QUASI-CLASSICAL EXPANSIONS AND THE PROBLEM OF QUANTUM CHAOS

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Abstract

In the present paper we comment some problems discussed in [1]. Following [1] we consider the distribution of eigenvalues E_{mn} of the Laplace-Beltrami operator on a two-dimensional revolution surface. We prove that the quasi-classical quantization rules give a correct asymptotic expansion for large E_{mn} and show that for the problem of quantum chaos two first terms of the quasi-classical expansion are essential. We specify a little bit the geometric problem studied in [1,5] to prove the Poisson distribution for the number ξ of the eigenvalues in the segment $[E, E+c]$, when $E \to \infty$, and show that the main theorem of [5] implies that for 'typical' revolution surfaces, $\xi = \xi^- + \xi^+$ where $\xi^- \equiv 0 \pmod 4$, $\xi^+ \equiv 0 \pmod 2$ and both $\frac{\xi^-}{4}$ and $\frac{\xi^+}{2}$ obey the Poisson distributions.

Contents

1.Introduction

The above-mentioned paper of Ya.G. Sinai [1] provides a general approach to the rigorous study of quantum chaos in the case when the underlying classical system is integrable. It contains many interesting results and beautiful problems and the aim of these notes is to specify and prove some of them. We begin with the eigenvalue problem (see [1])

$$v'' + (Ef^2 - n^2)v = 0 \tag{1.1}$$

with the periodic boundary conditions

$$v(s+h) = v(s). \tag{1.2}$$

Here $v = v(s)$, $s \in \mathbb{R}^1$, is an eigenfunction, $h > 0$ is a period, E is an eigenvalue, $f = f(s)$ is a given periodic, $f(s+h) = f(s)$, positive C^∞-smooth function on $\mathbb{R}^1$ and $n \in \mathbb{Z}$ is an integer external parameter of the problem. Because of the periodic boundary conditions we may consider Eq.(1.1) on the circle $S^1 = [0,h]$, $0 = h$. Now we shall present some results concerning the eigenvalue problem (1.1), (1.2).

Let $L^2(S^1; f^2(s)ds)$ be the Hilbert space of complex-valued functions $x(s)$ on $S^1 = [0,h]$, $0 = h$, with the scalar product

$$(x,y) = \int_0^h x(s)\overline{y(s)} f^2(s) ds.$$

Theorem 1. (see e.g. [2,3]). *There exists a countable set of real-valued C^∞-smooth eigenfunctions $v_m(s)$, $m = 0,1,2,\ldots$, of the problem (1.1), (1.2), which form an orthonormal basis in $L^2(S^1; f^2(s)ds)$. Corresponding eigenvalues E_m go to ∞ when $m \to \infty$.*

We shall assume that E_m are ordered, $0 \le E_0 \le E_1 \le \ldots$. Note that

$$E_0 \ge E_* \equiv \frac{n^2}{\max_{0 \le s \le h} f^2(s)}. \tag{1.3}$$

Oscillating properties of the eigenfunctions $v_m(s)$ are described in the following theorem.

Theorem 2. *All zeroes of $v_m(s)$ are simple. The number of zeroes of $v_m(s)$ is even and it is equal to $2[(m+1)/2]$.*

Remark that it is the 'periodic' version of a well-known result for the eigenvalue problem (1.1) with the Dirichlet boundary conditions (see e.g. [2,3]). With some alterations the proof given in [2] can be adapted to the periodic case (see Sect. 5 below).

2. Quasi-classical expansion

The quantum chaos concerns the distribution of eigenvalues $E_m = E_{m,n}$ when $E_{m,n} \to \infty$. Let

$$E = L^2\varepsilon, \quad n = L\nu, \tag{2.1}$$

where L is a large parameter. Then (1.1) is read as

$$v'' + L^2(\varepsilon f^2 - \nu^2)v = 0. \tag{2.2}$$

Now we construct the quasi-classical (QC) expansion for v. To this end we write

$$v = A\exp(iL\Phi), \tag{2.3}$$

where $L\Phi$ is the phase and A is the amplitude, the both being real, and substitute it into (2.2). Since

$$v'' = (A'' + 2A'iL\Phi' + AiL\Phi'' - AL^2\Phi'^2)\exp(iL\Phi),$$

we get that

$$A'' - AL^2\Phi'^2 + L^2(\varepsilon f^2 - \nu^2)A + i(2A'\Phi' + A\Phi'')L = 0,$$

or

$$A'' + AL^2(-\Phi'^2 + \varepsilon f^2 - \nu^2) = 0, \tag{2.4}$$

$$2A'\Phi' + A\Phi'' = 0. \tag{2.5}$$

Solving the last equation we get (up to a constant multiplier)

$$A = \frac{1}{\sqrt{\Phi'}}. \tag{2.6}$$

To solve (2.4) we rewrite it as

$$\Phi'^2 = \varepsilon f^2 - \nu^2 + \frac{1}{L^2}\frac{A''}{A}. \tag{2.7}$$

Denote

$$\Psi = \Phi'. \tag{2.8}$$

Then $A = \frac{1}{\sqrt{\Psi}}$ so that

$$v(s) = \frac{1}{\sqrt{\Psi(s)}}\exp(iL\int_{s_0}^{s}\Psi(s')ds'). \tag{2.9}$$

After the substitution of $A = \frac{1}{\sqrt{\Psi}}$ into (2.7) we get that

$$\Psi^2 = \varepsilon f^2 - \nu^2 + \frac{1}{L^2}\left(-\frac{\Psi''}{2\Psi} + \frac{3}{4}\frac{\Psi'^2}{\Psi^2}\right). \tag{2.10}$$

Expand Ψ into the asymptotic series

$$\Psi = \Psi_0 + \frac{1}{L^2}\Psi_1 + \frac{1}{L^4}\Psi_2 + ...$$

and equate in (2.10) the coefficients of the expansion in $\frac{1}{L^2}$-series. Then we get:

$$\Psi_0^2 = \varepsilon f^2 - \nu^2, \tag{2.11}$$

$$\Psi_1 = -\frac{\Psi_0''}{4\Psi_0^2} + \frac{3}{8}\frac{\Psi_0'^2}{\Psi_0^3} = -\left(\frac{\Psi_0'}{4\Psi_0^2}\right)' - \frac{1}{8}\frac{\Psi_0'^2}{\Psi_0^3}, \tag{2.12}$$

...

$$\Psi_k = \frac{1}{\Psi_0^3}Q_k(\Psi_0, ..., \Psi_{k-1}, \Psi_0', ..., \Psi_{k-1}', \Psi_0'', ..., \Psi_{k-1}''), \tag{2.13}$$

...

where Q_k is a polynomial of $3k$ variables. To get (2.13) one can rewrite (2.10) as

$$\Psi^4 - (\varepsilon f^2 - \nu^2)\Psi^2 = \frac{1}{L^2}\left(-\frac{1}{2}\Psi\Psi'' + \frac{3}{4}\Psi'^2\right)$$

and equate the coefficients at $\frac{1}{L^{2k}}$. Then one obtains that

$$4\Psi_0^3\Psi_k - (\varepsilon f^2 - \nu^2)2\Psi_0\Psi_k = \hat{Q}_k, \tag{2.14}$$

where $\hat{Q}_k$ is a polynomial of $\Psi_0, ..., \Psi_{k-1}''$. So by (2.11) one has that

$$2\Psi_0^3\Psi_k = \hat{Q}_k$$

or

$$\Psi_k = \frac{1}{\Psi_0^3}\frac{\hat{Q}_k}{2} \equiv \frac{1}{\Psi_0^3}Q_k, \tag{2.15}$$

what was stated.

Eqs. (2.11)-(2.13) determine Ψ_k recurrently. By (2.11)

$$\Psi_0 = \pm\sqrt{\varepsilon f^2 - \nu^2}. \tag{2.16}$$

Two cases are possible: (i) $\varepsilon f^2 - \nu^2 > 0$ for all $0 \le s \le h$; (ii) $\varepsilon f^2 - \nu^2 \le 0$ for some $0 \le s \le h$. The first case is simpler and we begin with it. Denote

$$\varepsilon^* = \frac{\nu^2}{\min_{0 \le s \le h} f^2(s)}. \tag{2.17}$$

Then the case (i) is characterized by the condition $\varepsilon > \varepsilon^*$, or

$$E > L^2\varepsilon^*. \tag{2.18}$$

For definiteness we shall consider the sign + in (2.16), i.e.

$$\Psi_0 = \sqrt{\varepsilon f^2 - \nu^2}. \tag{2.19}$$

Remark, that if $\varepsilon > \varepsilon^*$ then Ψ_0 is a smooth periodic function in s. Hence $\Psi_1, \Psi_2, ...$ are also smooth periodic functions because of the recurrent equations (2.13). Consider the finite series

$$\Psi^{(k)} = \sum_{j=0}^{k} \frac{1}{L^{2j}} \Psi_j \tag{2.20}$$

and put

$$v^{(k)} = \frac{1}{\sqrt{\Psi^{(k)}}} \exp(iL \int \Psi^{(k)} ds). \tag{2.21}$$

$v^{(k)}$ is called the QC solution of Eq. (1.1) of the k-th order. If we substitute $\Psi^{(k)}$ into (2.10) we get the equation

$$\Psi^{(k)2} = \varepsilon f^2 - \nu^2 + \frac{1}{L^2}\left(-\frac{\Psi^{(k)\prime\prime}}{2\Psi^{(k)}} + \frac{3}{4}\frac{\Psi^{(k)\prime 2}}{\Psi^{(k)2}}\right) + \frac{1}{L^{2k+2}} R_k \tag{2.22}$$

where

$$R_k = \frac{1}{\Psi^{(k)2}} P_k(\Psi_0, ..., \Psi_k''; \frac{1}{L}), \tag{2.23}$$

where P_k is a polynomial of $3k+4$ variables. It is worth to note for further use that $v^{(k)}$ satisfies the equation

$$v^{(k)\prime\prime} + L^2(\varepsilon f^2 - \nu^2 + \frac{1}{L^{2k+2}} R_k) v^{(k)} = 0. \tag{2.24}$$

The condition that $v^{(k)}$ is periodic in s is

$$L \int_0^h \Psi^{(k)}(s) ds = 2\pi m, \tag{2.25}$$

$m \in \mathbb{Z}$. For $k = 0$ it is reduced to

$$L \int_0^h \sqrt{\varepsilon f^2 - \nu^2} ds = \int_0^h \sqrt{E f^2 - n^2} ds = 2\pi m. \tag{2.26}$$

Eq. (2.25) is called the quantization condition of the k-th order.

