Non-Gaussian Energy Level Statistics for Some Integrable Systems

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The number of levels with energy less than E of an integrable quantum system with two degrees of freedom is equal to $\lambda E + sE^{1/4}$, where λ is a constant and s a fluctuating quantity with a non-Gaussian distribution. The probability distribution of s decreases roughly like $\exp(-s^4)$ when s is large. The number of levels between E and $E+z\sqrt{E}$ is equal to $\lambda z\sqrt{E}+rE^{1/4}$ where r is another fluctuating quantity. The distribution of r tends to a Gaussian distribution as $z \to 0$ and oscillates around some limiting non-Gaussian distribution as $z \to \infty$.

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The nature of the distribution of quantum energy levels for systems with integrable or chaotic classical Hamiltonians is an active field of study using both numerical and analytic methods [1-7]. These studies strongly indicate the universality of the local statistics of eigenvalues of generic quantum Hamiltonians: For integrable systems the local statistics is Poissonian, while for chaotic systems it is the Wigner statistics of the ensemble of Gaussian matrices. In this Letter we summarize new rigorous results about the statistics of levels of simple integrable Hamiltonians [8-12]. These are related to the distribution of integer lattice points inside a "random" region of the plane, a problem of independent interest in number theory [13,14].

To see the connection with the lattice problem, consider a free particle on a torus. The eigenvalues are, in suitable units, $E_{\mathbf{n}}=n_1^2+n_2^2$, with $\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2$. More generally we may consider integrable systems with eigenvalues

$$E_{\mathbf{n}}=I(n_1-\alpha_1,n_2-\alpha_2)\,, \qquad (1)$$

where $I(x_1,x_2)$ (maybe after some renormalization) is a smooth homogeneous function of second degree and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ is a point in the unit square [1]. The number of levels

$$N(E)=\{\mathbf{n}\,|\,E_{\mathbf{n}}\le E\}$$

is clearly the same as the number of lattice points inside the oval curve defined by $I(x_1-\alpha_1,x_2-\alpha_2)=E$. We are interested in the behavior of $N(E)$ and $N(E,S)=N(E+S)-N(E)$, the number of levels between E and $E+S$, for generic Hamiltonians in this class, e.g., for "typical" $\boldsymbol{\alpha}$ in (1) and also for individual Hamiltonians with given α. Thus if E is considered a random variable uniformly or otherwise smoothly distributed on the interval $[c_1T,c_2T]$, $c_2>c_1>0$, we can ask for the variance and distribution of $N(E)$ and $N(E,S)$ when the energy or T is very large compared to the average spacing between levels, which is of order 1; cf. Fig. 1 for the behavior of $N(E)-\pi E$ as a function of $R=\sqrt{E}$ when $I(x_1,x_2)=x_1^2+x_2^2$ and $\boldsymbol{\alpha}=(0.34367,0.43037)$. The local statistics of energy levels in a fixed interval S, which does not grow with energy, has been investigated numerically, and there is strong evidence that the statistics are Poissonian [1,4,15]. When S grows with E in such a way that $S\ge CE^{1/2}$, the levels are no longer "truly random" on this scale, and we can expect deviations from Poisson statistics [1,4].

We first describe the results informally in the language of energy levels and then give a brief indication of the proof in terms of lattice problems. Our statements are to be understood as referring to the limit $T\to\infty$ when E is uniformly (or otherwise smoothly) distributed in the interval $c_1T<E\le c_2T$, $c_2>c_1>0$. Let λ be the area of the interior of the oval curve γ defined by $I(x_1,x_2)=1$. We assume that γ is (at least 7 times) differentiable and has strictly positive radius of curvature everywhere.

