Mean square value of exponential sums related to representation of integers as sum of two squares

by

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1. Introduction. The problem we address here arises in the study of the error function in the shifted circle problem (see [BCDL]). Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) be a fixed point in a plane. Define

\[
N(R; \alpha) = \# \{ m \in \mathbb{Z}^2 : |m - \alpha| \leq R \}
\]

and

\[
F(R; \alpha) = \frac{N(R; \alpha) - \pi R^2}{\sqrt{R^2}}.
\]

A long-standing famous conjecture of Hardy (see [H]) is to prove that when \( R \to \infty \),

\[
F(R; \alpha) = O(R^\varepsilon), \quad \forall \varepsilon > 0
\]

(Hardy considered \( \alpha = 0 \)). In [BCDL] and [B] it was proved that the mean square limit

\[
D(\alpha) = \lim_{T \to \infty} \frac{1}{T} \int_1^T |F(R; \alpha)|^2 dR
\]

exists and is equal to

\[
D(\alpha) = (2\pi^2)^{-1} \sum_{n=1}^{\infty} n^{-3/2} |r_\alpha(n)|^2,
\]

where

\[
r_\alpha(n) = \sum_{k^2 + l^2 = n} e(\alpha_1 k + \alpha_2 l), \quad e(t) = \exp(2\pi it)
\]

(for \( \alpha = 0 \) this reduces to a classical result of Cramér [C]). The existence of a limit distribution \( p_\alpha(t)dt \) of \( F(R; \alpha) \) was shown in [BCDL] as well:

\[
\lim_{T \to \infty} \frac{1}{T} \int_{\{ R: 1 \leq R \leq T, a \leq F(R; \alpha) \leq b \}} \int_a^b p_\alpha(t) dt dR
\]

[71]
for every $a < b$ (for $\alpha = 0$ this result is due to Heath-Brown, see [H-B]). The density $p_\alpha(t)$ was proved to be an analytic function in $t$ which decays at infinity, roughly speaking, as $C \exp(-\lambda t^4)$.

In [BCDL] one of the key points in the proof was to evaluate the asymptotics of the series

\begin{equation}
S_\alpha(b) = \sum_{n=1}^{\infty} |r_\alpha(n)|^2 \exp(-n/b)
\end{equation}

when $b \to \infty$. This gives a mean square value of $|r_\alpha(n)|$ as $n \to \infty$. In the present work we show that $S_\alpha(b)$ has an unexpected wild behavior. Namely, $S_\alpha(b)$, as a function of $\alpha$, has a “bumpy” shape when $b \to \infty$, with a big bump at every rational point $\alpha$. This behavior of $S_\alpha(b)$ is closely related to the fact, discovered in [BD], that the mean square limit $D(\alpha)$ has a sharp local maximum at every rational point. We prove here the following theorems:

**Theorem 1.1.** For any fixed $\alpha$,

\begin{equation}
\liminf_{b \to \infty} (b^{-1}S_\alpha(b)) \geq \pi.
\end{equation}

**Theorem 1.2.** Except for an exceptional set of $\alpha$ of measure zero in $\mathbb{R}^2$,

\begin{equation}
S_\alpha(b) = \pi b + O(b^{3/4+\varepsilon}) \quad as \ b \to \infty.
\end{equation}

**Remark.** We prove in Theorem 1.5 that all rational $\alpha$ and all $\alpha$ sufficiently rapidly approximable by rationals belong to the exceptional set. Theorem 1.3 implies the weaker statement that

\begin{equation}
\lim_{b \to \infty} (b^{-1}S_\alpha(b)) = \pi
\end{equation}

for almost all $\alpha$. The power $3/4$ in Theorem 1.2 is best possible by virtue of (3.13) below.

**Theorem 1.3.** Assume that $\alpha = (\alpha_1, \alpha_2)$ is Diophantine, i.e.,

\begin{equation}
|\alpha_1 k + \alpha_2 l - n| > C(k^2 + l^2)^{-D}
\end{equation}

with some $C, D > 0$ for all integers $k, l, n$ with $k^2 + l^2 \neq 0$. Then (1.4) holds.