Since the function

$$I(\varepsilon) \equiv \int_0^h \sqrt{\varepsilon f^2 - \nu^2} ds$$

is increasing in ε, Eq. (2.26) has a unique solution $\varepsilon = \varepsilon_m^{(0)}$ for any $m > m^* \equiv L\mu^*$, where

$$\mu^* = \frac{1}{2\pi}\int_0^h \sqrt{\varepsilon^* f^2 - \nu^2}\, ds. \tag{2.27}$$

$E_m^{(0)} = L^2 \varepsilon_m^{(0)}$ is called the quasi-eigenvalue of the zeroth order. Since

$$I(\varepsilon_m^{(0)}) = \frac{2\pi m}{L},$$

then

$$\varepsilon_{m+1}^{(0)} - \varepsilon_m^{(0)} = \frac{2\pi}{I'(\varepsilon_m^{(0)})}\frac{1}{L} + O\left(\frac{1}{L^2}\right),$$

so

$$E_{m+1}^{(0)} - E_m^{(0)} = \frac{2\pi}{I'(\varepsilon_m^{(0)})}L + O(1), \tag{2.28}$$

i.e. the distance between neighbor quazi-eigenvalues is of order of L.

Compute $E_m^{(1)}$. It is determined from the equation

$$\frac{2\pi}{m}L = \int_0^h \Psi^{(1)}(s)ds = \int_0^h \Psi_0(s)ds + \frac{1}{L^2}\int_0^h \Psi_1(s)ds.$$

Substituting formula (2.12) into this equation we get that

$$\begin{aligned}\frac{2\pi m}{L} &= \int_0^h \sqrt{\varepsilon f^2 - \nu^2}\, ds - \frac{\varepsilon^2}{8L^2}\int_0^h \frac{(ff')^2}{(\varepsilon f^2 - \nu^2)^{5/2}} ds \\ &= I(\varepsilon) - \frac{1}{L^2}G(\varepsilon),\end{aligned} \tag{2.29}$$

where

$$G(\varepsilon) \equiv \frac{\varepsilon^2}{8}\int_0^h \frac{(ff')^2}{(\varepsilon f^2 - \nu^2)^{5/2}} ds. \tag{2.30}$$

Since $I'(\varepsilon) > 0$, the function $I(\varepsilon) - \frac{1}{L^2}G(\varepsilon)$ is increasing for large L, so Eq. (2.29) has a unique solution $\varepsilon = \varepsilon_m^{(1)}$ for large L. It can be written as

$$\varepsilon_m^{(1)} = \varepsilon_m^{(0)} + \frac{g(\varepsilon_m^{(0)})}{L^2} + O\left(\frac{1}{L^4}\right),$$

where

$$g(\varepsilon) = \frac{G(\varepsilon)}{I'(\varepsilon)} > 0. \tag{2.31}$$

Thus the quasi-eigenvalue of the first order is

$$E_m^{(1)} = L^2\varepsilon_m^{(1)} = L^2\varepsilon_m^{(0)} + g(\varepsilon_m^{(0)}) + O\left(\frac{1}{L^2}\right). \tag{2.32}$$

Similarly one can compute quasi-eigenvalues of higher order but subsequent corrections to $E_m^{(1)}$ are of order $O\left(\frac{1}{L^2}\right)$ and higher and they are not essential for the quantum chaos study.

The central problem now is: What is the correspondence between the eigenvalues E_m of the original problem (1.1), (1.2) and the quasi-eigenvalues $E_m^{(k)}$. Remark that the quasi-eigenvalues $E_m^{(k)}$ are twice degenerate. Namely, since Eq. (1.1) is real, both $v_m^{(k)}$ and

$$\overline{v_m^{(k)}} = \frac{1}{\sqrt{\Psi_m^{(k)}}} \exp(-iL \int \Psi_m^{(k)} ds), \quad \Psi_m^{(k)} \equiv \Psi^{(k)}|_{E=E_m^{(k)}},$$

are quasi-eigenfunctions. Now we can formulate the following main theorem.

Theorem 3. *$\forall k \geq 0$, $0.1 \geq \rho \geq 0$, $\lambda > 0$, $0.1 \geq \sigma > 0$, $C > 0$ $\exists M = M(k)$ and $\exists L_0 > 0$ such that $\forall L > L_0$ and $\forall E_m^{(0)}$ in the interval*

$$(\varepsilon_* + \lambda L^{-\rho})L^2 \leq E_m^{(0)} \leq CL^2 \tag{2.33}$$

the following estimates hold:

$$|E_{2m-1} - E_m^{(k)}|, \quad |E_{2m} - E_m^{(k)}| \leq \frac{1}{L^{2k-\rho M(k)-\sigma}}.$$

Proof of this theorem will be given below in Sect. 5. Choosing $k = 1$ and $\delta = 1/2$, $\rho = \frac{1}{2M(1)}$, we get from Theorem 3 and Eq. (2.32) that for large L,

$$\begin{aligned} |E_{2m-1} - L^2\varepsilon_m^{(0)} - g(\varepsilon_m^{(0)})| &\leq \frac{1}{L}, \\ |E_{2m} - L^2\varepsilon_m^{(0)} - g(\varepsilon_m^{(0)})| &\leq \frac{1}{L}. \end{aligned} \tag{2.34}$$

3.Quasi-classical Expansion in the Presence of Turning Points

Consider now the case (ii),

$$L^2\varepsilon_* < E < L^2\varepsilon^*,$$

where

$$\varepsilon_* = \frac{\nu^2}{\max_{0\leq s\leq h} f^2(s)}, \quad \varepsilon^* = \frac{\nu^2}{\min_{0\leq s\leq h} f^2(s)}. \tag{3.1}$$

We shall assume that $f(s)$ has a unique point of maximum $s_{\max}$ and a unique point of minimum $s_{\min}$, the both being non-degenerate, i.e.

$$f''(s_{\max}) < 0, \quad f''(s_{\min}) > 0,$$

and $f'(s) \neq 0$ if $s \neq s_{\min}, s_{\max}$. Without loss of generality we may assume that $s_{\min} = 0$. Consider for any $\varepsilon_* < \varepsilon < \varepsilon^*$ the points $0 < a = a(\varepsilon) < b = b(\varepsilon) < h$ such that

$$\varepsilon f^2(a) - \nu^2 = \varepsilon f^2(b) - \nu^2 = 0. \tag{3.2}$$

a and b are called the turning points (see e.g. [4]). We have :

$$\varepsilon f^2(s) - \nu^2 > 0, \quad \text{if} \quad a < s < b,$$
$$\varepsilon f^2(s) - \nu^2 > 0, \quad \text{if} \quad s < a \quad \text{or} \quad s > b.$$

Let $\delta > 0$ be a small number. Consider the intervals

$$I_\delta(a) = S^1 \setminus [b - \delta, b + \delta],$$
$$I_\delta(b) = S^1 \setminus [a - \delta, a + \delta]$$

on the circle $S^1 = [0, h]$, $0 = h$. We construct QC solutions of Eq. (1.1) in $I_\delta(a)$ and in $I_\delta(b)$ and then we use matching relations for these solutions on their common part $I_\delta(a) \cap I_\delta(b)$ to obtain the quantization conditions of quasi-eigenvalues. Since

$$v'' + L^2(\varepsilon f^2 - \nu^2)v = 0 \tag{3.3}$$

and $\varepsilon f^2 - \nu^2$ has a simple zero at $s = a$, it is natural to construct the QC solution around $s = a$ with the help of the Airey function $\mathrm{Ai}(x)$ which is the solution of the equation

$$\mathrm{Ai}''(x) - x\mathrm{Ai}(x) = 0,$$

such that

$$\lim_{x \to \infty} \mathrm{Ai}(x) = 0.$$

Put

$$v(s) = \frac{1}{\sqrt{\varphi'(s)}} \mathrm{Ai}(-L^{2/3}\varphi(s)). \tag{3.4}$$

Then Eq. (3.3) is reduced to

$$\varphi'^2\varphi = \varepsilon f^2 - \nu^2 + \frac{1}{L^2}\left(-\frac{\varphi'''}{2\varphi'} + \frac{3}{4}\frac{\varphi''^2}{\varphi'^2}\right). \tag{3.5}$$

