LIMIT GIBBS DISTRIBUTIONS FOR THE ISING MODEL ON HIERARCHICAL LATTICES

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UDC 519.21

1. Introduction

In the present paper we shall consider systems of binary dependent random variables which were first studied in the physical papers of Migdal [1] and Kadanoff [2], as a simplification of the widely known Ising model on the integral lattice Z^d. One should note that the Ising model on Z^d is a quite complcated system of random variables and for $d > 2$ the answer to many important questions for it such as the complete description of limit Gibbs distributions, the calculation of the singularities of thermodynamic functions in a neighborhood of a critical point, etc., is unknown. In connection with this many authors have introduced various modifications of the Ising model, mean field model, spherical model, Dyson's hierarchical model, etc. The results found for the simplified models let one advance essentially in the understanding of the general situation in the Ising model on Z^d.

The Ising model on hierarchical lattices is one of the most widely applied modifications of the classical model, since it preserves the property of the Ising model on Z^d of interaction of closest neighbors. This lets one assume that phase transition in the simplified model is similar in its properties to phase transition in the classical Ising model. On the other hand, such a model admits different generalizations, which are of great interest for applications, such as, for example, the Potts model, the XY-model, the gauge model.

The system of recurrence equations which were first introduced in Migdal [1] as an approximation to the system of renorm-group equations for the Ising model on Z^d, is at the base of the study of the Ising model on HL. In later papers of Berker and Ostlund for the Potts model [3] and Bleher and Zalys for the XY-model [4] it was shown simultaneously and independently that the approximate equations of the renorm group found by Migdal are exact for the corresponding models on a diamond-shaped hierarchical lattice (DHL). It is true that different terminology was used in these papers, but the name "hierarchical lattice" appeared in Griffiths and Kaufman [5], in which general definitions of HL were given and their most important properties were considered (cf. [6]). In particular, the concept of infinite HL was defined there and it was shown that there exists a continuum of inequivalent infinite HL. Recently a large number of papers [7-11] have appeared, in which various HL and spin models on them are considered as self-sufficient objects of study.

The basic goal of our paper is the proof of the existence of limit distributions for the ferromagnetic Ising model on infinite DHL. We prove that for low temperatures and zero external field, there exist exactly two extreme Gibbs limit distributions, and in other cases the Gibbs distribution is unique.

The definition of Ising model on HL is given in Sec. 2, and in Sec. 3 infinite HL are described, the limit Gibbs distributions on them are defined, and the basic theorem is formulated. We give the proofs of all assertions of the paper in Secs. 5 and 6, and Sec. 4 is devoted to the study of various recurrence equations and operators, which arise in the model considered. To conclude the paper we consider the possibility of generalizing the results found for the study of other models on hierarchical lattices.

M. V. Keldysh Institute of Applied Mathematics, Academy of Sciences of the USSR. Institute of Mathematics and Cybernetics, Academy of Sciences of the Lithuanian SSR. Translated from Litovskii Matematicheskii Sbornik (Lietuvos Matematikos Rinkinys), Vol. 28, No. 2, pp. 252-268, April-June, 1988. Original article submitted May 21, 1987.

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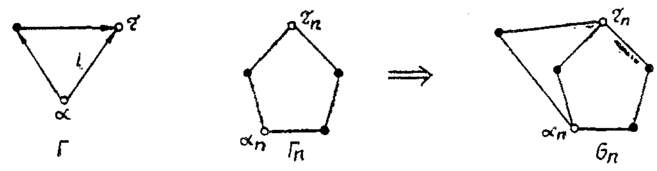

Fig. 1

2. The Ising Model on HL

In the most general sense an HL is a connected graph which is constructed recursively according to given rules. A zeroth order HL Γ_0 is two vertices joined by one edge. Let an oriented, connected graph Γ with two different distinguished vertices α and τ be fixed. These vertices are called outer and the graph Γ, generating. We write a recursive procedure for the construction of the graph Γ_{n+1} from the graph Γ_n.

Let us assume that on the graph Γ_n two outer vertices α_n and τ_n are distinguished, and we denote by $V(\Gamma)$, $V(\Gamma_n)$, and $L(\Gamma)$, $L(\Gamma_n)$ the sets of vertices and edges of the graphs Γ and Γ_n respectively.

We define the operation of "attaching" the graph Γ_n to the graph Γ along the edge $\ell \in L(\Gamma)$. Intuitively, this operation consists of replacing the edge ℓ by the graph Γ_n, where the first distinguished vertex of the graph Γ_n is identified with the beginning of the edge ℓ, and the second, with its end. Formally, as a result of the operation of attaching, there arises a graph $G_n = G_n(l, \Gamma_n, \Gamma)$ with vertices $v(p) \in V(G_n) = \{V(\Gamma) \cup V(\Gamma_n)\} \setminus \{s(l), f(l)\}$, where $s(\ell)$ and $f(\ell)$ are the beginning and end of the edge ℓ, and edges $l(g) \in L(G_n) = \{L(\Gamma) \cup L(\Gamma_n)\} \setminus \{l\}$, where edges which are incident in Γ to the vertices $s(\ell)$ and $f(\ell)$ (i.e., ones which go to or issue from these vertices) are considered incident in G_n to the vertices α_n and $\tau_n \in V(\Gamma_n)$ respectively (cf. Fig. 1).

Under the attaching operation the outer vertices of the graph Γ are considered to be the outer vertices of the graph G_n. An (n + 1)-st order HL is defined by successively attaching a graph Γ_n along all edges of the graph Γ (the result is independent of the successive choices of edges ℓ). As an example we consider the HL whose generating graph is a rhombus with opposite outer vertices (cf. Fig. 2). In what follows, to be definite, we shall basically consider precisely such lattices, called diamond shaped.

We define the Ising model on an HL in the usual way, by associating with each vertex $i \in V(\Gamma_n)$ a random variable $\sigma_i = \pm 1$ and defining the energy (Hamiltonian) of a configuration $\sigma^n = \{\sigma_i, i \in V(\Gamma_n)\}$ by

$$H_n(\sigma^n) = -\sum_{\langle i, j\rangle \in V(\Gamma_n)} \left(J\sigma_i\sigma_j + \frac{1}{2} h(\sigma_i + \sigma_j)\right), \tag{2.1}$$

where the summation is over all pairs of vertices i and j, joined by an edge (over all closest neighbors), and $J > 0$ and $h \in R$ are parameters of the model, the interaction constant and exterior field, respectively (the condition $J > 0$ means a ferromagnetic model).

The Gibbs distribution on an HL is the measure

$$\mu_n(\sigma^n; T, h) = \Xi_n^{-1}(T, h) \exp\left(-T^{-1} H_n(\sigma^n)\right), \tag{2.2}$$

where $T > 0$ is a basic parameter of the model, the temperature, and

$$\Xi_n(T, h) = \sum_{\sigma^n} \exp\left(-T^{-1} H_n(\sigma^n)\right)$$

is the large statistical sum.

3. Limit Gibbs Distributions

First we define infinite HL. By definition of the attaching operation, for each $\ell \in L(\Gamma)$ we have an imbedding $\pi_l : \Gamma_n \to \Gamma_{n+1}$. Suppose given an infinite sequence $I = \{l_1, l_2, \dots\}$, whose elements are edges of the generating graph Γ, i.e., $I: N \to L(\Gamma)$. We consider the sequence of

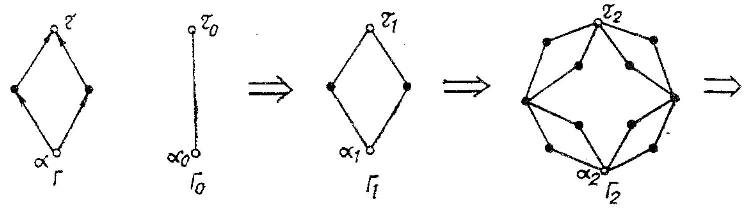

Fig. 2

imbeddings $\pi_{l_1}, \pi_{l_2}, \ldots : \Gamma_0 \to \Gamma_1 \to \ldots$ Then the inductive limit $\varinjlim_{n\to\infty} \Gamma_n = \Gamma_\infty(I)$ is called the infinite hierarchical lattice corresponding to the sequence I.