**Theorem 1.4.** Assume that

\begin{equation}
\alpha_1 p + \alpha_2 q - r = 0
\end{equation}

for some integer $p, q, r$ with coprime $p, q \neq 0$. Assume also that $\alpha_1$ is Diophantine, i.e.,

\begin{equation}
|k\alpha_1 - l| > C(k^2)^{-D},
\end{equation}

with some $C, D > 0$ for all integers $k, l$ with $k \neq 0$. Then

\begin{equation}
\lim_{b \to \infty} (b^{-1}S_\alpha(b)) = \pi(1 + \varepsilon(pq)(p^2 + q^2)^{-1}),
\end{equation}
where
\[ \varepsilon(n) = \begin{cases} 
1 & \text{if } n \text{ is even}, \\
2 & \text{if } n \text{ is odd}. 
\end{cases} \]

Remark. Our proof shows that (1.6) without the assumption that either \( \alpha_1 \) or \( \alpha_2 \) is Diophantine implies
\[ (1.9) \quad \liminf_{b \to \infty} (b^{-1}S_\alpha(b)) \geq \pi(1 + \varepsilon(pq)(p^2 + q^2)^{-1}). \]

Theorem 1.5. Suppose the vector \( \alpha \) is rational, i.e., there exists an integer \( Q \) such that
\[ (1.10) \quad 2Q\alpha_1 = n_1, \quad 2Q\alpha_2 = n_2 \]
are integers and \( \gcd(Q, n_1, n_2) = 1 \). Then
\[ (1.11) \quad (b \log b)^{-1}S_\alpha(b) = C(Qr(Q))^{-1} + O(\log^{-1} b) \quad \text{as } b \to \infty, \]
where
\[ (1.12) \quad r(Q) = \prod_{p|Q}(1 + p^{-1}), \]
with the product taken over primes \( p \) dividing \( Q \), and
\[ C = 3 \quad (Q \text{ even}), \quad C = 4 \quad (Q \text{ odd, } n_1 + n_2 \text{ even}), \]
\[ C = 2 \quad (Q \text{ odd, } n_1 + n_2 \text{ odd}). \]

Remark. We have \( Q < Qr(Q) \leq \sigma(Q) \), where \( \sigma(Q) \) is the sum of the divisors of \( Q \). According to Theorem 323 in [HW],
\[ \sigma(n) = O(n \log \log n). \]

Corollary 1. For fixed rational \( \alpha \), the mean-square value of \( r_\alpha(n) \) for \( n \) of the order of magnitude \( N \) is at least \( 2(\sigma(Q))^{-1}\log(\log(N/Q)). \)

Corollary 2. If \( \alpha \) is an almost-rational vector, i.e., if an infinite sequence \( \{Q_1, Q_2, \ldots\} \) of integers exists such that
\[ (1.14) \quad 2Q_j\alpha = P_j + \varepsilon_j, \]
with \( P_j \) integral vectors and
\[ (1.15) \quad (\sigma(Q_j))^{-1}\log(|\varepsilon_j|^{-1}) \to \infty \quad \text{as } j \to \infty, \]
then
\[ (1.16) \quad \limsup_{b \to \infty}(b^{-1}S_\alpha(b)) = \infty. \]

The Tauberian theorem of Hardy and Littlewood (see [HL]) enables us to derive from Theorems 1.2–1.5 the asymptotics of
\[ \sigma_\alpha(n) = n^{-1}\sum_{k=1}^{n}|r_\alpha(k)|^2 \]
as \( n \to \infty \). The theorem of Hardy and Littlewood is the following:
**Theorem HL.** If \( f(x) = \sum a_n x^n \) is a power series with positive coefficients, and
\[
f(x) \sim A (1 - x)^{-1} |\log(1 - x)|^\alpha \quad (x \to 1),
\]
where \( A > 0 \) and \( \alpha \geq 0 \), then
\[
a_1 + \ldots + a_n \sim An \log^n n.
\]
Define \( a_n = |r_\alpha(n)|^2 \), \( 1 - x = \exp(-b) \). Then we see from Theorems 1.2–1.5 and HL that for all Diophantine \( \alpha \),
\[
\lim_{n \to \infty} \sigma_\alpha(n) = \pi;
\]
for all \( \alpha \) satisfying (1.6) and (1.7),
\[
\lim_{n \to \infty} \sigma_\alpha(n) = \pi(1 + \varepsilon(pq)(p^2 + q^2)^{-1});
\]
and finally for all rational \( \alpha \),
\[
\lim_{n \to \infty} (\log n)^{-1} \sigma_\alpha(n) = C(Qr(Q))^{-1},
\]
with \( r(Q) \) and \( C \) defined in (1.12) and (1.13), respectively.