Expand φ into the asymptotic series in $\frac{1}{L^2}$,

$$\varphi = \varphi_0 + \frac{1}{L^2}\varphi_1 + \ldots$$

and equate in (3.5) the coefficients of the expansions in $\frac{1}{L^2}$-series. Then we get:

$$\varphi_0'^2\varphi_0 = \varepsilon f^2 - \nu^2, \tag{3.6}$$

$$2\varphi_1'\varphi_0'\varphi_0 + \varphi_0'^2\varphi_1 = -\frac{\varphi_0'''}{2\varphi_0'} + \frac{3}{4}\frac{\varphi_0''^2}{\varphi_0'^2}, \tag{3.7}$$

$$\dots$$

$$2\varphi_k'\varphi_0'\varphi_0 + \varphi_0'^2\varphi_k = \frac{1}{\varphi_0'^2}Q_k(\varphi_0, ..., \varphi_{k-1}'''), \tag{3.8}$$

$$\dots$$

where Q_k is a polynomial of $4k$ variables. Now we can solve these equations recurrently and find uniquely smooth $\varphi_0, \varphi_1, ...$ Remark, that Eqs. (3.6)-(3.8) can be rewritten as

$$\left(\frac{3}{2}\varphi_0^{3/2\prime}\right)^2 = \varepsilon f^2 - \nu^2,$$

$$\left(\varphi_0^{1/2}\varphi_1\right)' = \frac{1}{2\varphi_0^{1/2}\varphi_0'}\left(-\frac{\varphi_0'''}{2\varphi_0'} + \frac{3}{4}\frac{\varphi_0''^2}{\varphi'^2}\right),$$

$$\dots$$

$$\left(\varphi_0^{1/2}\varphi_k\right)' = \frac{1}{2\varphi_0^{1/2}\varphi_0'}\frac{1}{\varphi_0'^2}Q_k(\varphi_0, ..., \varphi_{k-1}'''),$$

$$\dots$$

The smooth solutions of these equations are

$$\varphi_0(s) = \left|\frac{2}{3}\int_a^s \sqrt{|\varepsilon f^2 - \nu^2|}ds'\right|^{2/3} \operatorname{sign}(s-a), \tag{3.9}$$

$$\varphi_1(s) = \frac{1}{|\varphi_0|^{1/2}}\int_a^s \frac{1}{2|\varphi_0|^{1/2}\varphi_0'}\left(-\frac{\varphi_0'''}{2\varphi_0'} + \frac{3}{4}\frac{\varphi_0''^2}{\varphi'^2}\right)ds', \tag{3.10}$$

$$\dots$$

$$\varphi_k(s) = \frac{1}{|\varphi_0|^{1/2}}\int_a^s \frac{1}{2|\varphi_0|^{1/2}\varphi_0'}\frac{1}{\varphi_0'^2}Q_k(\varphi_0, ..., \varphi_{k-1}''')ds' \tag{3.11}$$

$$\dots$$

Remark, that $\varphi_0, \varphi_1, ...$ are C^∞-smooth in $I_\delta(a)$ (including the point $s = a$!) and $\varphi_0'(a) \neq 0$.

Consider the finite series

$$\varphi^{(k)}(s) = \sum_{j=0}^{k}\frac{1}{L^{2j}}\varphi_j(s)$$

and put

$$v^{(k)}(s) = \frac{1}{\sqrt{\varphi^{(k)\prime}(s)}}\mathrm{Ai}(-L^{2/3}\varphi^{(k)}(s)).$$

The function $v^{(k)}(s)$ is called the QC approximation of the k-th order for Eq. (1.1). Substituting $\varphi^{(k)}(s)$ into Eq. (3.5) we get that

$$\varphi^{(k)\prime 2}\varphi^{(k)} = \varepsilon f^2 - \nu^2 + \frac{1}{L^2}\left(-\frac{\varphi^{(k)\prime\prime\prime}}{2\varphi^{(k)\prime}} + \frac{3}{4}\frac{\varphi^{(k)\prime\prime 2}}{\varphi^{(k)\prime 2}}\right) + \frac{1}{L^{2k+2}}R_k, \tag{3.12}$$

where

$$R_k = \frac{1}{\varphi^{(k)\prime 2}}P_k(\varphi_0, ..., \varphi_k'''; \frac{1}{L}), \tag{3.13}$$

where P_k is a polynomial of $4k+4$ variables. Eq. (3.12) implies that $v^{(k)}$ satisfies the equation

$$v^{(k)\prime\prime} + L^2(\varepsilon f^2 - \nu^2 + \frac{1}{L^{2k+2}}R_k)v^{(k)} = 0. \tag{3.14}$$

A similar QC expansion can be constructed in $I_\delta(b)$. A question arises how to match the expansions in $I_\delta(a)$ and $I_\delta(b)$. Consider a segment $[c,d]$ such that $a+\delta < c < d < b-\delta$, so that $[c,d] \subset I_\delta(a) \cap I_\delta(b) \cap [a,b]$. For large x we have the following QC asymptotics of $\mathrm{Ai}(-x)$:

$$\mathrm{Ai}(-x) = \frac{\mathrm{const}}{\sqrt{\zeta'(x)}}\cos(\zeta(x)),$$

where $\zeta(x)$ is the asymptotic series

$$\zeta(x) = -\frac{\pi}{4} + \frac{2}{3}x^{3/2} - \frac{5}{16}x^{-3/2} + ... = -\frac{\pi}{4} + \frac{2}{3}x^{3/2}\left(1 + \sum_{j=1}^{\infty}\alpha_j x^{-3j}\right). \tag{3.15}$$

It implies that for $s \in [c,d]$,

$$v(s) = \frac{\mathrm{const}}{\sqrt{\varphi'(s)\zeta'(x)}}\cos(\zeta(L^{2/3}\varphi(s))) = \frac{\mathrm{const}}{\sqrt{\Phi'(s)}}\cos(L\Phi(s)), \tag{3.16}$$

where

$$\begin{aligned}\Phi(s) = L^{-1}\zeta(L^{2/3}\varphi(s)) &= -\frac{\pi}{4L} + \frac{2}{3}\varphi^{3/2} - \frac{5}{16L^2}\varphi^{-3/2} + ... = \\ &= -\frac{\pi}{4L} + \frac{2}{3}(\varphi_0 + \frac{1}{L^2}\varphi_1 + ...)^{3/2} - \frac{5}{16L^2}(\varphi_0 + \frac{1}{L^2}\varphi_1 + ...)^{-3/2} + ... \\ &= -\frac{\pi}{4L} + \frac{2}{3}\varphi_0^{3/2} + \frac{1}{L^2}(\varphi_0^{1/2}\varphi_1 - \frac{5}{16}\varphi_0^{-3/2} +) ....\end{aligned}$$

Substituting the expressions (3.9), (3.11) for $\varphi_0, \varphi_1, ...$ into the last formula, we get that

$$\begin{aligned}\Phi(s) &= -\frac{\pi}{4L} + \int_a^s \sqrt{\varepsilon f^2 - \nu^2}ds' \\ &+ \frac{1}{L^2}\left[\int_a^s \frac{1}{\varphi_0^{1/2}}\left(-\frac{1}{4}\frac{\varphi_0'''}{\varphi_0'^2} + \frac{3}{8}\frac{\varphi_0''^2}{\varphi_0'3}\right)ds' - \frac{5}{72}\left(\int_a^s \sqrt{\varepsilon f^2 - \nu^2}ds'\right)^{-1}\right] + ... \\ &= -\frac{\pi}{4L} + \Phi_0(s) + \frac{1}{L^2}\Phi_1(s) + \frac{1}{L^4}\Phi_2(s) + ...\end{aligned} \tag{3.17}$$

Put

$$\Psi(s) \equiv \Phi'(s) = \Phi_0'(s) + \frac{1}{L^2}\Phi_1'(s) + \frac{1}{L^4}\Phi_2'(s) + ...$$
$$= \Psi_0(s) + \frac{1}{L^2}\Psi_1(s) + \frac{1}{L^4}\Psi_2(s) + ... \tag{3.18}$$

Remark, that for $s \in [c,d]$, $\varepsilon f^2 - \nu^2 > 0$, hence (3.16), (3.17) give a QC expansion of the form (2.9). Hence the coefficients $\Psi_0, \Psi_1, ...$ are defined by Eqs. (2.11), (2.13). In particular,

$$\Psi_0 = p \equiv \sqrt{\varepsilon f^2 - \nu^2},$$
$$\Psi_1 = -\left(\frac{p'}{4p^2}\right)' - \frac{1}{8}\frac{p'^2}{p^3}. \tag{3.19}$$

Remark, that Ψ_1 has a non-integrable singularity at $s = a$, $\Psi_1(s) \sim \frac{\text{const}}{(s-a)^{5/2}}$ because of $p \sim \text{const}(s-a)^{1/2}$. An analysis of (3.17) shows that

$$\Phi_1(s) = -\frac{p'(s)}{4p^2(s)} - \text{reg}\int_a^s \frac{p'^2}{8p^3}ds', \tag{3.20}$$

where the regularization of the last integral is understood as

$$\text{reg}\int_a^s \frac{p'^2}{p^3}ds' = \lim_{\delta \to +0}\left(\int_{a+\delta}^s \frac{p'^2}{p^3}ds' - \frac{C_0}{\delta^{3/2}} - \frac{C_1}{\delta^{1/2}}\right),$$

where the constants C_0, C_1 are chosen in such a way that the limit does exist. Formulae (3.16), (3.17) were obtained starting from the turning point $s = a$. Similar formulae can be obtained starting from $s = b$:

$$v(s) = \frac{\text{const}}{\sqrt{\Omega'(s)}}\cos(L\Omega(s)), \tag{3.21}$$

where

$$\Omega(s) = -\frac{\pi}{4L} + \int_s^b \sqrt{\varepsilon f^2 - \nu^2}ds'$$
$$+ \frac{1}{L^2}\left[\int_s^b \frac{1}{\varphi_0^{1/2}}\left(-\frac{1}{4}\frac{\varphi_0'''}{\varphi_0'^2} + \frac{3}{8}\frac{\varphi_0''^2}{\varphi_0'3}\right)ds' - \frac{5}{72}\left(\int_s^b \sqrt{\varepsilon f^2 - \nu^2}ds'\right)^{-1}\right] + ...$$
$$= -\frac{\pi}{4L} + \Omega_0(s) + \frac{1}{L^2}\Omega_1(s) + \frac{1}{L^4}\Omega_2(s) + ... \tag{3.22}$$

Moreover, similarly to (3.16), (3.20) we have that

$$\Omega'(s) = -\Psi = -\sqrt{\varepsilon f^2 - \nu^2} - \frac{1}{L^2}\Psi_1 - \frac{1}{L^4}\Psi_2 - ..., \tag{3.23}$$

and

$$\Omega_1(s) = \frac{p'(s)}{4p^2(s)} - \text{reg}\int_s^b \frac{p'^2}{8p^3}ds', \tag{3.24}$$

where the regularization is made at $s = b$.