The average number of levels per unit energy, $N(E)/E$,

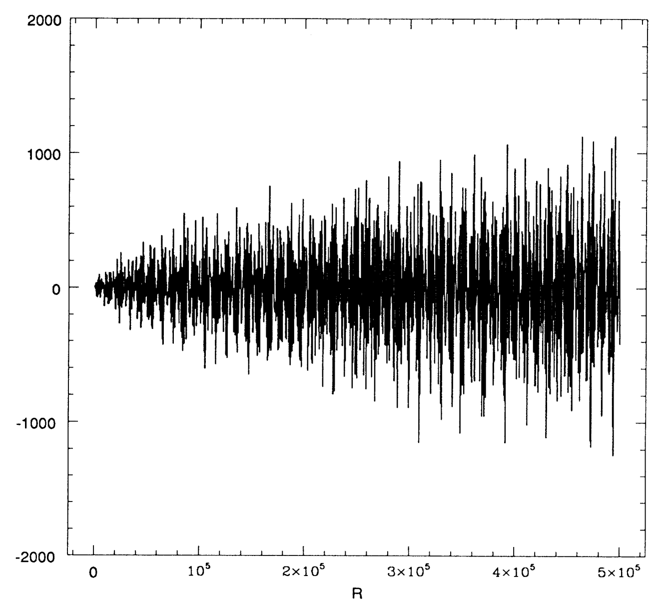

FIG. 1. The graph of the function $N_{\boldsymbol{\alpha}}(R^2)-\pi R^2=$ number of $\{\mathbf{n}\in\mathbb{Z}^2\,|\,|\mathbf{n}-\boldsymbol{\alpha}|\le R\}-\pi R^2$ for $\alpha=(0.34367,0.43037)$.

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is given asymptotically by $\lambda+O(E^{-1/2})$. The difference $N(E)-\lambda E$ has mean square fluctuation behaving like $\bar{V}E^{1/2}$ with a constant $\bar{V}$ as $E\to\infty$. This behavior can be understood from the fact that $N(E)$ counts the number of lattice points inside the curve $\sqrt{E}\gamma$, whose perimeter, around which the fluctuations occur, grows like $\sqrt{E}$. Furthermore the probability that the random variable $F(E)=[N(E)-\lambda E]/E^{1/4}$ takes on values in the interval between s and $s+ds$ has a limit $p(s)ds$ as E or $T\to\infty$. This density $p(s)$ is a real analytic function of s with mean zero and variance $\bar{V}$. The decay of $p(s)$ for large s is bounded above and below (roughly) by $\exp[-cs^4]$ where c and $\bar{V}$ depend on the shape of γ as well as on the shift $\boldsymbol{\alpha}$ in (1). This shows in particular that $p(s)$ cannot be Gaussian; cf. Fig. 2 where $p(s)$ is presented for

$$I(x_1,x_2)=(x_1-\alpha_1)^2/a_1^2+(x_2-\alpha_2)^2/a_2^2 \tag{2}$$

with $a_1/a_2=\pi/10$, $\alpha_1=\alpha_2=0$ in graph (a) and $a_1/a_2=\pi/10$, $\alpha_1=0.34367$, $\alpha_2=0.43037$ in graph (b). As these examples illustrate, the distribution $p(s)$ is usually skew and sometimes bimodal (the parameters in the second example were specially chosen to demonstrate a case of bimodal distribution).

Let us consider now the fluctuations in $N(E,S)$. These depend on the rate of growth of S when $E\to\infty$. When $S/E\to 0$ but $S/E^{1/2}\to\infty$, then $N(E+S)$ and $N(E)$ are asymptotically independent, so their variances add and $\langle[N(E,S)-\lambda S]^2\rangle\sim 2\bar{V}E^{1/2}$. The distribution of $[N(E,S)-\lambda S]/E^{1/4}$ then converges to the distribution of a difference of two independent identically distributed random variables, with distribution $p(s)$, the limit distribution of $F(E)$.

