearrangement of factors) be uniquely factored as a product of linear factors of the form $x - r_i$ where each root r_i is complex. Further, if $a_0, a_1, \ldots, a_{n-1}$ are *real* numbers, then the roots r_i occur in complex conjugate pairs. Hence, any polynomial of the form (1) for which $a_0, a_1, \ldots, a_{n-1}$ are real numbers can be factored into a product of linear and irreducible quadratic factors. (A quadratic factor $ax^2 + bx + c$, a, b, and c real, is said to be *irreducible* if it cannot be factored using only real numbers. Equivalently, it is irreducible if the discriminant $b^2 - 4ac < 0$.)

These results are great, but they are, of course, only theoretical: in practice you will want some practical tools for factoring polynomials by hand. Unfortunately there is no fixed set of rules you can follow which will always lead to a complete factorization of a polynomial. There is, however, one strategy that is often useful. It is based on two facts:

1. First, if the coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ in (2) are *integers*, and *if* (2) has any *rational* roots, then those roots are of the form
$$\pm \frac{\text{a factor of } a_0}{\text{a factor of } a_n}.$$
2. Second, if r is a root of (2), then $x - r$ is a factor of $p(x)$ — so $p(x) = (x - r)q(x)$ for some polynomial $q(x)$ of degree $n - 1$ — and to find the remaining roots of (2) it suffices to find the roots of the equation $q(x) = 0$.

By iterating these steps, we can often completely factor (1) and find the roots of (2).

Example: If $p(x) = 6x^3 - 11x^2 - 3x + 2$ then $a_0 = 2$ and $a_3 = 6$, so if $p(x)$ has any rational roots then they are of the form

$$\pm \frac{\text{a factor of } 2}{\text{a factor of } 6}.$$

Since the factors of 2 are 1 and 2, and the factors of 6 are 1, 2, 3, and 6, this implies that if $p(x)$ has any rational roots then they are in the (*finite*) set

$$\pm 1/1 = \pm 1, \quad \pm 1/2 = \pm 1/2, \quad \pm 1/3 = \pm 1/3, \quad \pm 1/6 = \pm 1/6$$

$$\pm 2/1 = \pm 2, \quad \pm 2/2 = \pm 1, \quad \pm 2/3 = \pm 2/3, \quad \pm 2/6 = \pm 1/3$$

or

$$\pm 1, \quad \pm 1/2, \quad \pm 1/3, \quad \pm 1/6, \quad \pm 2, \quad \pm 2/3.$$

Upon checking, we find that $p(1) \neq 0, p(-1) \neq 0$, and $p(1/2) \neq 0$. But $p(-1/2) = 0$! And since this says that $x - (-\frac{1}{2}) = x + \frac{1}{2}$ is a factor of $p(x)$, we divide $p(x)$ by $2x + 1$ to find that

$$\frac{p(x)}{2x+1} = 3x^2 - 7x + 2$$

or

$$p(x) = (2x + 1)(3x^2 - 7x + 2).$$

Repeating this process with $q(x) = 3x^2 - 7x + 2$ we find that $q(x) = (3x - 1)(x - 2)$, so that

$$p(x) = (2x + 1)(3x^2 - 7x + 2) = (2x + 1)(3x - 1)(x - 2).$$

If we had found that *none* of the possible roots of $q(x)$ actually *was* a root of $q(x)$, then we would know that $q(x)$ has no rational roots, so $p(x) = (2x + 1)(3x^2 - 7x + 2)$ would be as far as we can factor $p(x)$ using integer coefficients.

Task

Factor the following polynomials completely:

1. $x^2 - 7x + 12$	**5**. $x^3 + 2x^2 - x - 2$
2. $2x^2 + 5x - 3$	**6**. $4x^3 + 10x^2 - 6x - 18$
3. $6x^2 - x - 1$	**7**. $3x^3 - 4x^2 - 27x + 36$
4. $11x^2 - 54x + 63$	**8**. $9x^3 - 18x^2 - 4x + 8$