Since by (3.19), (3.23)

$$\Phi'(s) = -\Omega'(s) = \Psi(s),$$

then

$$\Phi(s) + \Omega(s) = \text{const}.$$

To match expansions (3.16) and (3.21) we have to choose here const $= \frac{\pi m}{L}$, i.e.

$$\Phi(s) + \Psi(s) = \frac{\pi m}{L}, \tag{3.25}$$

$m = 0, 1, \dots$. This is the quantization condition, which defines the QC eigenvalue. Substituting expansions (3.17), (3.22) of $\Phi(s)$, $\Psi(s)$ we get that

$$-\frac{\pi}{2L} + \int_a^b p ds - \frac{1}{L^2} \text{reg} \int_a^b \frac{p'^2}{8p^3} ds + \dots = \frac{\pi m}{L},$$

or

$$\int_a^b p ds - \frac{1}{L^2} \text{reg} \int_a^b \frac{p'^2}{8p^3} ds + \dots = \frac{\pi \left(m + \frac{1}{2}\right)}{L}, \tag{3.26}$$

where $p = \sqrt{\varepsilon f^2 - \nu^2}$. Put

$$\varepsilon_m = \varepsilon_m^{(0)} + \frac{1}{L^2} \varepsilon_{m1} + \frac{1}{L^4} \varepsilon_{m2} + \dots$$

and substitute this asymptotic expansion into the last equation. Then we can find recurrently the coefficients $\varepsilon_m^{(0)}$, ε_{mk}, $k \geq 1$. The zero coefficient $\varepsilon = \varepsilon_m^{(0)}$ is the solution of the equation

$$L \int_a^b \sqrt{\varepsilon f^2 - \nu^2} ds = \pi \left(m + \frac{1}{2}\right),$$

which is the famous Bohr-Sommerfeld quantization condition. It can be written also as

$$\int_a^b \sqrt{E f^2 - n^2} ds = \pi \left(m + \frac{1}{2}\right). \tag{3.27}$$

Next,

$$\varepsilon_{m1} = g(\varepsilon_m^{(0)}), \tag{3.28}$$

where

$$g(\varepsilon) = \frac{G(\varepsilon)}{F'(\varepsilon)},$$

$$F(\varepsilon) = \int_a^b p \, ds = \int_a^b \sqrt{\varepsilon f^2 - \nu^2} \, ds,$$

$$G(\varepsilon) = \text{reg} \int_a^b \frac{p'^2}{8p^3} \, ds.$$

Similarly one can compute $\varepsilon_{m2}, \varepsilon_{m3}, \dots$.

Now we can formulate the following main theorem.

Theorem 4. *$\forall k \geq 0$, $0.1 \geq \rho \geq 0$, $\lambda > 0$, $0.1 > \sigma > 0$, $\exists M = M(k)$ and $\exists L_0 > 0$ such that $\forall L > L_0$ and $\forall E_m^{(0)}$ in the interval*

$$L^2(\varepsilon_* + \lambda L^{-\rho}) < E_m^{(0)} < L^2(\varepsilon^* - \lambda L^{-\rho}), \tag{3.29}$$

which satisfies the Bohr-Sommerfeld quantization condition (3.27), the following estimate holds:

$$|E_m - E_m^{(k)}| < \frac{1}{L^{2k-\rho M(k)-\sigma}}.$$

Choosing $k = 1$ and $\sigma = 0.1$, $\rho = \frac{1}{2M(1)}$, we get from Th. 4 that for large L,

$$|E_m - L^2\varepsilon_m^{(0)} - g(\varepsilon_m^{(0)})| \leq \frac{1}{L}. \tag{3.30}$$

Proof of this theorem will be given in the next Section.

4. Proofs

Proof of Theorem 2. In [2] a similar theorem is proved for Dirichlet's boundary conditions. We show now how to adapt the proof given in [2] to periodic boundary conditions.

1. All zeroes of $v_m(s)$ are simple. It follows from the uniqueness theorem for the Cauchy problem for the Eq. (1.1).

2. The number of zeroes is even. Since zeroes are simple, each zero is the change of sign. By the periodicity the number of the sign changes is even, so the number of zeroes is the same.

3. The number of zeroes does not exceed $2\left[\frac{m+1}{2}\right]$. It is proved in [2] that the number of intervals between neighbor zeroes of the eigenfunction v_m does not exceed $m+1$. Since it is even, it does not exceed $2\left[\frac{m+1}{2}\right]$, so the number of zeroes of v_m does not exceed $2\left[\frac{m+1}{2}\right]$.

4. The number of zeroes is not less than $2\left[\frac{m+1}{2}\right]$. In [2], pp. 463–464, it is proved that the number of zeroes of v_m is not less than m. Since it is even, it is not less than $2\left[\frac{m+1}{2}\right]$. Theorem 2 is proved.

Proof of Theorem 3. Proof consists of 'spectral' and 'differential' parts. In the 'spectral' part we shall prove that near any quasi-eigenvalue $E_m^{(k)}$ at least two genuine eigenvalues lie. In the 'differential' part we shall prove on the contrary that if E is an eigenvalue which lies near $E_m^{(0)}$ then the corresponding eigenfunction has $2m$ zeroes. It enables us to prove Theorem 3.

'Spectral' part. Let

$$v_m^{(k)} = \frac{1}{\Psi_m^{(k)}} \exp(iL \int \Psi_m^{(k)} ds) \tag{4.1}$$

be the periodic quasi-eigenfunction of the k-th order with the quasi-eigenvalue $E_m^{(k)}$. Then by (2.24)

$$-v_m^{(k)\prime\prime} + (n^2 - E_m^{(k)} f^2) v_m^{(k)} = \frac{1}{L^{2k}} R_k v_m^{(k)}. \tag{4.2}$$

Let us show that R_k satisfies the estimate

$$|R_k| < \text{const } L^{\rho M(k)}. \tag{4.3}$$

To this end let us first prove that

$$|\Psi_k| < \text{const } L^{\rho M_0(k)}. \tag{4.4}$$

Let $\varepsilon = \varepsilon_m^{(k)}$. From (2.33) it follows that

$$\Psi_0 = \sqrt{\varepsilon f^2 - \nu^2} > \text{const } L^{-1.1\rho}.$$

It implies that

$$|\frac{d^j \Psi_0}{ds^j}| < \text{const } L^{M_1(j)\rho}.$$

Now the estimates (4.4), (4.3) are proved easily by induction in k. We omit the details.

Estimate (4.3) implies that

$$\begin{aligned} \| -v_m^{(k)\prime\prime} + (n^2 - E_m^{(k)} f^2) v_m^{(k)} \|_{L^2([0,h])} &\le \frac{1}{L^{2k}} \| R_k v_m^{(k)} \|_{L^2([0,h])} \\ &\le \frac{\text{const}}{L^{2k-\rho M(k)}} \| f v_m^{(k)} \|_{L^2([0,h])} . \end{aligned} \tag{4.5}$$

This inequality can be interpreted in the following way. Consider the differential operator

$$T : v \to \frac{1}{f^2}(-v'' + n^2 v),$$

which is self-adjoint in $L^2([0,h]; f^2(s)ds)$. Then (4.5) is equivalent to

$$\| T v_m^{(k)} - E_m^{(k)} v_m^{(k)} \| \le \frac{\text{const}}{L^{2k-\rho M(k)}} \| v_m^{(k)} \|, \tag{4.6}$$

where $\| \cdot \| = \| \cdot \|_{L^2([0,h]; f^2(s)ds)}$. Besides, we have the general spectral inequality

$$\| Tv - Ev \| \ge \| v \| \cdot \text{dist}(E, \text{Spec} T). \tag{4.7}$$

Comparing it with (4.6) we get that

$$\text{dist}(E_m^{(k)}, \{E_j, j = 0, 1, 2, ...\}) \le \frac{\text{const}}{L^{2k-\rho M(k)}},$$

or

$$|E_m^{(k)} - E_j| \leq \frac{\text{const}}{L^{2k-\rho M(k)}}$$

for some j.