Theorems 1.1 and 1.5 were proved in [BCDL]. Here we prove Theorems 1.2–1.4 and Corollary 2 of Theorem 1.5.

2. Preliminaries from [BCDL]. Here we recall some results from [BCDL]. The sum (1.1) may be written
\[
S_\alpha(b) = \sum_{m,m'} e(\alpha(m - m')) \exp(-m^2/b),
\]
summed over integer vectors \( m, m' \in \mathbb{Z}^2 \setminus \{0\} \) with \( m^2 = m'^2 \). As was shown in [BCDL], the sum (2.1) can be converted into an unrestricted sum,
\[
S_\alpha(b) = \frac{1}{2} \sum_{k,l,j,h} e(h(\lambda_1 - k\alpha_2)) \exp(-(k^2 + l^2)(j^2 + h^2)/(4b)),
\]
summed over all \( (j,k,l,h) \in \mathbb{Z}^4 \) satisfying
\[
h^2 + j^2 \neq 0, \quad k^2 + l^2 \neq 0,
\]
(2.3) either \( j \equiv h \equiv 0 \), or \( j \equiv h \equiv k \equiv l \equiv 1 \) (mod 2),
(2.4) and
(2.5) \( k, l \) are relatively prime,
which means that either \( |k| + |l| = 1 \), or \( \gcd(|k|, |l|) = 1 \).

According to the two possibilities in (2.4) we divide \( S_\alpha(b) \) into even and odd parts,
\[
S_\alpha(b) = S_e + S_o,
\]
where
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where the terms with \( j \) and \( h \) even are

\[
S_e = \frac{1}{2} \sum_{k,l} |F(w)F(0) - 1|,
\]

summed over integers \((k,l)\) satisfying (2.5), and

\[
S_o = \frac{1}{2} \sum_{k,l} G(w)G(0),
\]

summed over odd integers \( k \) and \( l \) satisfying (2.5). The functions \((F,G)\) are defined by

\[
\sum_x \exp(-x^2/a)e(xt) = F(t) \text{ or } G(t),
\]

where the sum is over integer \( x \) for \( F \) and over half-odd-integer \( x \) for \( G \). In (2.7)–(2.9) we have used the abbreviations \( w = 2(lo_1 - ko_2) \), \( a = b(k^2 + l^2)^{-1} \). By the Poisson summation formula, (2.9) gives

\[
F(t) = (\pi a)^{1/2} \sum_p \exp(-\pi^2 a(p + t)^2),
\]

\[
G(t) = (\pi a)^{1/2} \sum_p (-1)^p \exp(-\pi^2 a(p + t)^2).
\]

According to (2.9), the functions \( F \) and \( G \) are periodic with periods 1 and 2 respectively,

\[
F(w + 1) = F(w), \quad G(w + 1) = -G(w).
\]

For \( a \leq 1 \), (2.9) gives

\[
F(w) = 1 + O(\exp(-a^{-1})), \quad G(w) = O(\exp(-(4a)^{-1})).
\]

For \( a \geq 1 \), (2.10) gives

\[
F(w) = (\pi a)^{1/2}[\exp(-\pi^2 a \hat{w}^2) + O(\exp(-\pi/2)^2 a))],
\]

\[
G(w) = (-1)^w(\pi a)^{1/2}[\exp(-\pi^2 a \hat{w}^2) + O(\exp(-\pi/2)^2 a))],
\]

where \( \hat{w} \) is the distance of \( w \) from the nearest integer.