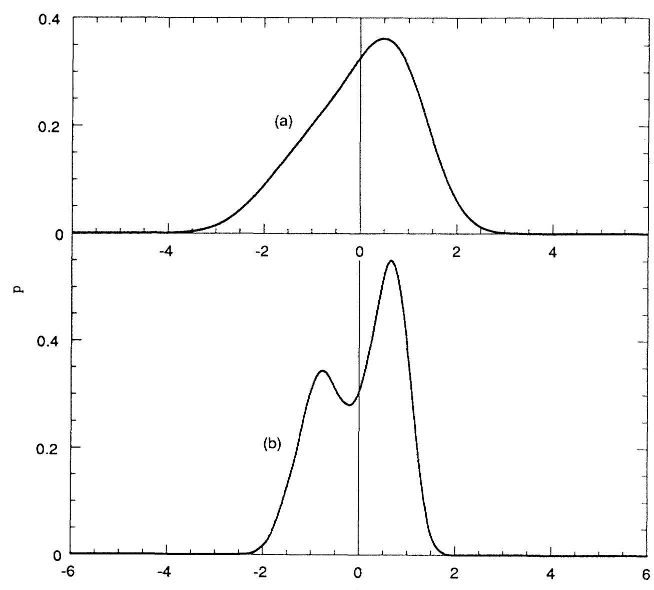

FIG. 2. The density $p(s)$ of the limit distribution of $F(E)=[N(E)-\lambda E]/E^{1/4}$ for the energy function $I(x_1,x_2)$ given in (2) with (a) $a_1/a_2=\pi/10$, $\alpha_1=\alpha_2=0$ and (b) $a_1/a_2=\pi/10$, $\alpha_1=0.34367$, $\alpha_2=0.43037$.

When $S/E^{1/2}\to z>0$, the variance has a scaling behavior,

$$\langle[N(E,S)-\lambda S]^2\rangle\sim E^{1/2}V(E^{-1/2}S)\,.$$

The scaling function $V(z)$ is an almost periodic function of z, so it is oscillating and has no limit at infinity; its average $(1/L)\int_0^L V(z)dz$ approaches, as $L\to\infty$, its value in the previous case, when $S/E^{1/2}\to\infty$, i.e., $2\bar{V}$. In the limit $z\to 0$ we show that in typical cases, corresponding to no systematic degeneracy of the $E_{\mathbf{n}}$,

$$V(z)\sim\lambda z\,. \tag{3}$$

This implies that when $z\to 0$, the variance of $N(E,S)$ becomes equal to its average value λS, which is consistent with (but does not imply) a Poisson distribution. The relation (3) is violated in degenerate cases. For instance, for the circle with center at the origin [or at any rational point $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$] the behavior of $V(z)$ is given by

$$V(z)\sim Cz|\ln z|\,,\quad z\to 0\,. \tag{4}$$

This anomalous behavior of $V(z)$ is related to an arithmetic degeneracy of the circle problem: For some integers k there are many representations of k as a sum of two squares, and on the average, their number grows as $\ln k$, which shows up in the log-correction to linear asymptotics of $V(z)$ as $z\to\infty$. For a circle with center at a very irrational (Diophantine) point (α_1,α_2), the behavior is normal, satisfying (3). For an ellipse centered on the origin with a transcendental ratio of the axes there is a fourfold degeneracy and $V(z)\sim 4\lambda z$.

We prove also the existence of a limit distribution of

$$F(E;S)=[N(E,S)-\langle N(E,S)\rangle]/\sqrt{\mathrm{Var}N(E,S)}$$

in the regime $S/E^{1/2}\to z$. The limit distribution is not Gaussian and in a generic case its density decays at infinity roughly as $\exp[-c(z)x^4]$. However, when $z\to 0$ this limit distribution converges to a standard Gaussian distribution, which can be taken as signaling an approach to a random regime.

When S stays fixed we expect to get a Poisson distribution with mean λS as $E\to\infty$. There is strong numerical and analytic evidence that this, and the corresponding exponential distribution of distances between neighboring levels, is indeed the case, for typical values of $\boldsymbol{\alpha}$ [15]. The only rigorous result in that direction, however, is due to Sinai and Major [16] for the number of lattice points in a narrow strip surrounding a typical "very random curve," so random in fact that it is not even twice differentiable, so that its relevance to real systems is questionable. For the type of smooth oval curves γ considered here all that can be shown at present is that the first and second moments of $N(E,S)$ are, after averaging over $\boldsymbol{\alpha}$, indeed given by λS, as they would be for a Poisson distribution [12].

To give a brief sketch of the proof we shall for simplicity consider the case of a circle centered at $\boldsymbol{\alpha}$. We then have $E=R^2$ and $N_{\boldsymbol{\alpha}}(R)$ is the number of lattice points in the shifted circle, $|\mathbf{n}-\boldsymbol{\alpha}|\le R$. The classical circle problem which goes back to Gauss is to prove a uniform bound for the fluctuations in $N_0(R)$. The best result in this direction at present [17], $|N_0(R)-\pi R^2|\le C_\epsilon \times R^{46/73+\epsilon}$, is quite far from the expected bound $R^{1/2+\epsilon}$ or even $R^{1/2}(\ln R)^\delta$ for some δ. (Numerical studies suggest $\delta<1$.)

