INVARIANT MANIFOLDS OF THE WILSON RENORMALIZATION GROUP

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A manifold of Hamiltonians invariant with respect to action of the Wilson renormalization group is constructed. The equations of bifurcation in 4 − d are written down, and their connection with the equations of the renormalization group in quantum field theory is discussed.

Key concepts in Wilson's theory of the renormalization group are those of the Hamiltonian, the renormalization transformation in the space of the Hamiltonians, the fixed point, etc. (see [1,2]). However, in the construction of a perturbation theory a certain jump occurs — the language of Hamiltonians is forgotten and one uses constructions such as the group of renormalizations, the Callan–Symanzik equation, etc. (see [3]). The aim of the present paper is to give a more transparent description of the connection between these two approaches. Strictly speaking, if we are dealing with the Hamiltonians of Gibbs fields in a finite volume, then no problems associated with divergences in the calculation of the Wilson renormalization-group transformation arise. But the equations for the fixed point become complicated and obscure (an idea of this is given by Chap. 7 of Ma's book [4]). In addition, difficulties arise with the treatment of the objects of the ε expansion when ε is not integral (here, ε = 4 − d, where d is the space dimension). In the present paper, we define a generalized Hamiltonian and find a manifold in the space of generalized Hamiltonians that is invariant with respect to the action of the renormalization group and on which its action reduces to a multiplicative renormalization.

1. The Wilson Renormalization Group in the Space of Generalized Hamiltonians

By the Hamiltonian of a Gibbs field in the ball $\Omega=\{\mathbf{k}: |\mathbf{k}|\leqslant R\}$ we understand an expression of the form

$$H(\sigma)=\sum_{m=1}^{\infty}\int_{\Omega^{2m}} h_{2m}(\mathbf{k}_1,\ldots,\mathbf{k}_{2m})\prod_{i=1}^{2m}\sigma(\mathbf{k}_i)\,d^d\mathbf{k}_i$$

(we immediately restrict ourselves to the case of even Hamiltonians). We shall assume C^∞ smoothness of the coefficient functions $h_{2m}(\mathbf{k}_1,\ldots,\mathbf{k}_{2m})$, and also that they are isotropic:

$$h_{2m}(U\mathbf{k}_1,\ldots,U\mathbf{k}_{2m})=h_{2m}(\mathbf{k}_1,\ldots,\mathbf{k}_{2m}),$$

for any orthogonal transformation U of the space $\mathbb{R}^d$.

The Wilson renormalization-group transformation is a composition of an operator of multiplicative renormalization $\mathcal{R}_\lambda^a$ and a restriction operator $\mathcal{P}_\Omega$. The operator $\mathcal{R}_\lambda^a$ acts on a coefficient function as follows:

$$\mathcal{R}_\lambda^a h(\mathbf{k}_1,\ldots,\mathbf{k}_{2m})=\lambda^{ma-2md+d}h(\mathbf{k}_1/\lambda,\ldots,\mathbf{k}_{2m}/\lambda).$$

The operator $\mathcal{R}_\lambda^a$ maps the Hamiltonian in the ball Ω to a Hamiltonian in the ball λΩ.

The restriction operator $\mathcal{P}_\Omega$ acts on the Hamiltonian in the ball λΩ and carries it to a Hamiltonian in the ball Ω. Let $H(\sigma) = H_0 + H_{\text{int}}(\sigma)$ be a Hamiltonian in the ball λΩ, where

$$H_0(\sigma)=\frac{1}{2}\int_{(\lambda\Omega)^2}|\mathbf{k}_1|^2\delta(\mathbf{k}_1+\mathbf{k}_2)\sigma(\mathbf{k}_1)\sigma(\mathbf{k}_2)\,d^d\mathbf{k}_1\,d^d\mathbf{k}_2$$