3. **Proof of Theorem 1.2.** For Theorem 1.2 we divide the sum (2.1) into two parts

\[
S_\alpha(b) = I(b) + R_\alpha(b),
\]

where \( I(b) \) consists of the terms with

\[
m = m',
\]

where the terms with \( j \) and \( h \) even are

\[
(2.7) \quad S_e = \frac{1}{2} \sum_{k,l} |F(w)F(0) - 1|,
\]

summed over integers \((k,l)\) satisfying (2.5), and

\[
(2.8) \quad S_o = \frac{1}{2} \sum_{k,l} G(w)G(0),
\]

summed over odd integers \( k \) and \( l \) satisfying (2.5). The functions \((F,G)\) are defined by

\[
\sum_x \exp(-x^2/a)e(xt) = F(t) \text{ or } G(t),
\]

where the sum is over integer \( x \) for \( F \) and over half-odd-integer \( x \) for \( G \). In (2.7)–(2.9) we have used the abbreviations \( w = 2(lo_1 - ko_2) \), \( a = b(k^2 + l^2)^{-1} \). By the Poisson summation formula, (2.9) gives

\[
F(t) = (\pi a)^{1/2} \sum_p \exp(-\pi^2 a(p + t)^2),
\]

\[
G(t) = (\pi a)^{1/2} \sum_p (-1)^p \exp(-\pi^2 a(p + t)^2).
\]

According to (2.9), the functions \( F \) and \( G \) are periodic with periods 1 and 2 respectively,

\[
F(w + 1) = F(w), \quad G(w + 1) = -G(w).
\]

For \( a \leq 1 \), (2.9) gives

\[
F(w) = 1 + O(\exp(-a^{-1})), \quad G(w) = O(\exp(-(4a)^{-1})).
\]

For \( a \geq 1 \), (2.10) gives

\[
F(w) = (\pi a)^{1/2}[\exp(-\pi^2 a \hat{w}^2) + O(\exp(-\pi/2)^2 a))],
\]

\[
G(w) = (-1)^w(\pi a)^{1/2}[\exp(-\pi^2 a \hat{w}^2) + O(\exp(-\pi/2)^2 a))],
\]

where \( \hat{w} \) is the distance of \( w \) from the nearest integer.
which are equal to the terms in (2.2) with \( h = 0 \). By (2.9) and (2.10),

\[
I(b) = \left( \sum_x \exp(-x^2/b) \right)^2 = \pi b + O(b \exp(-\pi^2 b)).
\]

By (3.1) and (3.3), Theorem 1.2 states that

\[
R_\alpha(b) = O(b^{3/4+\epsilon})
\]

except for a set of \( \alpha \) of measure zero.

Consider the integral

\[
J(b) = \int |R_\alpha(b)|^2 d\alpha,
\]

integrated over the square

\[
0 < \alpha_1 < 1, \quad 0 < \alpha_2 < 1.
\]

We represent \( R_\alpha(b) \) by the sum (2.2) with the condition \( (h \neq 0) \) replacing (2.3). It is convenient to restrict the sum to positive \( h \) and drop the factor 1/2. When (2.2) is inserted into (3.5), the result is an eight-fold sum over the integers \( (k, l, j, h, k', l', j', h') \). The integration over (3.6) eliminates all terms except those with

\[
hl = h'l', \quad hk = h'k'.
\]

Since \( h \) and \( h' \) are positive and the fractions \( k/l \) and \( k'/l' \) are reduced to their lowest terms by (2.5), (3.7) implies

\[
h = h', \quad k = k', \quad l = l'.
\]

The eight-fold sum collapses to a five-fold sum

\[
J(b) = \sum_{k,l,h,j,j'} \exp\left[-\frac{(k^2 + l^2)(2h^2 + j^2 + j'^2)}{4b}\right],
\]

with summations restricted only by

\[
(k, l) = 1, \quad h > 0, \quad \text{either } (j, j', h) \text{ all even or } (j, j', h, k, l) \text{ all odd.}
\]

When \( b \) is large, each of the sums over \( j \) and \( j' \) gives

\[
[\pi b/(k^2 + l^2)]^{1/2} + O(1),
\]

and the sum over \( h \) gives the same result multiplied by \( 2^{-3/2} \). Therefore (3.9) becomes

\[
J(b) = 2^{-3/2} \sum_{k,l} (c_k + c_l)[\pi b/(k^2 + l^2)]^{3/2} + O(b),
\]

where \( c_k = 0 \) for \( k \) even and \( c_k = 1 \) for \( k \) odd. The sum over \( (k, l) \) is convergent, so that

\[
J(b) = Bb^{3/2} + O(b),
\]
where $B$ is a calculable constant, namely
\begin{equation}
B = (2\pi)^{3/2}((3 + \sqrt{2})/7)\zeta(3/2)L(3/2)/\zeta(3),
\end{equation}
where $\zeta$ and $L$ are the Riemann and Dirichlet functions,
\begin{equation}
\zeta(s) = \sum_n n^{-s}, \quad L(s) = \sum_n (-1)^{n-1}(2n - 1)^{-s}.
\end{equation}