Our results show that the situation is much better when we abandon the attempt to prove uniform bounds and consider the statistical behavior of the fluctuations. Using the Poisson summation formula $\sum_{\mathbf{n}} f(\mathbf{n})=\sum_{\mathbf{n}}\tilde{f}(2\pi\mathbf{n})$, where $\tilde{f}$ is the two-dimensional Fourier transform of f and the summation goes over $\mathbf{n}\in\mathbb{Z}^2$, yields

$$F_{\boldsymbol{\alpha}}(R)=[N_{\boldsymbol{\alpha}}(R)-\pi R^2]/\sqrt{R}=\sum_{\mathbf{n}\neq 0}|\mathbf{n}|^{-1}e(\mathbf{n}\cdot\boldsymbol{\alpha})J_1(2\pi|\mathbf{n}|R)\sqrt{R}$$
$$=\pi^{-1}\sum_{\mathbf{n}\neq 0}|\mathbf{n}|^{-3/2}e(\mathbf{n}\cdot\boldsymbol{\alpha})\cos[2\pi|\mathbf{n}|R-3\pi/4]+O(R^{-1})\,, \qquad (5)$$

where $J_1(t)$ is the Bessel function and $e(t)=e^{2\pi i t}$.

The basic idea, due originally to Heath-Brown [14], is to rewrite (5) as a sum of terms representing a flow on an infinite-dimensional torus with incommensurate frequencies, R playing the role of time. The problem of finding a limiting distribution is then a problem in ergodic theory [18]. To do this we first group terms in (5) with commensurate frequencies, i.e., all those $\mathbf{n}\in\mathbb{Z}^2$ for which $|\mathbf{n}|=k\sqrt{m}, k=1,2,\ldots$, where $m>0$ is a fixed square-free natural number, i.e., $m\neq k^2 l$ with $k>1$. This gives

$$F_{\boldsymbol{\alpha}}(R)=\sum_{\text{square-free } m} f_m(\sqrt{m}R;\boldsymbol{\alpha})+O(R^{-1})\,, \qquad (6)$$

where

$$f_m(t;\boldsymbol{\alpha})=\pi^{-1}m^{-3/4}\sum_{k=1}^{\infty}k^{-3/2}\cos(2\pi kt-3\pi/4)\sum_{\mathbf{n}:|\mathbf{n}|=k\sqrt{m}}e(\mathbf{n}\cdot\boldsymbol{\alpha}) \qquad (7)$$

is periodic in t with period 1.

Since the $\sqrt{m}$'s with square-free m are linearly independent over the field of rational numbers we are led to the study of limit distributions of almost periodic functions of the form

$$F(t)=\sum_{n=1}^{\infty}a_n(\gamma_n t) \qquad (8)$$

with rationally independent γ_n. If the sum (8) were finite then ergodic theory would indeed imply the existence of a limiting distribution of $F(t)$ corresponding to the distribution of the finite series $\sum_{n=1}^{N}a_n(\theta_n)$, where the θ_n are independent random variables uniformly distributed on [0,1] (see, e.g., [19]). This means, roughly speaking, that the $\gamma_n t \bmod 1$, $n=1,\ldots,N$, behave like independent random variables uniformly distributed on [0,1], as t varies over a sufficiently large range. Our problem is that the series in (6) is infinite, and in fact only conditionally convergent, since $f_m(t;\boldsymbol{\alpha})\sim m^{-3/4}$. To prove the existence of a limit distribution of such a series we generalize ideas of Heath-Brown to a large class of almost periodic functions (the Besicovitch space B^1). This permits us to find the distribution of lattice points in regions bounded by smooth convex curves.