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is the massless free Hamiltonian, and

$$H_{\text{int}}(\sigma)=\sum_{|\alpha|\geq 1} u^{\alpha}H_{\alpha}(\sigma),$$

where $u = (u_1, \ldots, u_s)$, is a vector of small parameters $\alpha=(\alpha_1,\ldots,\alpha_s)$, $u^{\alpha}=u_1^{\alpha_1}\ldots u_s^{\alpha_s}$, $|\alpha|=\alpha_1+\ldots+\alpha_s$ and all Hamiltonians $H_\alpha(\sigma)$ are finite-particle Hamiltonians (i.e., for any α there exists $N = N(\alpha)$ such that $h_{\alpha,2m} \equiv 0$ if $m \geq N(\alpha)$). Then

$$\mathscr{S}_{\Omega}(H)(\sigma_0)=H_0(\sigma_0)+\ln\langle\exp(H_{\text{int}}(\sigma_0+\sigma_1))\rangle_{\mu(d\sigma_1)}, \tag{1}$$

where the integration is over the Gaussian measure $\mu(d\sigma_1)$ with correlation function $\delta(\mathbf{k}_1+\mathbf{k}_2)|\mathbf{k}_1|^2\chi_{\lambda\Omega\backslash\Omega}(\mathbf{k}_1)$ (by $\chi_\Lambda(\mathbf{k})$ we denote the characteristic function of the set Λ), σ_0 is a configuration in the ball Ω, and σ_1 is a configuration in the spherical layer $\lambda\Omega\backslash\Omega$. The typical term in the expansion (1) in a series in Feynman diagrams has the form

$$\int \delta(\mathbf{p}_1+\ldots+\mathbf{p}_n)\mathscr{F}_G(\mathbf{p}_1,\ldots,\mathbf{p}_n)\sigma(\mathbf{p}_1)\ldots\sigma(\mathbf{p}_n)d^d\mathbf{p}_1\ldots d^d\mathbf{p}_n,$$

where

$$\mathscr{F}_G(\mathbf{p}_1,\ldots,\mathbf{p}_n)=u^{\alpha}\int\prod_{v\in V}h_v(\mathbf{k},\mathbf{p})\prod_{l\in L}\Delta_l(\mathbf{p},\mathbf{k})\prod_{i=1}^{h}d^d\mathbf{k}_i, \tag{2}$$

in which V is the set of vertices of the graph G, L is the set of internal lines, h is the cyclomatic number, and $\mathbf{k}_1,\ldots,\mathbf{k}_h$ is a certain set of cyclic variables.

If the coefficient functions h_v, $v\in V$, are isotropic, then the amplitude $\mathscr{F}_G(\mathbf{p}_1,\ldots,\mathbf{p}_n)$ is also isotropic and, in addition, can be analytically continued with respect to the d into the complex plane (see [5,6]). The operation of multiplicative renormalization $\mathscr{R}_\lambda^a$ also maps isotropic Hamiltonians to isotropic Hamiltonians. This makes it possible to define an analytic continuation of the Wilson renormalization group with respect to d into the complex plane. Indeed, specification of the Hamiltonian is equivalent to specification of the sequence of its coefficient functions. The isotropic coefficient functions and the expressions of the type (2) obtained from them can be treated as functions of the scalar products $(\mathbf{k}_i,\mathbf{k}_j)$. Let $\mathbf{k}=(\mathbf{k}_1,\ldots,\mathbf{k}_n)$, $\mathbf{k}_i\in\mathbb{R}^d$. We define the mapping $\rho_{nd}:\mathbf{k}\to\{(\mathbf{k}_i,\ \mathbf{k}_j)_{i,j=1}^n\}$ (see [5,6]), which associates the vectors $(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ with their Gram matrix. Let $G_n=\{g=\{g_{ij}\}_{i,j=1}^n|\ g_{ij}=g_{ji}\}$ be the set of symmetric positive-definite matrices. Then the isotropic function $h(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ uniquely defines a function on $G_n:\varphi((\mathbf{k}_i,\ \mathbf{k}_j),\ i,j=1,\ldots,n)=h(\mathbf{k}_1,\ldots,\mathbf{k}_n)$.