It is noted in [5] that for different I and I′ the graphs $\Gamma_\infty(I)$ and $\Gamma_\infty(I')$ can be non-isomorphic. We discuss this question in more detail. It is easy to see that if $I=\{l_1, l_2, \ldots\}$ and $I'=\{l'_1, l'_2, \ldots\}$ coincide, starting from some place, i.e., $\ell_j = \ell'_j$ if $j \geq N$ for some $0 < N < \infty$, then the graphs $\Gamma_\infty(I)$ and $\Gamma_\infty(I')$ are isomorphic. Now we consider a DHL. On the rhombus Γ we choose one of the two paths leading from the vertex α to the vertex τ and we denote the edges which lie on this path by the numbers 0 and 1, and those on the other path, by 10 and 11. It is easy to see that if in the sequence $I=\{l_1, l_2, \ldots\}$ one replaces some edge $\ell_i = 0$ by 10 and conversely, then for the sequence I′ which one gets, the graphs $\Gamma_\infty(I)$ and $\Gamma_\infty(I')$ will be isomorphic. Since the analogous assertion is also valid upon replacing 1 by 11 and conversely, it suffices in what follows to consider only the sequence $I=\{l_i=0,\ 1;\ i=1,\ 2, \ldots\}$. If one defines the parity transformation $\gamma: I=\{l_1, l_2, \ldots\} \to I'=\{1-l_1, 1-l_2, \ldots\}$, then again, due to symmetry, $\Gamma_\infty(I)$ and $\Gamma_\infty(\Gamma)$ will be isomorphic. The following general result holds.

<u>Proposition 3.1.</u> If at an infinite number of places the sequence $I=\{l_i=0,\ 1;\ i=1,\ 2, \ldots\}$ does not coincide with $I'=\{l'_i=0,\ 1;\ i=1,\ 2, \ldots\}$, or with $\gamma I'$, then $\Gamma_\infty(I)$ and $\Gamma_\infty(I')$ are not isomorphic. The proof of this assertion is given in Sec. 6 (in [5] it is cited without proof).

We define the limit Gibbs distribution on an infinite HL. We note that it is the application of the general definition (cf. [12]) to the case of HL.

We set $V=V(\Gamma_\infty(I))$, $L=L(\Gamma_\infty(I))$, $V_N=V(\Gamma_N(I))$. Let $\hat\sigma^N=\{\hat\sigma_i, i\in V\setminus V_N\}$ be an arbitrary configuration on $V\setminus V_N$. The function

$$H_N(\sigma^N \mid \hat\sigma^N) = H_N(\sigma^N) - \sum_{\langle i,j\rangle:\ i\in V_N,\ j\in V\setminus V_N} J\sigma_i\hat\sigma_j, \tag{3.1}$$

where $H_N(\sigma^N)$ is defined in (2.1), is called the Hamiltonian with boundary conditions $\hat\sigma^N$. The Gibbs distribution with boundary conditions $\hat\sigma^N$ is the measure

$$\mu_N(\sigma^N \mid \hat\sigma^N) = \Xi_N^{-1}(\hat\sigma^N)\exp\{-T^{-1}H_N(\sigma^N, \hat\sigma^N)\}, \tag{3.2}$$

where $\Xi_N(\hat\sigma^N)$ is the large statistical sum of the model with boundary conditions.

<u>Definition.</u> By the limit Gibbs distribution we mean the limit $\mu = \lim_{N\to\infty}\mu_N(\cdot \mid \hat\sigma^N)$, if it exists for some sequence of boundary conditions $\hat\sigma^N$ (the limit is understood in the sense of convergence of finite-dimensional distributions).

We call the sequence $I=\{l_i=0,\ 1;\ i=1,\ 2, \ldots\}$ degenerate, if there exists an index $N > 0$ such that $\ell_i = 0$ or $\ell_i = 1$ for $i \geq N$ and nondegenerate otherwise. We denote by μ^+ and μ^- the limit Gibbs distributions (if they exist) with boundary conditions $\hat\sigma_i \equiv +1$ and $\hat\sigma_i \equiv -1$ respectively.

The following theorem is the basic result of the present paper.

<u>THEOREM.</u> Let I be any nondegenerate sequence. Then for all values $T > 0$ and h the limit distributions μ^+ and μ^- exist for the Ising model on the DHL $\Gamma_\infty(I)$. In addition, any limit Gibbs distribution μ is a convex combination of them $\mu = a\mu^+ + (1-a)\mu^-$, $0 \leqslant a \leqslant 1$.

There exists a critical point $T_c > 0$ such that for $h = 0$, μ^+ coincides with μ^- if $T \geq T_c$ and does not coincide with it if $T < T_c$. For $h \neq 0$ and any value $T > 0$ the limit Gibbs distribution is unique (i.e., μ^+ coincides with μ^-).

We prove this theorem in Sec. 5, and now we make the following remarks.

1. As follows from Aizenman [14] and Higuchi [15], the analog of the theorem formulated is valid for the classical Ising model on Z^2. A complete description of the limit distributions for d > 2 has not yet been found although there are various partial results here (cf. [16]).

2. If the sequence I is degenerate, then one outer vertex of the graph $\Gamma_\infty(I)$ has an infinite number of neighbors. Then in (3.1) the sum is divergent and the limit Gibbs distribution is not generally defined.

4. Recurrence Equations

We define the conditional statistical sum (CSS) of the Ising model on a DHL

$$Z_n(\sigma', \sigma'', T, h)=\sum_{\sigma^n:\sigma_{\alpha_n}=\sigma', \sigma_{\tau_n}=\sigma''} \exp\{-T^{-1}H_n(\sigma^n)\}, \tag{4.1}$$

where the summation is over all configurations of spins $\sigma^n=\{\sigma_i, i\in V_n\}$ under the condition that the values of the spins on the outer vertices α_n and τ_n are fixed. We set $P_n=Z_n(+1, +1), Q_n=Z_n(+1, -1)=Z_n(-1, +1), R_n=Z_n(-1, -1)$ (we note) that all of them depend on T and h, but later, for brevity, we shall not indicate this. We note that the DHL, Γ_{n+1} consists of four sublattices Γ_n "attached" along outer vertices. Two of these vertices are the outer vertices α_{n+1}, τ_{n+1} of the lattice Γ_{n+1}. We denote the other two by $\bar\alpha_{n+1}, \bar\tau_{n+1}$. We calculate the CSS $Z_{n+1}(\sigma', \sigma'')$ in two steps. First we calculate it under the condition that the random variables σ_i are fixed not only for $i=\alpha_{n+1}, \tau_{n+1}$, but also for $i=\bar\alpha_{n+1}, \bar\tau_{n+1}$. Then

$$Z_{n+1}(\sigma', \sigma'', \bar\sigma', \bar\sigma'')=Z_n(\sigma', \bar\sigma')\,Z_n(\sigma', \bar\sigma'')\,Z_n(\bar\sigma', \sigma'')\,Z_n(\bar\sigma'', \sigma'').$$

Afterwards we sum over $\bar\sigma'$, $\bar\sigma''$

$$Z_{n+1}(\sigma', \sigma'')=\sum_{\bar\sigma', \bar\sigma''} Z_{n+1}(\sigma', \sigma'', \bar\sigma', \bar\sigma'') = \sum_{\bar\sigma', \bar\sigma''} Z_n(\sigma', \bar\sigma')\,Z_n(\sigma', \bar\sigma'')\,Z_n(\bar\sigma', \sigma'')\,Z_n(\bar\sigma'', \sigma'').$$

Substituting the concrete values of $\sigma' = \pm1$ and $\sigma'' = \pm1$ into this equation we get the recurrence equations for P_n, Q_n, R_n

$$P_{n+1}=P_n^4+2P_n^2Q_n^2+Q_n^4=(P_n^2+Q_n^2)^2;$$
$$Q_{n+1}=Q_n^2(P_n+R_n)^2; \quad R_{n+1}=(Q_n^2+R_n^2)^2.$$

The initial lattice Γ_0 contains two vertices α_0, τ_0 in all, so

$$Z_0(\sigma', \sigma'')=\exp\left\{T^{-1}\left[J\sigma'\sigma''+\frac{1}{2}h(\sigma'+\sigma'')\right]\right\}.$$

Substituting $\sigma' = \pm1$, $\sigma'' = \pm1$ we get the initial conditions for these equations

$$P_0=\exp\left(T^{-1}(J+h)\right); \quad Q_0=\exp(-T^{-1}J); \quad R_0=\exp\left(T^{-1}(J-h)\right).$$

We see

$$z_n=\sqrt{R_n/P_n}, \quad t_n=Q_n/\sqrt{R_nP_n}, \tag{4.2}$$

and in these variables we write the recurrence equations of the renorm group for the Ising model on a DHL

$$z_{n+1}=\frac{z_n^2(z_n^2+t_n^2)}{1+z_n^2t_n^2}, \quad t_{n+1}=\frac{t_n^2(1+z_n^2)^2}{(z_n^2+t_n^2)(1+z_n^2t_n^2)} \tag{4.3}$$

with the initial conditions $z_0=\exp(-T^{-1}h)$, $t_0=\exp(-2T^{-1}J)$.