Let us show that there is also another eigenvalue but E_j which lies near $E_m^{(k)}$. We use here that $E_m^{(k)}$ is twice degenerate. Let

$$v_m^{(k)} = \sum_{i=1}^{\infty} a_i v_i = a_j v_j + v'.$$

Since $v_j(s)$ are real-valued, then

$$\overline{v_m^{(k)}} = \sum_{i=1}^{\infty} \overline{a_i} v_i = \overline{a_j} v_j + \overline{v'}.$$

Differentiating by parts we get that

$$(v_m^{(k)}, \overline{v_m^{(k)}}) \equiv \int_0^h \exp(2iL \int \Psi_m^{(k)} ds') \frac{f^2(s) ds}{\Psi_m^{(k)}} = O(\frac{1}{L^N}) \tag{4.8}$$

for any $N \geq 0$, so, in particular, for $N = 1$ we have that

$$(v_m^{(k)}, \overline{v_m^{(k)}}) = a_j^2 + (v', \overline{v'}) = O(\frac{1}{L}).$$

Since

$$|a_j|^2 + \| v' \|^2 = \| v_m^{(k)} \|^2 = \int_0^h \frac{f^2}{\Psi_m^{(k)}} ds$$

is of order of 1, it implies that for large L

$$\| v_m^{(k)} \| \leq 2 \| v' \| .$$

Now, by the general spectral inequality

$$\| (T - E_j) v' \| \geq \| v' \| \cdot \text{dist}(E_j, \{E_i, i \neq j\}).$$

Since

$$\| (T - E_j) v' \| = \| (T - E_j) v_m^{(k)} \| \leq \| (T - E_m^{(k)}) v_m^{(k)} \| + |E_m^{(k)} - E_j| \| v_m^{(k)} \|$$
$$\leq \frac{\text{const}}{L^{2k-\rho M(k)}} \| v_m^{(k)} \|,$$

it implies that

$$\text{dist}(E_j, \{E_i, i \neq j\}) \leq \frac{\text{const}}{L^{2k}} \cdot \frac{\| v_m^{(k)} \|}{\| v' \|} \leq 2 \frac{\text{const}}{L^{2k-\rho M(k)}}.$$

Hence there exists an eigenvalue $E_i, i \neq j$, such that

$$|E_i - E_m^{(k)}| \leq \frac{\text{const}}{L^{2k-\rho M(k)}}.$$

Thus we have shown that in $O\left(\frac{1}{L^{2k-\rho M(k)}}\right)$-neighborhood of $E_m^{(k)}$ at least two eigenvalues E_j, E_i lie.

'Differential part'. Consider now an arbitrary eigenvalue E in the interval

$$(\varepsilon_* + \lambda L^{-\rho}) \leq E \leq CL^2.$$

Assume that the corresponding eigenvalue has $2m$ zeroes. We shall prove that in such a case E is close to $E_m^{(k)}$. For simplicity of formulae we shall consider the case $\rho = 0$. The general case is treated in the same manner. Let

$$E_\pm = E \pm \frac{1}{L^{2k-\sigma}} \tag{4.9}$$

and $\Psi_\pm(s) = \Psi^{(k)}(s; E_\pm)$ be the QC phase function of the k-th order at $E = E_\pm$. Assume that s_0 is a zero of $v(s)$. Consider the QC real eigenfunctions

$$v_\pm(s) = \frac{1}{\sqrt{\Psi_\pm(s)}} \sin(L \int_{s_0}^{s} \Psi_\pm(s')ds'). \tag{4.10}$$

Remark that

$$v_\pm(s_0) = v(s_0) = 0. \tag{4.11}$$

We shall show now that any zero of $v(s)$ lies between the corresponding zeroes of $v_-(s)$ and $v_+(s)$.

Denote

$$\begin{aligned} q &= Ef^2 - n^2, \\ q_\pm &= E_\pm f^2 - n^2 + \frac{1}{L^{2k}} R_{k,\pm}, \end{aligned} \tag{4.12}$$

so that

$$\begin{aligned} &v'' + qv = 0, \\ &{v_\pm}'' + q_\pm v_\pm = 0. \end{aligned} \tag{4.13}$$

In virtue of (4.3) and (4.9),

$$0 < q_- < q < q_+ \tag{4.14}$$

for large L. Put

$$\frac{v}{v'} = \tan\phi, \quad \frac{v_\pm}{v'_\pm} = \tan\phi_\pm.$$

Then (4.13) implies that

$$\begin{aligned} \phi' &= q\sin^2\phi + \cos^2\phi, \\ \phi'_\pm &= q_\pm \sin^2\phi_\pm + \cos^2\phi_\pm. \end{aligned} \tag{4.15}$$

By (4.11)

$$\phi(s_0) = \phi_\pm(s_0) = 0. \tag{4.16}$$

Since for any ϕ,

$$q_- \sin^2\phi + \cos^2\phi \le q\sin^2\phi + \cos^2\phi \le q_+ \sin^2\phi + \cos^2\phi$$

we get from (4.15), (4.16) that

$$\phi_-(s) \le \phi(s) \le \phi_+(s) \tag{4.17}$$

for any $s > s_0$.

Since $q > 0$, $q_\pm > 0$, (4.15) implies that $\phi(s)$, $\phi_\pm(s)$ are increasing. The l-th zeroes of $\phi(s)$ and of $\phi_\pm(s)$ are the solutions of the equations

$$\begin{aligned} \phi(s_l) &= \pi l, \\ \phi_\pm(s_{l,\pm}) &= \pi l, \end{aligned}$$

respectively. Thus we get from (4.17) that for $l \ge 0$,

$$s_{l,-} \le s_l \le s_{l,+}.$$

We assume that v has $2m$ zeroes, therefore $s_{2m} = s_0 + h$, so

$$s_{2m,-} \le s_0 + h \le s_{2m,+}.$$

By (4.10)

$$L\int_{s_0}^{s_{2m,\pm}} \Psi^{(k)}(s; E_\pm)ds = 2\pi m,$$

so the last inequality implies that

$$L\int_{s_0}^{s_0+h} \Psi^{(k)}(s; E_-)ds \le 2\pi m \le L\int_{s_0}^{s_0+h} \Psi^{(k)}(s; E_+)ds.$$

Since $E_m^{(k)}$ is defined by the equation

$$L\int_{s_0}^{s_0+h}\Psi^{(k)}(s;E_m^{(k)})ds = 2\pi m$$

and $\int_{s_0}^{s_0+h}\Psi^{(k)}(s;E)ds$ is increasing with E, we get that $E_- \le E_m^{(k)} \le E_+$, or

$$|E_m^{(k)} - E| \le \frac{1}{L^{2k-\sigma}}. \tag{4.18}$$

Thus we have prove that if E lies in the interval

$$(\varepsilon_* + \lambda)L^2 \le E \le CL^2. \tag{4.19}$$

and v has $2m$ zeroes then (4.18) holds.

Summing up the both parts of the proof we get that for large L:

(i) In $\frac{1}{L^{2k-\sigma}}$-neighborhood of any $E_m^{(k)}$ in the interval (4.19) at least two eigenvalues E_j, E_i lie;

(ii) If E_j lies in the interval (4.19) and v_j has $2m$ zeroes then $|E_j - E_m^{(k)}|\frac{1}{L^{2k-\sigma}}$.

Since by Theorem 2 v_{2m-1}, v_{2m} have $2m$ zeroes and by (2.28) $E_{m+1} - E_m >$ const L, (i), (ii) imply that

$$|E_{2m-1} - E_m^{(k)}|, \quad |E_{2m} - E_m^{(k)}| < \frac{1}{L^{2k-\sigma}}.$$

This proves Theorem 3.

Proof of Theorem 4. For simplicity we shall consider the case $\rho = 0$. The general case is considered in the same manner with minor modifications. Let

$$\mu_*(\lambda) = \int_{a(\varepsilon_*(\lambda))}^{b(\varepsilon_*(\lambda))}\sqrt{\varepsilon_*(\lambda)f^2 - \nu^2}ds,$$
$$\mu^*(\lambda) = \int_{a(\varepsilon^*(\lambda))}^{b(\varepsilon^*(\lambda))}\sqrt{\varepsilon^* f^2 - \nu^2}ds,$$

where $\varepsilon_*(\lambda) = \varepsilon_* + \lambda$, $\quad \varepsilon^*(\lambda) = \varepsilon^* - \lambda$. Put

$$m_*(\lambda) = L\mu_*(\lambda) - \frac{1}{2}, \quad m^*(\lambda) = L\mu^*(\lambda) - \frac{1}{2}.$$

Then the condition (3.29) is equivalent to

$$m_*(\lambda) \le m \le m^*(\lambda). \tag{4.20}$$

Consider QC approximations in the segment $I_\delta(a) = S^1 \setminus [b-\delta, b+\delta]$,

$$v^{(k)} = \frac{1}{\sqrt{\varphi^{(k)\prime}}}\mathrm{Ai}(-L^{2/3}\varphi^{(k)}), \tag{4.21}$$
$$\varphi^{(k)} = \sum_{j=0}^{k} \frac{1}{L^{2j}}\varphi_j,$$

and in the segment $I_\delta(b) = S^1 \setminus [a-\delta, a+\delta]$,

$$w^{(k)} = \frac{1}{\sqrt{\omega^{(k)\prime}}}\mathrm{Ai}(-L^{2/3}\omega^{(k)}), \tag{4.22}$$
$$\omega^{(k)} = \sum_{j=0}^{k} \frac{1}{L^{2j}}\omega_j,$$

and try to arrange from them a unified function on S^1. Let

$$m_1 = \left[\frac{m}{2}\right], \quad m_2 = m - m_1 + 1. \tag{4.23}$$

We state that ε exists in the interval

$$|\varepsilon - \varepsilon_m^{(k)}| < \frac{1}{L^{2k+2-\sigma}} \tag{4.24}$$

such that the m_1-th zero of $v^{(k)}$ (counting out from $s = a$) coincides with the m_2-th zero of $w^{(k)}$ (counting out from $s = b$).