We shall find it convenient to generalize the concept of a Hamiltonian somewhat. We shall call a sequence of functions with label ν: $(\varphi;\ \nu)=(\varphi_2(g_2),\ \varphi_4(g_4),\ldots;\nu)$, where $g_{2n}\in G_{2n}$, $\nu\in\mathbb{C}$, a generalized Hamiltonian. For natural $\nu = d$, the generalized Hamiltonian $(\varphi;\ d)$ can be regarded as an ordinary Hamiltonian. The coefficient functions of this Hamiltonian are given by the relations $h_{2n}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n})=\varphi_{2n}((\mathbf{k}_i,\ \mathbf{k}_j)_{i,j=1}^{2n})$. For such a Hamiltonian, the actions of the operators of restriction, $\mathscr{S}_\Omega$ and multiplicative renormalization, $\mathscr{R}_\lambda^a$, are well defined. We consider the question of analytic continuation with respect to the parameter ν.

The factors $\lambda^{na-2nd+d}$, which arise on application of the operator $\mathscr{R}_\lambda^a$ to the coefficient functions h_{2n}, admit a natural analytic continuation with respect to d into the complex plane. It follows from the results of [5,6] that an arbitrary Feynman amplitude $\mathscr{F}_G(\mathbf{p}_1,\ldots,\mathbf{p}_{2n};\ d)$ of the form (2) can be analytically continued with respect to d into the complex plane and defines a function $\varphi(g_{2n};\ \nu)$ on G_{2n} that depends analytically on ν. We denote this mapping by $A_\nu:\varphi(g_{2n};\ \nu)=A_\nu(\mathscr{F}_G)$.

We can now define the action of the Wilson renormalization group on the space of generalized Hamiltonians: $\mathscr{R}_{\chi,\lambda}^a\ (\varphi;\ \nu)=(A_\nu(\mathscr{S}_\Omega\mathscr{R}_\lambda^a(\varphi;\ d));\ \nu)$, where by $A_\nu(\mathscr{S}_\Omega(\varphi,\ d))$ we understand the analytic continuation with respect to d of all the Feynman amplitudes (2). In what follows, we shall for convenience retain the usual notation of the Hamiltonians but understand them as generalized (for this we shall also give the label with them).

We write the renormalization-group parameter a in the form $a=6-\delta$ and the parameter ν of the "fractional (complex) dimension" in the form $\nu = 4 - \varepsilon$. We also denote the Hamiltonians

$$H_0(\sigma)=\frac{1}{2}\int|\mathbf{k}_1|^2\delta(\mathbf{k}_1+\mathbf{k}_2)\sigma(\mathbf{k}_1)\sigma(\mathbf{k}_2)\,d^\nu\mathbf{k}_1\,d^\nu\mathbf{k}_2,$$

$$H_1(\sigma)=\int \delta(\mathbf{k}_1+\ldots+\mathbf{k}_4)\sigma(\mathbf{k}_1)\ldots\sigma(\mathbf{k}_4)d^\nu\mathbf{k}_1\ldots d^\nu\mathbf{k}_4,\quad H_2(\sigma)=\int \delta(\mathbf{k}_1+\mathbf{k}_2)\sigma(\mathbf{k}_1)\sigma(\mathbf{k}_2)d^\nu\mathbf{k}_1\, d^\nu\mathbf{k}_2.$$