For h = 0 for all n, $z_n \equiv 1$ and

$$t_{n+1}=\Psi(t_n)\equiv 4t_n^2/(1+t_n^2)^2. \tag{4.4}$$

The map $t \to \Psi(t)$ has two stable fixed points t = 0 and t = 1 and one unstable fixed point $t_c \approx 0,2955977\ldots$ on the segment [0, 1]. In addition the sequence $t_{n+1}=\Psi(t_n)$ converges to t = 0, if $t_0 < t_c$ and to t = 1, if $t_0 > t_c$.

The recurrence method of constructing HL also lets one get an equation which connects the Gibbs measures for lattices of different orders. Let $\bar\sigma^N$ be fixed arbitrary boundary conditions on $\Gamma_\infty(I)\setminus\Gamma_N(I)$ (I is a fixed nondegenerate sequence), and $\mu_N(\sigma^N\mid\bar\sigma^N)$ is the Gibbs distribution with these boundary conditions, defined by (3.1) and (3.2). For any $n < N$ we choose a sublattice $\Gamma_n(I) \subset \Gamma_N(I)$, we denote by $\mu_n(\sigma^n)$ the Gibbs measure on $\Gamma_n(I)$ with zero boundary conditions, and we rewrite the Hamiltonian (3.1) as a sum

$$H_N(\sigma^N \mid \hat{\sigma}^N) = H_n(\sigma^n) + \hat{H}_{n,N}(\sigma^n, \sigma^{N \setminus n} \mid \hat{\sigma}^N),$$

where only interactions between spins defined on V_n occur in the first summand, and all the rest in the second. It follows from this that

$$\mu_N(\sigma^N \mid \hat{\sigma}^N) = L_{n,N}\mu_n(\sigma^n)\exp\{-T^{-1}\hat{H}_{n,N}(\sigma^n, \sigma^{N \setminus n} \mid \hat{\sigma}^N)\}. \tag{4.5}$$

Summing both sides of the last equation over all spin variables defined on $V_N \setminus V_n$ we get the basic formula for proving the existence of limit distributions,

$$\mu_{n,N}(\sigma^n \mid \hat{\sigma}^N) = L_{n,N}\mu_n(\sigma^n) F_{n,N}(\sigma^n \mid \hat{\sigma}^N), \tag{4.6}$$

where

$$F_{n,N} = S_{n+1}^{(i_{n+1})} S_{n+2}^{(i_{n+2})} \ldots S_N^{(i_N)} F_{N,N}, \quad i_k = 0,\ 1, \tag{4.7}$$

and the functions

$$F_{N,N}(\sigma^N) = F_{N,N}(\sigma^N \mid \hat{\sigma}^N) = L(\hat{\sigma}^N)\exp\Big(T^{-1} \sum_{\langle i,j\rangle:\, i \in V_N,\, j \in V \setminus V_N} J\sigma_i\hat{\sigma}_j\Big). \tag{4.8}$$

We consider (4.6) and (4.7) in more detail. Since on DHL the interior spins are connected with the boundary conditions only though the outer vertices α_N and τ_N the function $F_{N,N}$ depends only on σ_{α_N}, σ_{τ_N} and can assume four values in all: $F_{N,N}(+,+)$, $F_{N,N}(+,-)$, $F_{N,N}(-,+)$, $F_{N,N}(-,-)$, which lets one consider it as a four-dimensional vector of values.

In order to explain how the operators $S_n^{(i_n)}$, are constructed, we set $n = N - 1$ and we rewrite the Hamiltonian from (4.5)

$$\hat{H}_{N-1,N}(\sigma^{N-1}, \sigma^{N \setminus N-1} \mid \hat{\sigma}^N) = H_{N-1,N}(\sigma^{N-1}, \sigma^{N \setminus N-1}) + J \sum_{\langle i,j\rangle:\, i \in V_N,\, j \in V \setminus V_N} \sigma_i\hat{\sigma}_j$$

so that in $H_{N-1,N}$ only constraints on spins from $V_N \setminus V_{N-1}$ occur. Then, as in (4.6)-(4.8), we get

$$F_{N-1,N} = L_{N-1,N} \sum_{\sigma^{N \setminus N-1}} \exp\{-T^{-1}H_{N-1,N}(\sigma^{N-1}, \sigma^{N \setminus N-1})\} F_{N,N}.$$

Since $H_{N-1,N}(\sigma^{N-1}, \sigma^{N \setminus N-1})$ is the sum of Hamiltonians of three sublattices of order $N - 1$ (except for $\Gamma_{N-1}(I)$, which belongs to the $\Gamma_n(I)$ we have chosen), one can represent the last equation as

$$F_{N-1,N}(\sigma_{\bar{\alpha}_{N-1}}, \sigma_{\bar{\tau}_{N-1}}) = L_{N-1,N} \sum_{\sigma_{\alpha_{N-1}},\, \sigma_{\tau_{N-1}}} F_{N,N}(\sigma_{\alpha_N}, \sigma_{\tau_N}) \prod_{i=1}^{3} \sum_{\sigma^{N-1}}{}' \exp\{-T^{-1}H_{N-1}(\sigma^{N-1,i})\},$$

where the first sum is taken over all pairs of values of spins at the vertices α_{N-1} and τ_{N-1}, which do not coincide with the outer vertices $\bar{\alpha}_{N-1}$ and $\bar{\tau}_{N-1}$ of the sublattice $\Gamma_{N-1}(I)$. and the second is over spins σ_i of one of the three sublattices with fixed boundary spins. Then

$$L_{N-1,N}^{-1} F_{N-1,N}(\bar{\alpha}_{N-1}, \bar{\tau}_{N-1}) = \sum_{\alpha_{N-1},\, \tau_{N-1}} Z_{N-1}(\alpha_{N-1}, \tau_{N-1})\ Z_{N-1}(\tau_{N-1}, \bar{\alpha}_{N-1}) Z_{N-1}(\alpha_{N-1}, \bar{\tau}_{N-1}) F_{N,N}(\bar{\alpha}_{N-1}, \alpha_{N-1}). \tag{4.9}$$

Here, for brevity we write $\alpha_{N-1}, \tau_{N-1}, \ldots$ instead of $\sigma_{\alpha_{N-1}}, \ldots$. (4.9) is valid for $\ell_N = 0$. If in the sequence I, $\ell_N = 1$, then it is necessary to replace $F_{N,N}(\bar{\alpha}_{N-1}, \alpha_{N-1})$ by $F_{N,N}(\bar{\tau}_{N-1}, \tau_{N-1})$ in the formula.

It is convenient to write the recurrence formula for the "vectors of boundary conditions" $F_{N,N}$ in matrix form: $F_{N-1,N} = S_N F_{N,N}$, where S_N is a square matrix of order 4. If $\ell_N = 0$, then up to a numerical factor the matrix S_N has the form

$$S_N^{(0)} = \begin{pmatrix} 1 + z_N^2 t_N^2 & z_N^2 t_N^2 + z_N^4 t_N^2 & 0 & 0 \\ z_N t_N + z_N^3 t_N^3 & z_N^3 t_N + z_N^5 t_N & 0 & 0 \\ 0 & 0 & z_N t_N + z_N^3 t_N & z_N^3 t_N^3 + z_N^5 t_N \\ 0 & 0 & z_N^2 t_N^2 + z_N^4 t_N^2 & z_N^4 t_N^2 + z_N^6 \end{pmatrix}, \tag{4.10}$$

where z_N and t_N are calculated from (4.3). For $\ell_N = 1$ the matrix S_N has analogous form, but with the second and third rows and columns interchanged, i.e., in this case S_N coincides up to a numerical factor with the matrix $S_N^{(1)} = KS_N^{(0)}K$, where

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Analogously, for all $m = n+1, \ldots, N$; $F_{m-1,N} = S_m F_{m,N}$, where $S_m = S_m^{(i_m)}$, $i_m = 0,1$ (we write the formulas for S_m up to a numerical factor (cf. the Remark in Sec. 5)), from which (4.7) follows.