The nature of the limiting distribution $p(s)$ for large $|s|$ is determined by the large m behavior of the f_m in (7). The proof is based on the following result [8,9,11]. Let θ_m, $m=1,2,\ldots$, be independent uniformly distributed random variables in [0,1] and consider the distribution of $\sum_m a_m(\theta_m)$ where the $a_m(\theta)$ are continuous functions of period one with

$$\int_0^1 a_m(\theta)d\theta=0,\quad \sum_{m=1}^{\infty}\int_0^1|a_m(\theta)|^2d\theta<\infty \qquad (9)$$

and

$$c_1 m^{-2\nu}>\sup_\theta|a_m(\theta)|^2$$
$$\ge\int_0^1|a_m(\theta)|^2d\theta>c_2 m^{-2\nu}\,, \qquad (10)$$

with $c_1,c_2>0$ and $0<\nu<1$. Then

$$C'\exp[-c's^{1/(1-\nu)}]>\Pr\left\{\left|\sum_m a_m(\theta_m)\right|>s\right\}$$
$$>C''\exp[-c''s^{1/(1-\nu)}] \qquad (11)$$

for some $C',C'',c',c''>0$. For the f_m in (7), $\nu=3/4$, and so we obtain the bound $\exp(-\lambda s^4)$ on $p(s;\boldsymbol{\alpha})$, for every $\boldsymbol{\alpha}$.

While the limiting distribution $p(s;\boldsymbol{\alpha})$ of $F(R;\boldsymbol{\alpha})$ is independent of the details of the distribution of R in the interval $[c_1T,c_2T]$, it has a very nonsmooth dependence on $\boldsymbol{\alpha}$. In particular, the variance $\bar{V}_{\boldsymbol{\alpha}}$ is nondifferentiable at every rational point $\boldsymbol{\beta}=(p_1/q_1,p_2/q_2)$, behaving like $\bar{V}_{\boldsymbol{\alpha}}\sim\bar{V}_{\boldsymbol{\beta}}+C|\boldsymbol{\alpha}-\boldsymbol{\beta}|\ln|\boldsymbol{\alpha}-\boldsymbol{\beta}|$ as $\boldsymbol{\alpha}\to\boldsymbol{\beta}$ (see [10]). The variance $\bar{V}_{\boldsymbol{\alpha}}$ can be obtained either from the second moment of the limiting distribution $p(s)$ or directly from the limit

$$\langle F_{\boldsymbol{\alpha}}^2(R)\rangle=\lim_{T\to\infty}(1/T)\int_0^T F_{\boldsymbol{\alpha}}^2(R)dR\,.$$

The validity of this interchange of limits is an open problem for higher moments of $F_{\boldsymbol{\alpha}}(R)$. In fact while the decay of $p(s;\boldsymbol{\alpha})$ guarantees the existence of all moments $\int s^k p(s;\boldsymbol{\alpha})ds$, we do not even know whether $\langle |F_{\boldsymbol{\alpha}}(R)|^k\rangle$ exists for $k>9$ (cf. [14]).

The analysis of the distribution of $N(E,S)$ proceeds in a similar way. Considering the variance $v_{\boldsymbol{\alpha}}(\delta)$ of $[N_{\boldsymbol{\alpha}}(R+cR^{\delta})-N_{\boldsymbol{\alpha}}(R)]/\sqrt{R}$ we find that $v_{\boldsymbol{\alpha}}(\delta)\to 2\bar{V}$ for $0<\delta<1$, and $v_{\boldsymbol{\alpha}}(\delta)\to V_{\boldsymbol{\alpha}}(z)$ for $c=z/2\pi$ and $\delta=0$.

Our results depend strongly on the assumption that the curvature of γ does not vanish anywhere. The situation can be different in the presence of inflection points [20]. In the extreme case when γ is a rectangle of unit area with a very irrational orientation with respect to the x axis, e.g., $\tan\theta=\sqrt{2}$, Beck [21] has shown that the fluctuations in $N_{\boldsymbol{\alpha}}(R)-R^2$ have a variance which grows like $\ln R$. After normalization by $(\ln R)^{1/2}$ the distribution of fluctuations converges to a Gaussian distribution.

If we keep γ fixed and rescale the lattice $\mathbb{Z}^2$ by R^{-1}, we may think of $R^{-2}[N_{\boldsymbol{\alpha}}(R)-\lambda R^2]$ as the difference between the Riemann sum and the integral for the function $\psi(\mathbf{r})=1$ inside γ. Our results generalize to the case where $\psi(\mathbf{r})$ is a general smooth function of $\mathbf{r}$ which does not vanish on γ. The main contribution to the deviation of the Riemann sum from the integral then comes from the boundary of γ and will be of order $R^{-3/2}$ when the lattice spacing R^{-1} tends to zero.

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