It can be directly verified that there exists a "Gaussian" branch of generalized Hamiltonians invariant with respect to the actions of the renormalization group: $\mathscr{R}^{6-\varepsilon}_{\chi,\lambda}$ $(H_0;\, 4-\varepsilon)=(H_0;\, 4-\varepsilon)$. For $\varepsilon = 0$ there appears on this branch in the spectrum of the differential of the renormalization group an eigen Hamiltonian with eigennumber 1. This eigen Hamiltonian has the form $(:H_1:_{\Delta\chi};\, 4)$, where $::_{\Delta\chi}$ is the Wick operation with respect to the Gaussian measure with propagator $\Delta\chi\equiv|\mathbf{k}|^2\chi(\mathbf{k})$. One can therefore expect that for $\varepsilon = 0$ a new non-Gaussian branch will separate from the Gaussian branch of fixed points of the renormalization group. This non-Gaussian branch of fixed points was first discovered and analyzed in the first two orders of perturbation theory in [7]. In the present paper, we investigate the complete perturbation-theory series for the non-Gaussian branch of fixed points.

PROPOSITION 1. Let $H = H_0 + H_{int}$ and $\chi(\mathbf{k})$ be the characteristic function of the ball Ω, $\chi_\lambda(\mathbf{k})=\chi(\mathbf{k}/\lambda)$. Then the action of the renormalization group on the generalized Hamiltonian $(H;\, 4 - \varepsilon)$ is given by

$$\mathscr{R}^{6-\delta}_{\chi,\lambda}(H;4-\varepsilon)=(H_0-:\exp(-\mathscr{R}^{6-\delta}_{\lambda}H_{int}-(\lambda^{\varepsilon-\delta}-1)H_0):_{-\Delta(\chi_\lambda-\chi)};4-\varepsilon), \tag{3}$$

where $:^{c}_{-\Delta(\chi_\lambda-\chi)}$ is the connected Wick operation with propagator $-\Delta(\chi_\lambda-\chi)\equiv-|\mathbf{k}|^2(\chi_\lambda(\mathbf{k})-\chi(\mathbf{k}))$.

In calculations in accordance with (3) it is convenient to use as $\chi(\mathbf{k})$ a function of the class C_0^∞. In this case, formula (3) determines a smoothed renormalization group.

By analogy with [8], we shall seek the non-Gaussian branch of the fixed points of the renormalization group in the space of projection Hamiltonians. We recall the corresponding definitions. Let $H = H_0 + H_{int}$ be the Hamiltonian of a Gibbs field in the complete space $\mathbb{R}^d$. By the projection of H onto Ω we understand the Hamiltonian on Ω defined by

$$\mathscr{P}_\Omega(H)=H_0-:\exp(-H_{int}):^{c}_{-\Delta(1-\chi)}, \tag{4}$$

where $:^{c}_{-\Delta(1-\chi)}$ is the connected Wick operation with propagator $-\Delta(1-\chi)\equiv|\mathbf{k}|^2(1-\chi(\mathbf{k}))$. The operation of analytic continuation with respect to the dimension parameter makes it possible to consider projection Hamiltonians in spaces of complex dimension d.

PROPOSITION 2. The action of the renormalization group on the generalized projection Hamiltonian (4) is given by

$$\mathscr{R}^{6-\delta}_{\chi,\lambda}(H_0-:\exp(-H_{int}):^{c}_{-\Delta(1-\chi)};4-\varepsilon)=(H_0-:\exp(-\mathscr{R}^{6-\delta}_{\lambda}H_{int}-(\lambda^{\varepsilon-\delta}-1)H_0):^{c}_{-\Delta(1-\chi)};4-\varepsilon).$$

Thus, a projection Hamiltonian is carried by the action of the renormalization-group transformation to a projection Hamiltonian.

2. Invariant Manifolds of the Wilson Renormalization Group

The projection Hamiltonians of the form (4) contain in their expansion in a series with respect to Feynman diagrams divergent integrals, as is well known. If the Hamiltonian H_{int} is a linear combination of the Hamiltonians H_0, H_1, H_2, these divergences can be controlled by means of some scheme of renormalization theory. We shall consider the scheme of dimensional renormalization with minimal subtractions. We merely note that when applied to generalized Hamiltonians we shall understand the procedure of dimensional renormalization in the following way.