For the rest of the paper we make a comment. If one replaces all factors $S_m^{(1)}$ in (4.7) by $KS_m^{(0)}K$, then since $K^2 = E$ we get the product of matrices $S_m^{(0)}$, between which, matrices K will be inserted, in the places where $l_m \neq l_{m+1}$. We combine each operator K with the operator $S_m^{(0)}$ which precedes it, and we denote $S_m^{(0)}K$ by $S_m^{(1)}$ again. Then we again get in (4.7) the product of operators $S_m^{(i_m)}$, where $i_m = 0$ if $l_m = l_{m+1}$, and $i_m = 1$ if $l_m \neq l_{m+1}$ (if $l_{N+1} = 1$, then before all the operators $S_m^{(i_m)}$ there will stand one operator K but it has no influence on the rest of the discussion connected with the proof of convergence of the vectors $F_{n,N}$). For reference we give the form of the matrix $S_N^{(1)} = S_N^{(0)}K$

$$S_N^{(1)} = \begin{pmatrix} 1 + z_N^2 t_N^2 & 0 & z_N^2 t_N^2 + z_N^4 t_N^2 & 0 \\ z_N t_N + z_N^3 t_N^3 & 0 & z_N^3 t_N + z_N^5 t_N & 0 \\ 0 & z_N t_N + z_N^3 t_N & 0 & z_N^3 t_N^3 + z_N^5 t_N \\ 0 & z_N^2 t_N^2 + z_N^4 t_N^2 & 0 & z_N^4 t_N^2 + z_N^6 \end{pmatrix}. \tag{4.11}$$

We consider some properties of the matrices (4.10) and (4.11) and their limits as $N \to \infty$

First let $h \neq 0$, $T > 0$. Then for variables z_N and t_N which satisfy (4.3), one has the following:

<u>LEMMA 4.1.</u> If $z_0 < 1$, $0 < t_0 \leq 1$ then $\lim\limits_{N\to\infty} (t_N, z_N) = (1, 0)$. Here: $|t_N - 1| + z_N \leq C \exp(-\gamma 2^N)$, where C, $\gamma > 0$.

It follows from this assertion that for $h \neq 0$, $\lim\limits_{N\to\infty} S_N^{(0)} = \lim\limits_{N\to\infty} S_N^{(1)} = S$, where for the limit matrix all elements except $s_{11} = 1$ are equal to zero.

For $h = 0$ (i.e., $z_N \equiv 1$) we set

$$\lim_{N\to\infty} S_N^{(i_N)} = S^{(i)}(t_0) = \begin{cases} S^{(i)}(0) & \text{for} \quad t_0 < t_c, \\ S^{(i)}(c) & \text{for} \quad t_0 = t_c, \\ S^{(i)}(1) & \text{for} \quad t_0 > t_c. \end{cases} \tag{4.12}$$

From (4.10) and (4.11) it is easy to find the form of these limit matrices. For $t_0 < t_c$, $S^{(0)}(0) = S^{(1)}(0) = S(0)$ is the matrix for which $s_{11}^{(0)} = s_{44}^{(0)} = 1$, and the other elements are equal to zero. For $t_0 > t_c$ there are two limit matrices $S^{(0)}(1)$ and $S^{(1)}(1)$, for which all the elements different from zero are equal. For $t_0 = t_c$ we also have two matrices $S^{(0)}(c)$ and $S^{(1)}(c)$, which are obtained from (4.10) and (4.11) for $z_N = 1$, $t_N = t_c$. In what follows we shall use the matrices $S^{(0)}(c)$ and $S^{(1)}(c)$, divided by $4t_c$. In this case $S^{(0)}(c)$ will have two eigenvalues $\lambda_1^{(0)} = \lambda_3^{(0)} = 1$ and $0 < \lambda_2^{(0)}, \lambda_4^{(0)} < 1$, and $S^{(1)}(c)$ will have one $\lambda_1^{(1)} = 1.0 < \lambda_2^{(1)}, \lambda_3^{(1)} < 1$ and $|\lambda_4| < 1$. All eigenvectors $f_i^{(0)}$ and $f_i^{(1)}$, corresponding to these values have the form (x, y, y, x) or (x, y, −y, −x); here $f_1^{(0)} = f_1^{(1)} = f_1$ and $f_2^{(0)} = f_2^{(1)} = f_2$. We note that the vectors of the transposed matrices $(S^{(0)}(c))^*$ and $(S^{(1)}(c))^*$, corresponding to the eigenvalues $(\lambda_1^{(0)})^* = (\lambda_1^{(1)})^* = 1$, are equal to $(f_1^{(0)})^* = (f_1^{(1)})^* = f_1^* = (1, t_c, t_c, 1)$. To describe the action of the product of the matrices $S^{(0)}(c)$ and $S^{(1)}(c)$ on a four-dimensional vector g, we need the following:

<u>LEMMA 4.2.</u> Among the terms of the sequence $\{i_n = 0, 1; n = 1, 2, \ldots\}$, let there be an infinite number of ones. Then

$$\lim_{N\to\infty} \left(\prod_{k=1}^{n} S^{(i_k)}(c) \right) g = \frac{(g, f_1^*)}{(f_1, f_1^*)} f_1.$$

To conclude this section we make some comments.

1. As already noted, the "vector of boundary conditions" is the four-dimensional vector of values $\mathbf{F}_{N,N}=(F_{N,N}(+,+), F_{N,N}(+,-), F_{N,N}(-,+), F_{N,N}(-,-))$. where each component is

$$F_{N,N}(i,j)=L_N(\hat{\sigma}^N)\exp\left\{J\left(i\sum_{k=1}^{2^N}\sigma_k^{(1)}+j\sum_{k=1}^{2^N}\sigma_k^{(2)}\right)\right\}, \tag{4.13}$$

where $\sigma_k^{(1)}, \sigma_k^{(2)}$ mean the values of the spins from $V\backslash V_N$ connected with the outer vertices α_N and τ_N, respectively.

2. In the course of the proof we shall use normalized boundary condition vectors, choosing the normalization so that $\|\mathbf{F}_{N,N}\|=\sum_{i=1}^{4}|\alpha_i|=1$. The norm of matrices used in the proof is $\|\mathbf{A}\|=\sum_{i<j}|a_{ij}|$; here all the matrices and vectors considered are bounded in this norm. Everywhere in what follows e_i denotes the unit vector with one in the i-th place.

3. One should note that the matrices $\mathbf{S}_m^{(i_m)}$ (cf. (4.10), (4.11)) have nonnegative coefficients and have the following property: if $x_i\geqslant 0$, $\sum_{i=1}^{4}x_i>0$ and

$$\mathbf{S}_m^{(i_m)}(x_1, x_2, x_3, x_4)=(y_1, y_2, y_3, y_4),$$

then $y_i\geqslant 0, \sum_{i=1}^{4}y_i>0$.

5. Proof of the Basic Theorem

All the arguments of this section will be given for the Ising model on an infinite DHL $\Gamma_\infty(I)$, where I is a fixed nondegenerate sequence so we shall not indicate the dependence on I. We divide the assertion of the basic theorem into two propositions, from whose validity the basic result follows.

Proposition 5.1. For $T < T_c$ and $h = 0$, there exist two extreme limit Gibbs distributions μ^+ and μ^- and any Gibbs limit distribution μ is a linear combination of μ^+ and μ^-.

Proposition 5.2. For $T \geq T_c$ and $h = 0$, and also for $h \neq 0$ and all values $T > 0$ there exists a unique limit Gibbs distribution.

The proof of these propositions reduces to the proof of convergence of the finite-dimensional distributions $\mu_{n,N}$ to a limit as $N \to \infty$ or in other words, the convergence of the functions $F_{n,N}(\sigma^n|\hat{\sigma}^N)$ to a nonzero limit (cf. (4.7)).