The asymptotic expansion of the m_1-th zero s_{m_1} of $v^{(k)}$ in $\frac{1}{L^2}$-series,

$$s_{m_1} = s_{m_1,0} + \frac{1}{L^2}s_{m_1,1} + \frac{1}{L^4}s_{m_1,2} + \dots,$$

can be found from the equation

$$L\Phi(s) \equiv \zeta(L^{2/3}\varphi(s)) = -\frac{\pi}{2} + m_1\pi. \tag{4.25}$$

Similarly, the m_2-th zero of $w^{(k)}$ can be found from the equation

$$-L\Omega(s) \equiv \zeta(-L^{2/3}\omega(s)) = -\frac{\pi}{2} + m_2\pi. \tag{4.26}$$

For

$$\varepsilon_m = \varepsilon_m^{(0)} + \sum_{j=1}^{\infty} L^{-2j}\varepsilon_{mj},$$

which is determined by the quantization condition (3.25), the equality (4.25) implies the one (4.26), so in that case the m_1-th zero of v coincides with the m_2-th zero of w. It implies that if $s_{m_1}^{(k)}$ is the m_1-th zero of $v^{(k)}(s;\varepsilon_m^{(k)})$, and $\tilde{s}_{m_2}^{(k)}$ is the m_2-th zero of $w^{(k)}(s;\varepsilon_m^{(k)})$, where

$$\varepsilon_m^{(k)} = \varepsilon_m^{(0)} + \sum_{j=1}^{k} L^{-2j}\varepsilon_{mj},$$

then

$$|s_{m_1}^{(k)} - \tilde{s}_{m_2}^{(k)}| < CL^{-2k-2}. \tag{4.27}$$

Next, $s = s_{m_1}^{(0)}$ is the solution of the equation

$$L\int_a^s \sqrt{\varepsilon f^2 - \nu^2}\,ds' = \pi(m+\frac{1}{2})$$

and it is easy to see that

$$\frac{ds_{m_1}^{(0)}}{d\varepsilon} > C > 0,$$

so for large L

$$\frac{ds_{m_1}^{(k)}}{d\varepsilon} = \frac{ds_{m_1}^{(0)}}{d\varepsilon} + O\left(\frac{1}{L^2}\right) > C_0 > 0.$$

Similarly,

$$\frac{d\tilde{s}_{m_2}^{(k)}}{d\varepsilon} < -C_0 < 0.$$

Therefore because of (4.27), ε exists in the interval (4.24) such that $s_{m_1}^{(k)} = \tilde{s}_{m_2}^{(k)}$, which was stated. Denote this ε by $\hat{\varepsilon}_m^{(k)}$ and $s_{m_1}^{(k)} = \tilde{s}_{m_2}^{(k)}$ by $\hat{s}_m^{(k)}$. Let $\hat{E}_m^{(k)} = L^2\varepsilon_m^{(k)}$. By (4.24)

$$|\hat{E}_m^{(k)} - E_m^{(k)}| < \frac{1}{L^{2k-\sigma}}. \tag{4.28}$$

Let $\chi(s) \in C^\infty(S^1)$ be such a function that $\chi(s) \equiv 0$ in a neighborhood of $s = 0$ and $\chi(s) \equiv 1$ in a neighborhood of the segment $[a,b]$, where $a = a(\hat{\varepsilon}_m^{(k)})$, $b = b(\hat{\varepsilon}_m^{(k)})$. Put

$$\begin{aligned}\hat{v}_m^{(k)}(s) = \quad & v_m^{(k)}(s)\chi(s) \quad \text{if} \quad 0 \le s \le \hat{s}_m^{(k)}, \\ & C_m^{(k)} w_m^{(k)}(s)\chi(s) \quad \text{if} \quad \hat{s}_m^{(k)} \le s \le h,\end{aligned} \tag{4.29}$$

where the constant $C_m^{(k)}$ is taken in such a way that $\frac{d\hat{v}_m^{(k)}}{ds}(s)$ has no jump at $s = \hat{s}_m^{(k)}$. It is easy to see that $C_m^{(k)} = (-1)^m + O(L^{-2k})$. We view $v_m^{(k)}(s)$ as a unique QC approximation of Eq. (1.1) on S^1. Remark, that $\frac{d^2\hat{v}_m^{(k)}}{ds^2}$ can have a jump at $s = \hat{s}_m^{(k)}$ but it does not essential in what follows.

'Spectral' part of the proof. Let c, d be such that $0 < c < a < b < d < h$ and $\chi(c) = \chi(d) = 1$. Consider three regions:

$$D_1 = \{c \le s \le s_m^{(k)}\},$$
$$D_2 = \{s_m^{(k)} \le s \le d\},$$
$$D_3 = \{J0 \le s \le c \quad \text{or} \quad d \le s \le h\}.$$

In D_1 $\hat{v}_m^{(k)}$ coincides with $v_m^{(k)}$ and it satisfies the equation (3.14), which implies that

$$\| \hat{v}_m^{(k)\prime\prime} + L^2(\hat{\varepsilon}_m^{(k)} f^2 - \nu^2)\hat{v}_m^{(k)} \|_{L^2(D_1)} \le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \|_{L^2(D_1)} \le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \|_{L^2([0,h])} .$$

Similarly, in D_2 $\hat{v}_m^{(k)}$ coincides with $w_m^{(k)}$ and it satisfies the estimate

$$\| \hat{v}_m^{(k)\prime\prime} + L^2(\hat{\varepsilon}_m^{(k)} f^2 - \nu^2)\hat{v}_m^{(k)} \|_{L^2(D_2)} \le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \|_{L^2(D_2)} \le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \|_{L^2([0,h])} .$$

At last, in D_3 $v_m^{(k)}$ is exponentially small because of $\text{Ai}(x)$ is exponentially decreasing as $x \to \infty$, so

$$\| \hat{v}_m^{(k)\prime\prime} + L^2(\hat{\varepsilon}_m^{(k)} f^2 - \nu^2)\hat{v}_m^{(k)} \|_{L^2(D_3)} \le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \|_{L^2([0,h])} .$$

Since $\hat{v}_m^{(k)}$ is smooth at $s \neq \hat{s}_m^{(k)}$ and $\hat{v}_m^{(k)}$, $\frac{d\hat{v}_m^{(k)}}{ds}$ have no jump at $s = \hat{s}_m^{(k)}$, we get from the last three estimates that

$$\begin{aligned} \| \hat{v}_m^{(k)\prime\prime} + L^2(\hat{\varepsilon}_m^{(k)} f^2 - \nu^2)\hat{v}_m^{(k)} \|_{L^2([0,h])} &\le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \|_{L^2([0,h])} \\ &\le \frac{\text{const}}{L^{2k}} \| f\hat{v}_m^{(k)} \|_{L^2([0,h])}, \end{aligned} \tag{4.30}$$

or

$$\| (T - \hat{E}_m^{(k)})\hat{v}_m^{(k)} \| \le \frac{\text{const}}{L^{2k}} \| \hat{v}_m^{(k)} \| . \tag{4.31}$$

By the general spectral inequality (4.7) it implies that

$$|\hat{E}_m^{(k)} - E_j| \le \frac{\text{const}}{L^{2k}} \tag{4.32}$$

for some E_j. By (4.28)

$$|E_m^{(k)} - E_j| < \frac{2}{L^{2k-\sigma}}. \tag{4.33}$$

'Differential' part of the proof. Assume that for some quasi-eigenvalue of the zeroth order $E_m^{(0)}$ in the interval (3.29) (recall that we suppose $\rho = 0$) we have an eigenvalue E of the problem (1.1), (1.2) such that

$$|E_m^{(0)} - E| \le L^{0.1}. \tag{4.34}$$

Let v be the eigenfunction with the eigenvalue E. We shall prove that the number of zeroes of v is not less than $m-2$ and is not greater than $m+3$.

Consider the quasi-eigenfunctions $v_- \equiv v^{(0)}_{m-1}$ and $v_+ \equiv v^{(0)}_{m+1}$. The functions v, v_-, v_+ satisfy the equations

$$v'' + qv = 0,$$

$$v''_{\pm} + q_{\pm}v_{\pm} = 0,$$

where

$$q = Ef^2 - n^2,$$

$$q_{\pm} = E^{(0)}_{m\pm1}f^2 - n^2 + R_{m\pm1,0}.$$

Sinse $|E^{(0)}_{m\pm1} - E^{(0)}_m| \sim \text{const } L$ and $|R_{m\pm1,0}| \leq \text{const}$, (4.34) implies that

$$q_- < q < q_+.$$

Moreover, let $q_{\pm}(a_{\pm}) = q_{\pm}(b_{\pm}) = q(a) = q(b) = 0$. Then

$$[a_-, b_-] \subset [a, b] \subset [a_+, b_+].$$

We shall use the following lemma.

Lemma 4.1. *Between any two successive zeroes $s_l, s_{l+1} \in [a,b]$ of v at least one zero s^+_k of v_+ lies. Similarly, between any two successive zeroes $s^-_l, s^-_{l+1} \in [a_-, b_-]$ of v_- at least one zero s_k of v lies. Besides, v has at most one zero in $[b,a] \equiv \{s \in S^1 | \ s \leq a \quad or \quad s \geq b\}$.*

Proof: Put

$$\frac{v}{v'} = \tan\,\phi, \quad \phi(s_l) = 0, \quad \phi(s_{l+1}) = \pi,$$
$$\frac{v_+}{v'_+} = \tan\,\phi_+, \quad 0 \leq \phi_+(s_l) < \pi.$$

Then ϕ, ϕ_+ satisfy Eqs. (4.15), so

$$\phi(s) < \phi_+(s) \quad \text{if} \quad s_l < s,$$

so $\pi < \phi_+(s_{l+1})$, so $\phi_+(s') = 0$ for some $s' \in (s_l, s_{l+1})$, so $v_+(s') = 0$, which was stated. The second statement of Lemma is proved in the same way. Prove the third one. Assume that

$$v(s') = v(s'') = 0, \quad s', s'' \in (b, a) \equiv S^1 \setminus [a, b].$$

Since $q < 0$ in the segment $[s', s'']$,

$$0 = \int_{s'}^{s''} (v'' + qv)vdv = \int_{s'}^{s''} (-v'^2 + qv^2)dv < 0.$$

This contradiction completes the proof of Lemma 4.1.