Dimensional renormalization of a Feynman amplitude usually reduces to three stages: 1) the Feynman amplitude is expanded in a Laurent series in ε; 2) a recursive procedure of subtractions is made (for the primitively divergent diagrams, it reduces simply to subtraction of the leading part of the Laurent series in ε); 3) ε is set equal to zero. For renormalization of a generalized projection Hamiltonian, we shall omit the final stage. We shall denote the procedure of dimensional renormalization by the letters D. R.

LEMMA (counterterm formula):

$$(\mathrm{D.R.}:\exp(-(u_0H_0+u_1H_1+u_2H_2)):^{c}_{-\Delta(1-\chi)};4-\varepsilon)=(:\exp(-(\widetilde{w}_0H_0+\widetilde{w}_1H_1+\widetilde{w}_2H_2)):^{c}_{-\Delta(1-\chi)};4-\varepsilon),$$

where

$$\widetilde{w}_0=(1+2u_0)w_0(u_1(1+2u_0)^{-2})+u_0,\quad \widetilde{w}_1=(1+2u_0)^2w_1(u_1(1+2u_0)^{-2}),\quad \widetilde{w}_2=u_2w_2(u_1(1+2u_0)^{-2}).$$

Here

$$w_i(z)=\sum_{n=0}^{\infty} a_n^{(i)} z^n,\quad i=0,1,2,$$

where $a_n^{(i)}$, $n=0, 1,\ldots,$ are polynomials in ε^{-1} determined from the relations

$$\Lambda(:H_1{}^n:{}^{c}_{-\Delta(1-\chi)})=a_n^{(0)}H_0+a_n^{(1)}H_1,\quad a_0^{(0)}=a_1^{(0)}=a_0^{(1)}=0,$$

$$\Lambda(:H_1{}^nH_2:_{-\Delta(1-\chi)})=a_n^{(2)}H_2,\quad a_1^{(1)}=a_0^{(2)}=1,$$

where Λ is the operator of "last subtraction" (see [9]).

The proof of the lemma follows from the counterterm formula (see [9]) and simple combinatorics.

We denote for brevity the renormalized generalized projection Hamiltonian $(H_0-\text{D.R.}:\exp(-(u_0H_0+u_1H_1+u_2H_2)):{}^{c}_{\Delta(1-\chi)};\ 4-\varepsilon)$ by $H(u_0, u_1, u_2;\ \varepsilon)$. Let $\lambda = \exp\tau$.

THEOREM. The action of the renormalization-group transformation $\mathcal{R}^{6-\delta}_{\chi,\exp\tau}$ on the manifold of Hamiltonians $H(u_0, u_1, u_2;\ \varepsilon)$ is given by

$$\mathcal{R}^{6-\delta}_{\chi,\exp\tau}H(u_0,u_1,u_2;\varepsilon)=H(u_0(\tau),u_1(\tau),u_2(\tau);\varepsilon), \tag{5}$$

where

$$\frac{du_i}{d\tau}=\rho_i(u_0,u_1,u_2;\varepsilon),\quad u_i(0)=u_i,\quad i=0,1,2,\quad \rho_0={}^1/{}_2(1+2u_0)(\varepsilon-\delta-\alpha_0),$$
$$\rho_1=2u_1(\varepsilon-\delta-\alpha_0)+(1+2u_0)^2\alpha_1,\quad \rho_2=u_2(2+\varepsilon-\delta-\alpha_2). \tag{6}$$

Here

$$\alpha_0=\varepsilon\frac{w_1w_0'}{w_1'(w_0+{}^1/{}_2)-2w_1w_0'},\quad \alpha_1=\varepsilon\frac{w_1(w_0+{}^1/{}_2)}{w_1'(w_0+{}^1/{}_2)-2w_1w_0'},$$

$$\alpha_2=\frac{w_2'}{w_2}\alpha_1,\quad w_i=w_i(u_1(1+2u_0)^{-2}),\quad i=0,1,2.$$