First we prove that one can choose a sequence of normalizing factors $\hat{L}_{n,N}>0$ so that

$$\lim_{N\to\infty}\hat{L}_{n,N}F_{n,N}(\sigma^n|\hat{\sigma}^N)=\hat{F}_{n,\infty}(\sigma^n)\neq 0$$

exists. Since $\mu_{n,N}(\sigma^n|\hat{\sigma}^N)$ is a probability measure,

$$\sum_{\sigma^n}\mu_{n,N}(\sigma^n|\hat{\sigma}^N)=\hat{L}_{n,N}^{-1}\sum_{\sigma^n}\mu_n(\sigma^n)\hat{L}_{n,N}F_{n,N}(\sigma^n|\hat{\sigma}^N)=1.$$

Consequently, the positive limit

$$\lim_{N\to\infty}\hat{L}_{n,N}=\lim_{N\to\infty}\left(\sum_{\sigma^n}\hat{L}_{n,N}F_{n,N}(\sigma^n|\hat{\sigma}^N)\mu_n(\sigma^n)\right)^{-1}=\hat{L}_{n,\infty}<\infty$$

exists; it follows from this that the limit

$$\lim_{N\to\infty}F_{n,N}(\sigma^n|\hat{\sigma}^N)=\lim_{N\to\infty}\hat{L}_{n,N}^{-1}\cdot\lim_{N\to\infty}\hat{L}_{n,N}F_{n,N}(\sigma^n|\hat{\sigma}^N)=\hat{L}_{n,\infty}^{-1}\hat{F}_{n,\infty}(\sigma^n)$$

exists and is not equal to zero. The arguments cited above permit us in what follows to choose an arbitrary normalization of the vectors $\mathbf{F}_{n,N}$ and operators $\mathbf{S}_n^{(i_n)}$. We shall denote all the factors which arise here by the same symbol $L_{k,j}$ and if nothing special is said, the proof is carried out "up to normalizing factors".

For the operators $\mathbf{S}_n^{(i_n)}$, $i_n=0, 1$, and the limit operators $\mathbf{S}^{(i)}(t_0)$, defined by (4.9), (4.10), and (4.12), we formulate the following assertion.

LEMMA 5.1. For $0 < z_n \le 1$ and $0 < t_n \le 1$ one has the following estimates:

1) $\| S_n^{(i_n)} - S^{(i)}(t_0) \| \le \varkappa^n$ for $n > n_0 > 0$, where $0 < \varkappa < 1$;

2) $\| S_{N+1}^{(i_{N+1})} S_{N+2}^{(i_{N+2})} \dots S_{N+n}^{(i_{N+n})} \| \le C$ for all n and N;

3) there exists an index m_0 such that for $N > m_0$,

$$\sup_n \| S_{N+1}^{(i_{N+1})} S_{N+2}^{(i_{N+2})} \dots S_{N+n}^{(i_{N+n})} - \underbrace{S^{(i)}(t_0) \dots S^{(i)}(t_0)}_{n} \| \le \nu^{N/2},$$

where $0 < \nu < 1$, $i_k = 0, 1$.

We shall not prove the validity of these estimates. We only note that 1) follows from (4.3) and the estimates of the rate of convergence of the numbers z_N and t_N to a limit as $N \to \infty$ and 2) and 3), from Lemma 3.2 of [13].

In the proof of the propositions, we shall consider the case of "coexistence of phases" (i.e., $h = 0$, $T < T_c$) in more detail. In the case of uniqueness of the Gibbs distribution, the proofs go by an analogous scheme.

Proof of Proposition 5.1. The proof of the existence of the limit Gibbs distribution, as already noted, is equivalent to the proof of the existence of

$$\lim_{N\to\infty} F_{n,N}(\sigma^n \mid \hat{\sigma}^N) = F_{n,\infty}(\sigma^n) \ne 0$$

under a sequence of boundary conditions $\hat{\sigma}^N$ we choose boundary conditions $\hat{\sigma}_i \equiv +1$, we denote the boundary condition vectors by F^+, and we prove the Cauchy criterion

$$\lim_{N', N'' \to\infty} \| F^+_{n,N''} - F^+_{n,N'} \| = 0, \quad N'' > N'. \tag{5.1}$$

Let $M = N'/2$. We consider the inequality

$$\| F^+_{n,N''} - F^+_{n,N'} \| \le \| S_{n+1}^{(i_{n+1})} \dots S_M^{(i_M)} \| \{ \| S_{M+1}^{(i_{M+1})} \dots S_{N'}^{(i_{N'})} -$$

$$- S^{N'-M} \| \| F^+_{N',N'} \| + \| S_{M+1}^{(i_{M+1})} \dots S_{N''}^{(i_{N''})} - S^{N''-M} \| \| F^+_{N'',N''} \| + \| S^{N''-M} F^+_{N'',N''} - S^{N'-M} F^+_{N',N'} \| \},$$

where $S = S^{(0)}(0) = S^{(1)}(0)$. Since $S^k = S$, from the assertions of Lemma 5.1 we get

$$\| F^+_{n,N''} - F^+_{n,N'} \| \le 2C\nu^{N'/2} + \| S(F^+_{N'',N''} - F^+_{N',N'}) \|.$$

From (4.13), considering Remark 2 of the preceding section, we have that for all $N > m_0$,

$$\| F^+_{N,N} - e_1 \| \le \nu_0^N, \quad 0 < \nu_0 < 1. \tag{5.2}$$

Hence $\| S F^+_{N,N} - e_1 \| = \| S(F^+_{N,N} - e_1) \| \le C_0 \nu_0^N$, and consequently,

$$\| F^+_{n,N''} - F^+_{n,N'} \| \le 2C\nu^{N'/2} + 2C_0\nu_0^{N'} \le C'\nu^{N'/2}, \tag{5.3}$$

from which (5.1) follows, i.e., the limit of the $F^+_{n,\infty}$ exists. Now we show that $F^+_{n,\infty} \ne 0$ (this is an important question, since we are not tracking the normalization of the vectors $F_{n,N}$). Using Lemma 5.1 and (5.2), we have

$$\| F^+_{n,N} - e_1 \| = \| (S_{n+1}^{(i_{n+1})} \dots S_N^{(i_N)} - S^{N-n}) F^+_{N,N} + S^{N-n}(F^+_{N,N} - e_1) \| \le \nu^{n/2} + C\nu_0^N,$$

whence, as $N \to \infty$,

$$\| F^+_{n,\infty} - e_1 \| \le \nu^{n/2}, \tag{5.4}$$

consequently, $\| F^+_{n_0,\infty} \| \ne 0$ for sufficiently large n_0. But since $F^+_{n,\infty} = S_{n+1}^{(i_{n+1})} \dots S_{n_0}^{(i_{n_0})} F^+_{n_0,\infty}$. Considering Remark 3 of Sec. 4, we have that $\| F^+_{n,\infty} \| > 0$.

Thus, we have proved the existence of the finite-dimensional distributions $\mu^+_{n,\infty}(\sigma^n)$ of the limit Gibbs distribution μ^+. Analogously one establishes the existence of the finite-dimensional distributions $\mu^+_{n,\infty}$. We note that μ^+ goes to μ^- under the map $\sigma_i \to -\sigma_i$. Just as one gets (5.4), one can get

$$\| F^-_{n,\infty} - e_4 \| \le \nu^{n/2},$$

showing that $F^+_{n,\infty} \ne F^-_{n,\infty}$ for sufficiently large n, so the finite-dimensional distributions $\mu^+_{n,\infty}$ and $\mu^-_{n,\infty}$ do not coincide, and $\mu^+ \ne \mu^-$.

To conclude we show that if for some sequence of vectors $\mathbf{F}_{N,N}$ a nonzero limit $\mathbf{F}_{n,\infty}$ exists, then

$$\mathbf{F}_{n,\infty}=a\mathbf{F}^{+}_{n,\infty}+(1-a)\mathbf{F}^{-}_{n,\infty},\quad 0\leqslant a\leqslant 1. \tag{5.5}$$

In the proof of (5.5), the following lemma is key.

LEMMA 5.2. Let $\mathbf{F}_{N,N}=\sum_{i=1}^{4}x_i^{(N)}\mathbf{e}_i$, where $x_i^{(N)}\geqslant 0$, $\sum_{i=1}^{4}x_i^{(N)}=1$. Then there exist numbers $L_{N-3,N}>0$ and $0\leqslant a_{N-3}\leqslant 1$ (depending on $\mathbf{F}_{N,N}$) and $N_0 > 0$ such that for $N > N_0$

$$\|L^{-1}_{N-3,N}\mathbf{F}_{N-3,N}-a_{N-3}\mathbf{e}_1-(1-a_{N-3})\mathbf{e}_4\|\leqslant\nu_1^N,\quad 0<\nu_1<1.$$

Proof. For simplicity of exposition we shall consider only the case when in the product $\mathbf{S}_{N-2}^{(i_{N-2})}\mathbf{S}_{N-1}^{(i_{N-1})}\mathbf{S}_N^{(i_N)}$ all the i are equal to zero. First let $\mathbf{F}_{N,N}=\mathbf{e}_i$.