By the construction $v_{m-1}^{(0)}$ has $(m-1)$ zeroes in $[a_-, b_-] \subset [a,b]$, therefore Lemma 4.1 implies that v has at least $(m-2)$ zeroes. Similarly, $v_{m+1}^{(0)}$ has by the construction $(m+1)$ zeroes in $[a_+, b_+] \supset [a,b]$, so by Lemma 4.1 v has at most $(m+2)$ zeroes in $[a,b]$. Since v has at most one zero in (b,a) the total number of zeroes of v does not exceed $(m+3)$, which was stated. This finishes the differential part of the proof.

Thus we have proved that

(i) In $L^{0.1}$-neighborhood of each quasi-eigenvalue $E_m^{(0)}$ in the interval (3.29) at least one eigenvalue E_j lies;

(ii) The number of zeroes of any eigenfunction v_j such that $|E_j - E_m^{(0)}| \leq L^{0.1}$ is between $m-2$ and $m+3$.

Let $E_{m_0}^{(0)}$ be the minimal quasi-eigenvalue in the interval (3.29) and $E_{m_1}^{(0)}$ be the maximal one. Then by (4.20)

$$m_0 = [L\mu_*(\lambda) - \frac{1}{2}] + 1,$$

$$m_1 = [L\mu^*(\lambda) - \frac{1}{2}],$$

so $m_1 - m_0 \sim \mathrm{const}\, L$. Consider the eigenvalues E_{j_0}, E_{j_1} such that

$$|E_{m_0}^{(0)} - E_{j_0}| < L^{0.1},$$

$$|E_{m_1}^{(0)} - E_{j_1}| < L^{0.1}.$$

Then v_{j_0} has at least $m_0 - 2$ zeroes, so by Theorem 2

$$j_0 \geq m_0 - 3.$$

Similarly, v_{j_1} has at most $m_1 + 3$ zeroes, so

$$j_1 \leq m_1 + 3.$$

It implies that near each $E_m^{(0)}$ in the interval (3.29) except at most 6 of them exactly one eigenvalue E_j, $j = j(m)$, lies. Let us show that $j(m) = m$.

Consider arbitrary $E_m^{(0)}$ near which exactly one E_j lies. We have:

$$(T - E_j)v_j = 0,$$

and by (4.31), (4.32)

$$\| (T - E_j)v \| \leq \frac{\mathrm{const}}{L^{2k}},$$

where

$$v = \frac{\hat{v}_m^{(k)}}{\| \hat{v}_m^{(k)} \|}.$$

Let

$$v = \alpha v_j + v_\perp,$$

where $(v_j, v_\perp) = 0$. Then

$$\| (T - E_j)v_\perp \| = \| (T - E_j)v \| \leq \frac{\text{const}}{L^{2k}}.$$

On the other hand by the general spectral inequality

$$\| (T - E_j)v_\perp \| \geq \text{dist}(E_j, \{E_i, i \neq j\}) \cdot \| v_\perp \| \geq \text{const } L \| v_\perp \|,$$

so we get that

$$\| v_\perp \| \leq \frac{\text{const}}{L^{2k+1}}.$$

It implies that

$$\| v_\perp \|_{L^2(S^1)} \leq \frac{\text{const}}{L^{2k+1}}. \tag{4.35}$$

Moreover, since

$$\| v_\perp'' + (E_j f^2 - n^2) v_\perp \|_{L^2(S^1)} = \| (T - E_j)v_\perp \| \leq \frac{\text{const}}{L^{2k}},$$

we get that

$$\| v_\perp'' \|_{L^2(S^1)} \leq \frac{\text{const}}{L^{2k-1}}. \tag{4.36}$$

The inequalities (4.35), (4.36) implies that

$$\| v_\perp \|_{C^1(S^1)} \leq \frac{\text{const}}{L^{2k-1}}. \tag{4.37}$$

Now, $v = \hat{v}_m^{(k)}$ has in $[a,b]$ m zeroes and it is strongly oscillating there:

$$|v'| = |\frac{1}{\sqrt{\varphi'}} \text{Ai}'(-L^{2/3}\varphi) L^{2/3}(-\varphi') + \ldots| \geq \text{const } L^{2/3}$$

at the zero points, so (4.37) implies that $\alpha v_j = v - v_\perp$ and hence v_j have also m zeroes in $[a,b]$.

Now, according to Lemma 4.1 v_j has at most one zero outside of $[a,b]$, so the total number of its zeroes is equal to m or $m+1$, but it is even so it is equal to $2[(m+1)/2]$. By Theorem 2 it implies that $j = m$. Thus we have proved that for all m between $m_*(\lambda)$ and $m^*(\lambda)$ except maybe 6 of them

$$|E_m^{(k)} - E_m| \leq \frac{1}{L^{2k-\sigma}}. \tag{4.37}$$

Remark that $E_m^{(k)}$ and E_m are increasing with m and $E_{m+1}^{(k)} - E_m^{(k)} \sim$ const L. It implies that those 6 (at most) exceptional m's are extreme, i.e. they lie near m_* or m^*. Changing λ by $\lambda/2$ in the definition of m_*, m^* we get (4.37) also for them. Theorem 4 is proved.

5. The Problem of Quantum Chaos

In [1] a beautiful geometrical problem was considered which enables to study the distribution of quasi-eigenvalues of the Laplace operator on 2-dimensional revolution surface. In present Section we specify a little the problem to consider the distribution of eigenvalues of this operator. The main specifications concern the following two aspects of the problem:

(i) To consider the distribution of eigenvalues it is neccessary to take into account not only the zeroth term of the QC expansion but also the first one because it is of order of 1;

(ii) The quantization rules are somewhat different in the absence of the turning points and in their presence and it has some consequences for the eigenvalue distribution.

Recall some definitions from [1]. Let $f(x)$ be a smooth periodic function of $x \in \mathbb{R}^1$, $f(x+h_0) = f(x)$, $h_0 > 0$, and S be the revolution surface which arises when the graph of function $f(x)$ on the segment $[0, h_0]$ is rotated around the x-axis, the points $(0, \varphi)$ being identified with (h_0, φ), $0 \le \varphi \le 2\pi$. Topologically S is a torus. Consider a Riemannian metrics on S,

$$ds^2 = f^2(x) d\varphi^2 + (1 + (f'(x))^2) dx^2,$$

which is induced by the Euclidean metrics in $\mathbb{R}^3$. The Laplace-Beltrami operator on S is

$$\Delta u = \frac{1}{f\sqrt{1+(f')^2}} \frac{\partial}{\partial x}\left(\frac{f}{\sqrt{1+(f')^2}}\frac{\partial u}{\partial x}\right) + \frac{1}{f^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

To simplify this formula introduce a new variable $s = s(x)$ such that

$$\frac{ds}{dx} = \frac{\sqrt{1+(f')^2}}{f}, \quad \text{or} \quad s = \int_0^x \frac{\sqrt{1+(f')^2}}{f} dx'.$$

Then we get

$$\Delta u = \frac{1}{f^2}\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial \varphi^2}\right),$$

$$u(s+h, \varphi) = u(s, \varphi), \quad h = s(h_0).$$

The equation for the eigenfunctions of the operator $-\Delta$ has the form $-\Delta u = Eu$, or

$$-\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial \varphi^2}\right) = Ef^2 u.$$

Look for u in the form

$$u(s,\varphi) = v(s)\exp(in\varphi).$$

Then we get for $v(s)$ the equation

$$\frac{d^2v}{ds^2} + (Ef^2 - n^2)v = 0,$$

which coincides with (1.1). The problem of quantum chaos, or maybe it is better to say of 'quantum order' since we started with the integrable classical system, is connected with the study of the distribution of the eigenvalues $E = E_{mn}$, when $E_{mn} \to \infty$. The QC formulae for E_{mn} enable to reduce this study to a geometrical problem.

Consider for any $\mu, \nu \geq 0$ the equation

$$\frac{1}{\pi}\int_a^b \sqrt{\varepsilon f^2(s) - \nu^2}\,ds = \mu, \tag{5.1}$$

where a, b are the turning points,

$$\varepsilon f^2(s) - \nu^2|_{s=a,b} = 0, \quad \text{if} \quad \mu \leq k^*\nu, \tag{5.2}$$

and

$$a = 0, \quad b = h, \quad \text{if} \quad \mu \geq k^*\nu, \tag{5.2'}$$

where

$$k^* = \frac{1}{\pi}\int_a^b \sqrt{\frac{f^2(s)}{f^2_{\min}} - 1}\; ds. \tag{5.3}$$

As concerns the solvability of Eq. (5.1), note that the function

$$I(\varepsilon) \equiv \frac{1}{\pi}\int_a^b \sqrt{\varepsilon f^2(s) - \nu^2}\; ds$$

is increasing,

$$I'(\varepsilon) > 0 \quad \text{for} \quad \varepsilon > \varepsilon_*,$$

and $I(\varepsilon_*) = 0$, $I(\infty) = \infty$, so Eq. (5.1) has a unique solution $\varepsilon = \varepsilon(\mu,\nu)$. One can see easily from Eq. (5.1) that

$$\varepsilon(t\mu, t\nu) = t^2\,\varepsilon(\mu,\nu) \quad \forall t > 0. \tag{5.4}$$

Consider the level lines of the function $\varepsilon(\mu,\nu)$,

$$\gamma_R = \{\mu,\nu|\; \varepsilon(\mu,\nu) = R^2\}.$$

Eq. (5.4) implies that

$$\gamma_R = R\gamma_1. \tag{5.5}$$

Remark, that γ_1 is given by the equation

$$\frac{1}{\pi}\int_a^b \sqrt{f^2(s)-\nu^2}\, ds = \mu,$$

i.e. it is the graph of the function

$$\mu = \mu(\nu) \equiv \frac{1}{\pi}\int_a^b \sqrt{f^2(s)-\nu^2}\, ds.$$

The function $\mu(\nu)$ possesses the following properties:

(i) $\mu(\nu)$ is even;

(ii) $\mu'(\nu) < 0$ if $\nu > 0$;

(iii) $\mu(0) = \frac{1}{\pi}\int_a^b f(s)\, ds$, $\mu'(0) = 0$, $\mu(f_{\max}) = 0$, $\mu'(f_{\max}) = -\sqrt{\frac{f_{\max}}{f''(s_{\max})}} < 0$;

(iv) $\mu(\nu)$ is smooth at $\nu \neq f_{\min}$ and $\mu''(\nu) \sim C_\pm |\nu - f_{\min}|^{-1/2}$ as $\nu \to f_{\min} \pm 0$.