For $\alpha_0, \alpha_1, \alpha_2$ we have the representations

$$\alpha_0(z)=z^2f(z),\quad f(z)=\sum_{n=0}^{\infty}p_nz^n,\quad \alpha_1(z)=z(\varepsilon-g(z)),\quad g(z)=\sum_{n=1}^{\infty}q_nz^n,\quad \alpha_2(z)=\sum_{n=1}^{\infty}r_nz^n,$$

where the coefficients p_n, q_n, r_n are constants that do not depend on ε or $z = u_1(1 + 2u_0)^{-2}$.

The proof of this theorem is similar to the proof of the basic theorem in [8].

In accordance with (6), the equations for the fixed point have the form

$$\rho_i=0,\ i=0,\ 1,\ 2. \tag{7}$$

Without loss of generality, we can assume that $u_0 = 0$, since the parameter u_0 occurs in the equations only in the combination $u_1(1 + 2u_0)^{-2}$ and merely renormalizes the variable u_1 (we introduced u_0 into the theory to give the dynamical equations a more natural form).

Solving Eqs. (7), we obtain

1) $u_1 = 0$, $u_2 = 0$, $\varepsilon = \delta$. This solution corresponds to the Gaussian branch of fixed points;

2) $u_2 = 0$, $u_1 = u_1^*$ is a solution of the equation $\varepsilon=g(u_1)$, $\delta=\varepsilon-\alpha_1(u_1^*)$. This solution describes the non-Gaussian branch.

In conclusion, we compare our notation with the notation that is usually employed in the approach to the description of fixed points based on the equations of the field renormalization group or the Callan–Symanzik equations, which have a similar structure. We give here the correspondence between the quantities that determine the dynamics in the representation of the Wilson renormalization group and the quantities that specify the coefficients of the renormalization-group equation for the vertex functions in the traditional formalism of quantum field theory for the variant of dimensional renormalization with minimal subtractions (for more details about these equations, see [3]). Up to the replacement of u_1 by $u_1/4!$, the following relations hold:

$w_1(1 + 2w_0)^{-2}$ is identical to the renormalized coupling constant;

$1 + 2w_0$ is identical to Z_φ, the renormalization of the wave function;

$-\alpha_1(u_1)$ is identical to the Gell-Mann–Low function $\beta(u_1)$;

$-\alpha_0(u_1)$ is identical to $\gamma_\varphi(u_1)=\beta(u_1)\dfrac{\partial \ln Z_\varphi}{\partial u_1}$;

$\alpha_2(u_1)$ is identical to $\gamma_{\varphi^2}(u_1)=-\beta(u_1)\dfrac{\partial \ln Z_{\varphi^2} Z_\varphi}{\partial u_1}$,

where Z_{φ^2} is the renormalization constant of the wave operator φ^2 (as regards the notation, see [3]).

We also note that $-\alpha_0(u_1^*) = \eta$ is twice the anomalous dimension for the non-Gaussian fixed point. The leading eigenfunction of the differential of the renormalization group $\mathscr{D}^{6-\delta}_{\chi,\lambda}$ on the non-Gaussian branch of fixed points has the form

$$(\text{D. R.}: H_2 \exp(-u_1^*(\varepsilon) H_1): {}^{\varepsilon}_{-\Delta(1-\chi)}; 4-\varepsilon).$$

The highest eigenvalue of the differential of the renormalization group is $\lambda^{2+\alpha_0(u_1^*)-\alpha_2(u_1^*)}$ and in accordance with the theory of the Wilson renormalization group the value of the critical exponent ν is $(2+\alpha_0(u_1^*)-\alpha_2(u_1^*))^{-1}$. The remaining exponents can be found from scaling theory.

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