If $\mathbf{F}_{N,N}=\mathbf{e}_1$, then by Lemma 5.1,

$$\|\mathbf{F}_{N-3,N}-\mathbf{e}_1\|=\|(\mathbf{S}^{(0)}_{N-2}\mathbf{S}^{(0)}_{N-1}\mathbf{S}^{(0)}_{N}-\mathbf{S}^3)\mathbf{e}_1\|\leqslant C\nu^N.$$

Analogously, if $\mathbf{F}_{N,N}=\mathbf{e}_4$, then $\|\mathbf{F}_{N-3,N}-\mathbf{e}_4\|\leqslant C\nu^N$. We consider the case $\mathbf{F}_{N,N}=\mathbf{e}_2$ and $\mathbf{F}_{N,N}=\mathbf{e}_3$. Calculations according to (4.10) show that as $t_N\to 0$,

$$\mathbf{S}^{(0)}_{N-2}\mathbf{S}^{(0)}_{N-1}\mathbf{S}^{(0)}_{N}\mathbf{e}_2=2t_N^2\mathbf{e}_1+O(t_N^3),$$
$$\mathbf{S}^{(0)}_{N-2}\mathbf{S}^{(0)}_{N-1}\mathbf{S}^{(0)}_{N}\mathbf{e}_3=2t_N^2\mathbf{e}_4+O(t_N^3).$$

It follows from the estimates found that

$$\left\|\mathbf{S}^{(0)}_{N-2}\mathbf{S}^{(0)}_{N-1}\mathbf{S}^{(0)}_{N}\left(\sum_{i=1}^{4}x_i^{(N)}\mathbf{e}_i\right)-(x_1^{(N)}+2t_N^2x_2^{(N)})\mathbf{e}_1-(x_4^{(N)}+2t_N^2x_3^{(N)})\mathbf{e}_4\right\|\leqslant$$
$$\leqslant(x_1^{(N)}+x_4^{(N)})C\nu^N+(x_1^{(N)}+x_3^{(N)})C_0t_N^3\leqslant C_1(\nu^N+t_N^3)(x_1^{(N)}+2t_N^2x_2^{(N)}+2t_N^2x_3^{(N)}+x_4^{(N)}),$$

where $C_1=\max\{C,C_0\}$. We set

$$L_{N-3,N}=x_1^{(N)}+2t_N^2x_2^{(N)}+2t_N^2x_3^{(N)}+x_4^{(N)};\quad a_{N-3}=L^{-1}_{N-3,N}(x_1^{(N)}+2t_N^2x_2^{(N)}).$$

Then one can write the last inequality in the form

$$\left\|L^{-1}_{N-3,N}\mathbf{S}^{(0)}_{N-2}\mathbf{S}^{(0)}_{N-1}\mathbf{S}^{(0)}_{N}\left(\sum_{i=1}^{4}x_i^{(N)}\mathbf{e}_i\right)-a_{N-3}\mathbf{e}_1-(1-a_{N-3})\mathbf{e}_4\right\|\leqslant\nu_0^N,$$

which is what had to be proved. The general case (different indices i) can be obtained analogously. Lemma 5.2 is proved.

Now let

$$\mathbf{F}^{(1)}_{n,N}=\mathbf{S}_{n+1}^{(i_{n+1})}\dots\mathbf{S}_{N-3}^{(i_{N-3})}\mathbf{e}_1,\quad \mathbf{F}^{(4)}_{n,N}=\mathbf{S}_{n+1}^{(i_{n+1})}\dots\mathbf{S}_{N-3}^{(i_{N-3})}\mathbf{e}_4.$$

It follows from Lemma 5.2 that for $N > N_0$,

$$\|L_{N-3,N}\mathbf{F}_{n,N}-a_{N-3}\mathbf{F}^{(1)}_{n,N}-(1-a_{N-3})\mathbf{F}^{(4)}_{n,N}\|=\|\mathbf{S}_{n+1}^{(i_{n+1})}\dots\mathbf{S}_{N-3}^{(i_{N-3})}\left(L_{N-3,N}\mathbf{F}_{N-3,N}-a_{N-3}\mathbf{e}_1-(1-a_{N-3})\mathbf{e}_4\right)\|\leqslant\nu_1^N.$$

Moreover, by Lemma 5.1 and (5.2),

$$\|\mathbf{F}^{+}_{n,N-3}-\mathbf{F}^{(1)}_{n,N}\|=\|\mathbf{S}_{n+1}^{(i_{n+1})}\dots\mathbf{S}_{N-3}^{(i_{N-3})}(\mathbf{F}^{+}_{N-3,N-3}-\mathbf{e}_1)\|\leqslant\nu_0^N$$

and analogously, $\|\mathbf{F}^{-}_{n,N-3}-\mathbf{F}^{(4)}_{n,N}\|\leqslant\nu_0^N$ for $N > N_0$. Thus

$$\|L_{N-3,N}\mathbf{F}_{n,N}-a_{N-3}\mathbf{F}^{+}_{n,N-3}-(1-a_{N-3})\mathbf{F}^{-}_{n,N-3}\|\leqslant\nu_1^N.$$

Further, from (5.3) for $N = N'$ and $N''\to\infty$, we get that

$$\|\mathbf{F}^{\pm}_{n,N-3}-\mathbf{F}^{\pm}_{n,\infty}\|\leqslant C'\nu^{N/2},$$

so

$$\|L_{N-3,N}\mathbf{F}_{n,N}-a_{N-3}\mathbf{F}^{+}_{n,\infty}-(1-a_{N-3})\mathbf{F}^{-}_{n,\infty}\|\leqslant\nu_1^N,\quad N>N_0.$$

From the sequence $a_{N-3}\in[0,1]$ we choose a convergent subsequence $\lim_{k\to\infty}a_{N_k-3}=a$. Then the limits exist over this sequence and

$$L\mathbf{F}_{n,\infty}=a\mathbf{F}^{+}_{n,\infty}+(1-a)\mathbf{F}^{-}_{n,\infty}.$$

Hence,

$$\mu_{n,\infty}(\sigma^n)=L_{n,\infty}\mu_n(\sigma^n)\,\mathbf{F}_{n,\infty}(\sigma^n)=$$
$$=\frac{L_{n,\infty}}{L}\,\mu_n(\sigma^n)\left(a\,\mathbf{F}^+_{n,\infty}(\sigma^n)+(1-a)\,\mathbf{F}^-_{n,\infty}(\sigma^n)\right)=L_n\left(a\,\mu^+_{n,\infty}(\sigma^n)+(1-a)\,\mu^-_{n,\infty}(\sigma^n)\right).$$

Summing both sides over σ^n we get that $1=L_n(a+1-a)=L_n$, so $\mu_{n,\infty}=a\,\mu^+_{n,\infty}+(1-a)\,\mu^-_{n,\infty}$ for any n, which is what had to be proved. Proposition 5.1 is proved.

Proof of Proposition 5.2. Let $\mathbf{F}^{(1)}_{n,N}$ and $\mathbf{F}^{(2)}_{n,N}$ be vectors which correspond to two different boundary conditions. Acting according to the same scheme as in the case of coexistence of phases, we get the estimate

$$\|\mathbf{F}^{(1)}_{n,N'}-\mathbf{F}^{(2)}_{n,N''}\|\leqslant 2C\nu^{N'/2}+\|\underbrace{\mathbf{S}^{(i)}(t_0)\ldots\mathbf{S}^{(i)}(t_0)}_{N''-M}\mathbf{F}^{(1)}_{N'',N''}-\underbrace{\mathbf{S}^{(i)}(t_0)\ldots\mathbf{S}^{(i)}(t_0)}_{N'-M}\mathbf{F}^{(2)}_{N',N'}\|,$$

where M = N'/2, N" > N' so to prove Proposition 5.2 it suffices to show that in all cases the limit $\lim_{N\to\infty}\underbrace{\mathbf{S}^{(i)}(t_0)\ldots\mathbf{S}^{(i)}(t_0)}_{N}\mathbf{F}$ exists, is not equal to zero, and is independent of the vector F (up to normalization factor).

For T = T_c, h = 0, this follows from Lemma 4.2.

For T > T_c, h = 0, the product $\underbrace{\mathbf{S}^{(i)}(1)\ldots\mathbf{S}^{(i)}(1)}_{N}=\mathbf{U}$, provided not all i = 0. Here U is a matrix with all elements identical, which carries any vector to a vector parallel to $\mathbf{e}=\sum_{i=1}^{4}\mathbf{e}_i$.