Following [1] let us write γ_1 in the polar coordinates r, α, $\mu = r\cos\alpha$, $\nu = r\sin\alpha$:

$$\gamma_1 = \{r, \alpha \mid r = G(\alpha)\}.$$

Then

$$\gamma_R = R\gamma_1 = \{r, \alpha \mid r = RG(\alpha)\}.$$

Put

$$\alpha^* = \operatorname{artan} k^* = \operatorname{artan}\left\{\frac{1}{\pi}\int_a^b \sqrt{\frac{p^2(s)}{f^2_{\min}} - 1}\, ds\right\}. \tag{5.6}$$

Then according to (5.2), (5.2'), (5.3) the condition of the existence of the turning points is $\alpha > \alpha^*$.

Fix now $E, c > 0$ and consider eigenvalues E_{mn} in the interval$E \leq E_{mn} < E + c$. Denote $\alpha(m,n) = \operatorname{artan}\frac{n}{m}$. The QC formulae for E_{mn} are a little bit different for $\alpha(m,n) < \alpha^*$ and $\alpha(m,n) > \alpha^*$. Introduce the sectors

$$V_\delta^+ = \{r, \alpha \mid \frac{\pi}{2} - \delta > \alpha > \alpha^* + \delta\},$$
$$V_\delta^- = \{r, \alpha \mid \alpha^* - \delta > \alpha > 0\}, \quad \delta > 0,$$
$$V^\pm = V_\delta^\pm|_{\delta=0}.$$

Let

$$\xi_\delta^\pm(E,c) = \sharp\{E_{mn} \mid (m,n) \in V_\delta^\pm,\ E \leq E_{mn} \leq E + c\},$$
$$\xi^\pm = \xi_\delta^\pm|_{\delta=0},$$

where $\sharp X$ denotes the number of elements of X. Fix two positive numbers a_1, a_2, $a_1 < a_2$ and consider the sets

$$A^{\pm}_{k,\delta}(L,c) = \{R \mid a_1 L \le R \le a_2 L, \quad \xi^{\pm}_{\delta}(R^2, c) = k\},$$
$$A^{\pm}_{k}(L,c) = A^{\pm}_{k,\delta}(L,c)|_{\delta=0}.$$

Put

$$p^{\pm}_{k,\delta}(L,c) = \frac{l(A^{\pm}_{k,\delta}(L,c))}{L(a_2-a_1)},$$
$$p^{\pm}_{k}(L,c) = p^{\pm}_{k,\delta}|_{\delta=0},$$

where $l(X)$ denotes the Lebesque measure of $X \subset \mathbb{R}^1$.

In [6] some convincing (although non-rigorous) arguments were given in the favor of the following general conjecture: If the underlying classical system is integrable then the eigenvalues of the quantum system in large spectral intervals $a_1 L < \sqrt{E} < a_2 L$ in the limit $L \to \infty$ are in some sense independent and they can be considered as a realization of a Poisson process. Different exact formulations of this conjecture are possible. In [1] for the model under consideration it was formulated as a statement on the limit Poisson distribution of the random variable $\xi(E,c) = \sharp\{E_{mn} \mid E \le E_{mn} \le E + c\}$, where $\sqrt{E}$ is uniformly distributed on the segment $[a_1 L, a_2 L]$, when $L \to \infty$. It is neccessary to make here several remarks.

Remark first that by (1.1) $E_{mn} = E_{m,-n}$, so each eigenvalue is twice degenerate, so $\xi(E,c)$ takes only even values. Next, since QC expansions are different for $(m,n) \in V^{\pm}$ it is convenient to consider separately the distributions of E_{mn} for $(m,n) \in V^-$ and for $(m,n) \in V^+$, or even for $(m,n) \in V^-_{\delta}$ and $(m,n) \in V^+_{\delta}$ taking next the limit $\delta \to 0$. Theorems 3,4 enable to study the distributions of the QC eigenvalues instead of the ones of the eigenvalues themselves.

For $(2m,n) \in V^-_{\delta}$ with the $O\left(\frac{1}{L}\right)$-precision we have by Theorem 3 (see also (2.34)) that

$$E_{2m-1,n} = E_{2m,n}, \tag{5.7}$$

$$E \le E_{2m,n} \le E + c \iff (2m,n) \in \Sigma_{\sqrt{E}}, \tag{5.8}$$

where

$$\Sigma_R = \left\{ r, \alpha \mid G(\alpha)\left(R + \frac{\zeta(\alpha)}{2R}\right) \le r \le G(\alpha)\left(R + \frac{\zeta(\alpha)+c}{2R}\right)\right\}. \tag{5.9}$$

$$\zeta(\alpha) = g\left(\frac{1}{G(\alpha)}\right). \tag{5.10}$$

Here $r = G(\alpha)$ is the equation of γ_1 and the function $g(\varepsilon)$ was defined in (2.31). Note that (5.7) implies that each eigenvalue in V^-_{δ} is (asymptotically for $L \to \infty$) four times degenerate, hence $\xi^-_{\delta}(E,c)$ takes values only of the form $k = 4l$.

In [1] QC eigenvalues of the zeroth order were considered which corresponds to $\zeta(\alpha) \equiv 0$ in (5.9). All the technique of [1,5] can be applied with minor modifications to the study of the distribution of $\sharp\{(2m, n) \in \Sigma_{\sqrt{E}} \cap V_\delta^-\}$ and it gives that

$$\lim_{k\to\infty} p^-_{4k,\delta}(L, c) = \exp(-\lambda_-)\frac{\lambda_-^k}{k!},$$
$$\lambda_- = \frac{c}{4}\int_0^{\alpha^*-\delta} G^2(\alpha)d\alpha,$$

for any 'typical' curve γ_1 (see [1,5] for exact formulations). Taking the limit $\delta \to 0$ one gets a similar result for the distribution of $\xi^-(E, c)$.

In the sector $(m, n) \in V_\delta^+$ we have by Theorem 4 that with $O\left(\frac{1}{L}\right)$-precision

$$E \le E_{mn} \le E + c \Leftrightarrow (m + \frac{1}{2}, n) \in \Sigma_{\sqrt{E}},$$

where Σ_R is defined by (5.9), (5.10) with $g(\varepsilon)$ in (5.10) given by (3.28). In the sector V_δ^+ each eigenvalue is twice degenerate and the technique of [1,5] enables to prove that

$$\lim_{k\to\infty} p^+_{2k,\delta}(L, c) = \exp(-\lambda_+)\frac{\lambda_+^k}{k!},$$
$$\lambda_+ = \frac{c}{2}\int_{\alpha^*+\delta}^{\frac{\pi}{2}-\delta} G^2(\alpha)d\alpha$$

for any 'typical' curve γ_1. Taking the limit $\delta \to 0$ one obtains similar result for $\delta = 0$.

Thus in the limit $L \to \infty$ we have that $\xi(E, c) \to \xi(c)$ which is the sum of $\xi^-(c)$ and $\xi^+(c)$ where $\xi^-(c)$ takes the values $k = 4l$ and $\Pr\{\xi^-(c) = 4l\} = \exp(-\lambda_-)\frac{\lambda_-^l}{l!}$, $\lambda_- = \frac{c}{4}\int_0^{\alpha^*} G^2(\alpha)d\alpha$, and $\xi^+(c)$ takes values $k = 2l$ and $\Pr\{\xi^+(c) = 2l\} = \exp(-\lambda_+)\frac{\lambda_+^l}{l!}$, $\lambda_+ = \frac{c}{2}\int_{\alpha^*}^{\pi/2} G^2(\alpha)d\alpha$. It is worth to note that various aspects of the spectrum degeneracy in the problem of quantum chaos were discussed in [7].

It is neccessary to mention here that our considerations above had a small defect: We tacitly assumed that there is a distribution of random functions $f(x)$ which secures the fulfilment of the conditions 1°-3° of [1] on the random curve γ_1. We think that it is true but at the moment we have not full proof of this assumption.

Remark finally that in [8] some properties of the eigenvalues distribution were studied for revolution surfaces which are topologically isomorphic to sphere. An interesting problem is to extend the results of [1,5] (with some specifications of the present paper) to that case. Note, in particular, that for that case the segment $[0, h]$ is substituted by the whole axis $\mathbb{R}^1$ and the spectral problem (1.1) is stated on infinite line. Moreover turning points always exist in that case and so the quantization conditions have the form (3.25), (3.27).

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