For h ≠ 0, T > 0, in view of the symmetry of h and −h, realized by the substitution $\sigma_i\to-\sigma_i$ one can assume h > 0. Analogously to the proof of Lemma 5.2 for any vector F of boundary conditions, we have

$$\mathbf{S}^{(i_{n+1})}_{n+1}\ldots\mathbf{S}^{(i_N)}_{N}\mathbf{F}=C\left(\mathbf{e}_1+O(\nu^{M-n})\right),$$

where M is the index of the last one in the indices i of the operators $\mathbf{S}^{(i_{n+1})}_{n+1}\ldots\mathbf{S}^{(i_N)}_{N}$. This relation shows that for a nondegenerate sequence I the limit $\lim_{N\to\infty}\mathbf{S}^{(i_{n+1})}_{n+1}\ldots\mathbf{S}^{(i_N)}_{N}\mathbf{F}$ exists, is not equal to zero, and is independent of F. The rest of the proof is analogous to the proof of Proposition 5.1. Thus, the basic theorem is proved.

6. Proofs of the Auxiliary Assertions

Proof of Proposition 3.1. We let $\Gamma_{n,\infty}(I)=\ldots\pi_{l_{n+2}},\ \pi_{l_{n+1}}\Gamma_n\subset\Gamma_\infty(I)$ be the image of Γ_n in Γ_∞(I). Let $J_n=\{k_{n+1},\ldots,k_{n+m}\}$, where k_i = 0, 1; m = 1, 2, ... We let $\Gamma_{n,\infty}(I,J_n)=\ldots\pi_{l_{n+m+2}}\pi_{l_{n+m+1}}\ldots\pi_{k_{n+1}}\Gamma_n\subset\Gamma_\infty(I)$. e shall call the subgraphs $\Gamma_{n,\infty}(I,J_n)$ generalized edges of order n. Let us assume that there exists an isomorphism φ of the graphs Γ_∞(I) and Γ_∞(I'). We assert that

1) for any n there exists a J_n such that $\varphi(\Gamma_{n,\infty}(I))=\Gamma_{n,\infty}(I',J_n)$, i.e., the image of the subgraph Γ_n is a generalized edge of order n;

2) $\varphi(\Gamma_{n,\infty}(I))\neq\Gamma_{n,\infty}(I')$, if $l_n+l_{n+1}\neq l'_n+l'_{n+1}\pmod 2$.

Assertion 1) is proved by induction. For the rhombus $\Gamma_{1,\infty}(I)$ it can be verified directly. Let us assume that it is valid for $\Gamma_{n-1,\infty}(I)$. The graph $\Gamma_{n,\infty}(I)$ consists of four subgraphs $\Gamma_{n-1,\infty}(I,J_{n-1})$. By the inductive hypothesis, each of these subgraphs goes into a generalized edge of order (n − 1) in Γ_∞(I).

We denote by ψ_{N-1} the transformation over the graph Γ_∞(I) which consists of replacement of each generalized edge of order (n − 1) by an ordinary edge. Then $\psi_{n-1}\Gamma_{n,\infty}(I)$ is an ordinary rhombus and the isomorphism φ induces a map of this rhombus into $\psi_{n-1}\Gamma_\infty(I')$. As a result, there arises a rhombus in $\psi_{n-1}\Gamma_\infty(I')$, to which there corresponds in Γ_∞(I'), a generalized edge $\Gamma_{n,\infty}(I',J_n)$ of order n. Thus, we have that $\varphi:\Gamma_{n,\infty}(I)\to\Gamma_{n,\infty}(I,J_n)$, which is what had to be proved.

One verifies assertion 2) by direct calculation of the number of edges issuing from outer vertices of the graphs $\Gamma_{n,\infty}(I)$ and $\Gamma_{n,\infty}(I')$.

It follows from assertions 1) and 2) that $L\left(\varphi\left(\Gamma_{n,\infty}(I)\right)\right) \cup L\left(\Gamma_{n,\infty}(I')\right) \neq 0$ and consequently, $L\left(\varphi\left(\Gamma_{k,\infty}(I)\right)\right) \cap L\left(\Gamma_{n,\infty}(I')\right)=0$ for all $k \le n$ since $\Gamma_{k,\infty}(I) \subset \Gamma_{n,\infty}(I)$. If there exist an infinite number of indices n such that $l_n + l_{n+1} \neq l'_n + l'_{n+1} \pmod 2$, then from the last relation we have that $L\left(\varphi\left(\Gamma_{k,\infty}(I)\right)\right) \bigcap L\left(\Gamma_\infty(I)\right)=0$, where $L\left(\Gamma_\infty(I')\right)=\bigcup_n L\left(\Gamma_{n,\infty}(I)\right)$,. but this contradicts the assertion that φ is an isomorphism. The proposition is proved.

Proof of Lemma 4.1. It follows directly from (4.3) that $0 < t_n \le 1$ and

$$\frac{z_{n+1}}{z_n^2} = \frac{z_n^2+t_n^2}{1+z_n^2 t_n^2} < 1, \quad \text{if} \quad 0 \leqslant z_0 < 1.$$

Consequently, $z_n \leqslant z_{n-1}^2 \leqslant \ldots \leqslant z_0^{2^n}$, so $\lim_{n\to\infty} z_n = 0$. We rewrite (4.3) in the form

$$t_{n+1} = \frac{1+2z_n^2+z_n^4}{1+z_n^4+z_n^2 t_n^2+z_n^2/t_n^2} = \frac{\eta_1(z_n,\ t_n)}{1+z_n^2/t_n^2}, \tag{6.1}$$

$$z_{n+1} = z_n^2 t_n^2\ \frac{1+z_n^2/t_n^2}{1+z_n^2 t_n^2} = z_n^2 t_n^2 \left(1+\frac{z_n^2}{t_n^2}\right)\eta_2(z_n,\ t_n). \tag{6.2}$$

It is easy to verify that η_1, $\eta_2 \to 1$ as $n \to \infty$. We choose an N such that $1/2 < \eta_1,\ \eta_2 < 2,\ z_n < 1/4$ for $n \ge N$ and we show that there exists an $n \ge N$ such that $z_n/t_n \le 1$.

Let us assume that for all $n \geqslant N$, $z_n/t_n \geqslant 1$. Then

$$z_{n+1} = z_n^4\left(1+\frac{t_n^2}{z_n^2}\right)\eta_2 < 4z_n^4; \quad t_{n+1} = t_n^2\ \frac{\eta_1}{z_n^2+t_n^2} \geqslant t_n^2.$$

It follows from this that the sequence z_n decreases much faster than t_n. In fact,

$$2z_n \leqslant (2z_{n-1})^4 \leqslant (2z_{n-2})^{16} \leqslant \ldots \leqslant (2z_n)^{4^{n-N}},$$
$$t_n \geqslant t_{n-1}^2 \geqslant t_{n-2}^4 \geqslant \ldots \geqslant t_N^{2^{n-N}},$$

from which

$$\frac{2z_n}{t_n} \leqslant \frac{(2z_n)^{4^{n-N}}}{t_N^{2^{n-N}}} \leqslant \left(\frac{(1/2)^{2^{n-N}}}{t_N}\right)^{2^{n-N}}.$$

We choose an n such that $(1/2)^{2^{n-N}} < t_N$. Then $2z_n/t_n < 1$ which contradicts the assumption that $z_n/t_M \ge 1$ for all $n > N$.

Thus, we have found that there exists an index $M \ge N$ such that $z_M/t_M \le 1$. We show that this is true for all $n \ge M$.

Let us assume that $z_n/t_n \le 1$ for some $n \ge M$ and we show that then $z_{n+1}/t_{n+1} \leqslant 1$. Dividing (6.2) by (6.1), we have

$$\frac{z_{n+1}}{t_{n+1}} = z_n^2 t_n^2 \left(1+\frac{z_n^2}{t_n^2}\right)^2 \frac{\eta_2}{\eta_1} \leqslant 16 z_n^2. \tag{6.3}$$

But $16z_n^2 \le 1$ for $n \ge N$, which proves the proposition. Simultaneously, from (6.3) it follows that $\lim_{n\to\infty} \frac{z_n}{t_n} = 0$, since $\lim_{n\to\infty} z_n = 0$. Here it follows from (6.1) that $\lim_{n\to\infty} t_n = 1$, which proves the lemma.

We note that an estimate of the rate of convergence of $(t_n,\ z_n)$ to (1, 0) also follows from the proof, namely, $z_n \leqslant z_0^{2^n}$, $|t_n - 1| \leqslant C z_0^{2^n}$, where C is an absolute constant.

Proof of Lemma 4.2. We let Π^e and Π^o be the subspaces of vectors of the form (x, y, y, x) and (x, y, −y, −x) respectively. Then $\Pi^e \perp \Pi^o$, $\Pi^e \oplus \Pi^o = R^4$ and they are invariant relative to the operators $S^{(0)}(c)$ and $S^{(1)}(c)$, while on Π^e these operators coincide. We represent the vector

$$\mathbf{g} = \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 + \mathbf{g}',$$

where $\mathbf{f}_1, \mathbf{f}_2 \in \Pi^e$ are eigenvectors of the operators $S^{(0)}(c)$ and $S^{(1)}(c)$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 < 1$, and $\mathbf{g}' \in \Pi^o$. Then $\alpha_1 = (\mathbf{g}, \mathbf{f}_1^*)/(\mathbf{f}_1, \mathbf{f}_1^*)$ and

$$\left(\prod_{k=1}^{n} S^{(i_k)}(c)\right)\mathbf{g} = \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 \lambda_2^n + \left(\prod_{k=1}^{n} S^{(i_k)}(c)\right)\mathbf{g}',$$

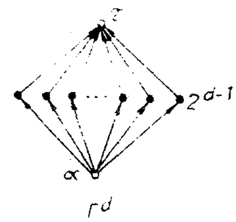

Fig. 3

so

$$\lim_{n\to\infty}\Big(\prod_{k=1}^{n} S^{(i_k)}(c)\Big)\, g=\alpha_1 f_1+\lim_{n\to\infty}\Big(\prod_{k=1}^{n} S^{(i_k)}(c)\Big)\, g'.$$

It remains to prove that the second summand is equal to zero. Let $f_3^{(0)}, f_4^{(0)}\in\Pi^o$ be eigenvectors of the matrix $S^{(0)}(c)$ with eigenvalues $\lambda_3 = 1$ and $0 < \lambda_4 < 1$. We write the action of the operators $S^{(0)}(c)$ and $S^{(1)}(c)$ on the space Π^o in the basis $u_1=Cf_3^{(0)}$, $u_2=f_4^{(0)}$, where we choose the constant $C > 0$ below. Then

$$S^{(0)}(c)\mapsto\Lambda=\begin{pmatrix}1 & 0\\ 0 & \lambda_4\end{pmatrix},\quad S^{(1)}(c)\mapsto D=\begin{pmatrix}s_{11} & Cs_{12}\\ C^{-1}s_{21} & s_{22}\end{pmatrix},$$

where the coefficients s_{ij} are found from the equations

$$S^{(1)}(c)\,f_3^{(0)}=s_{11}\Big(1,\ \frac{1+t_c^2}{2}\Big)+s_{12}\Big(1,\ -\frac{1}{t_c}\Big),$$

$$S^{(1)}(c)\,f_4^{(0)}=s_{21}\Big(1,\ \frac{1+t_c^2}{2}\Big)+s_{22}\Big(1,\ -\frac{1}{t_c}\Big),$$

and, as calculations show, one can choose the constant C so that $\|Dw\|\leqslant\varepsilon\|w\|$, where $\varepsilon<1$, $w=w_1u_1+w_2u_2$, $\|w\|=\sqrt{w_1^2+w_2^2}$. We write $g'=\beta_1u_1+\beta_2u_2$. Then

$$\Big(\prod_{k=1}^{n} S^{(i_k)}(c)\Big)\, g'=\beta_1^{(n)}u_1+\beta_2^{(n)}u_2,$$

where $\sqrt{(\beta_1^{(n)})^2+(\beta_2^{(n)})^2}\leqslant\varepsilon r_n\sqrt{\beta_1^2+\beta_2^2}$, and r_n is the number of ones among the numbers $i_1, \ldots, i_n$. This estimate is explained by the fact that application of the operator $S^{(0)}(c)$ (i.e., multiplication by the matrix Λ), does not increase the norm of the vector $(\beta_1^{(j)}, \beta_2^{(j)})$. and application of $S^{(1)}(c)$ (multiplication by the matrix D) decreases it ϵ times. Since the number of ones among the numbers i_1, i_2, ... is infinite, we get that $\lim_{n\to\infty}(\beta_1^{(n)},\beta_2^{(n)})=(0,\ 0)$. which proves the lemma.

7. Conclusion

The DHL considered in the present paper corresponds to the simplified Migdal equations for the two-dimensional Ising model. To the Ising model on a d-dimensional integral lattice Z^d corresponds an HL with generating graph Γ^d (cf. Fig. 3), in which the number of inner vertices is equal to 2^{d-1}. The theorem proved in the paper Fig. 3 carries over without any special changes to an HL with generating graph Γ^d. As to generalizations to HL with other generating graphs, they are connected with the study of the recurrence equations corresponding to the given generating graph: the set of limit Gibbs distributions may depend essentially on the properties of these equations.

Another direction of generalizations is related to change of type of random variables. Apparently it is straightforward to carry the theorem proved over to the Potts model, in which the random variables assume q = 1, 2, ..., N values (cf. [7, 8, 11] on the Potts model on DHL).

It is much more difficult to study limit Gibbs distributions in models with continuous symmetry, for example, the XY-model, where the random variables $\sigma_i\in S^1\subset R^2$. This is connected with the fact that for these models the solution of the recurrence equations at a critical point is unknown. Using the results of [4], one can study the limit distributions for them in the domains of very small and very large values of T.

The recurrence equations found also let us calculate various thermodynamic characteristics of the Ising model on HL: free energy, magnetization, critical indices, zeros of statsum, etc. Due to lack of space we are unable to give these results here, but we hope to return to the description of the calculations we have made in a separate paper.

LITERATURE CITED

1. A. A. Migdal, "Recursion equations in gauge field theories," Zh. Éksp. Teor. Fiz., 69, No. 9, 810-822; No. 10, 1457-1465 (1975).
2. L. P. Kadanoff, "Notes on Migdal's recursion formulas," Ann. Phys., 100, No. 1-2, 359-394 (1976).
3. A. N. Berker and S. Ostlund, "Renormalization group calculations of finite systems," J. Phys. C., 12, 4961-4975 (1979).
4. P. M. Bleher and E. Žalys, "Existence of long-range order in the Migdal recursion equations," Commun. Math. Phys., 67, No. 1, 17-42 (1979).
5. R. B. Griffiths and M. Kaufman, "Spin systems on hierarchical lattices. Introduction and thermodynamic limit," Phys. Rev. B, 26, 5022-5039 (1982).
6. R. B. Griffiths and M. Kaufman, "Spin systems on hierarchical lattices. Some examples of soluble models," Phys. Rev. B, 30, 244-249 (1984).
7. M. Kaufman and R. B. Griffiths, "First-order transitions in defect structures at a second-order critical point for the Potts model on hierarchical lattices," Phys. Rev. B, 26, 5282-5284 (1982).
8. M. Kaufman and M. Kardar, "Pseudodimensional variation and tricriticality of Potts models," Phys. Rev. B, 30, 1609-1611 (1984).
9. B. Derrida, L, De Seze, and C. Ytzykson, "Fractal structure of zeros in hierarchical models," J. Stat. Phys., 30, No. 3, 559-570 (1983).
10. B. Derrida, C. Ytzykson, and J. M. Luck, "Oscillatory critical amplitudes in hierarchical models," Commun. Math. Phys., 94, No. 1, 115-127 (1984).
11. M. Kaufman, "Duality and Potts critical amplitudes on a class of hierarchical lattices," Phys. Rev. B, 30, 413-414 (1984).
12. K. Preston, Gibbs States on Countable Sets [Russian translation], Nauka, Moscow (1975).
13. P. Bleher, "Construction of non-Gaussian self-similar random fields with hierarchical structure," Commun. Math. Phys., 84, No. 4, 557-578 (1982).
14. M. Aizenman, "Translation invariance and instability of phase coexistence in the two-dimensional Ising system," Commun. Math. Phys., 73, No. 1, 83-96 (1980).
15. Y. Higuchi, "On the absence of nontranslation invariant Gibbs states for the two-dimensional Ising model," Random Fields, 27, No. 1, 517-534 (1981).
16. R. L. Dobrushin and S. B. Shlosman, "The problem of translation invariant states in statistical mechanics," Soviet Sci. Rev., C5, 54-160 (1985).