We need to prove from (3.5) and (3.13) that (3.4) holds except for a set of $\alpha$ of measure zero. But (3.4) does not follow from (3.13) alone. We need in addition the fact that $R_{\alpha}(b)$ is a smoothly-varying function of $b$, so that it cannot become large at isolated peaks without violating (3.13). To prove (3.4) we require bounds on all the derivatives of $R_{\alpha}(b)$. It is convenient to use the notations
\begin{equation}
D = b^{-2}(d/db),
\end{equation}
\begin{equation}
J_p(b) = \int |D^p R_{\alpha}(b)|^2 d\alpha,
\end{equation}
integrated over (3.6). The same analysis that led to (3.9) now gives
\begin{equation}
J_p(b) = 4^{-2p} \sum_{k,l,h,j,j'} \exp\left[-(k^2 + l^2)(2h^2 + j^2 + j'^2)/(4b)\right] \times (h^2 + j^2)^p(h^2 + j'^2)^p(k^2 + l^2)^{2p}.
\end{equation}
The sums over $(h, j, j')$ give
\begin{equation}
A_p(b/(k^2 + l^2))^{2p+3/2},
\end{equation}
plus terms of lower order in $b$, with a numerical constant $A_p$. Inserting (3.19) into (3.18) gives
\begin{equation}
J_p(b) = A_p b^{2p+3/2} \sum_{k,l} (k^2 + l^2)^{-3/2} = B_p b^{2p+3/2} + O(b^{2p+1}).
\end{equation}
Thus $D^p R_{\alpha}(b)$ has the root-mean-square order of magnitude
\begin{equation}
b^{p+3/4}.
\end{equation}
We have to prove that this same order of magnitude estimate holds pointwise, for almost all $\alpha$, as $b \to \infty$ for fixed $\alpha$.

We use an induction on $p$, working downward from $p + 1$ to $p$. Our inductive hypothesis says that
\begin{equation}
|D^p R_{\alpha}(b)| < Ab^{p+3/4 + f(p)},
\end{equation}
with some positive $f(p)$ depending only on $p$, with $A$ depending on $p$ and $\alpha$ but not on $b$, except for a set of $\alpha$ of measure zero. We assume that (3.22) holds for $p + 1$ and find for which $f(p)$ it will hold for $p$. Let $(b_1, b_2, \ldots)$ be a sequence of numbers tending to infinity, for example
\begin{equation}
b_j = j^m,
\end{equation}
with an exponent $m$ to be chosen later, such that
\[(3.24) \quad |b_{j+1} - b_j| < Ab_j^{1-1/m}.\]
The inductive hypothesis together with (3.24) implies that for every $b$ in the range
\[(3.25) \quad b_j \leq b < b_{j+1},\]
we have
\[(3.26) \quad |D^p R_\alpha(b) - D^p R_\alpha(b_j)| < Ab_j^{p+3/4 + f(p+1) - 1/m}.\]
Comparing (3.26) with (3.22), we see that if
\[(3.27) \quad f(p+1) < f(p) + 1/m,\]
then (3.22) holds for all $b$ if and only if
\[(3.28) \quad |D^p R_\alpha(b_j)| < Ab_j^{p+3/4 + f(p)}\]
holds for all $j$ and some $A$ depending on $\alpha$, with the usual exception of a set of $\alpha$ of measure zero. Therefore, to complete the induction it is only necessary to prove (3.28).

Let $m_{jp}(A)$ be the measure of the set of $\alpha$ for which (3.28) is false for a particular $j$. Comparing (3.28) with (3.17) and (3.20), we see that
\[(3.29) \quad m_{jp}(A) < B_p A^{-2} b_j^{-2f(p)} (1 + O(b_j^{-1/2})).\]
Therefore
\[(3.30) \quad \sum_j m_{jp}(A) < C_p A^{-2},\]
where $C_p$ is the sum of the coefficients on the right of (3.29). The series (3.30) converges and the sum is finite by (3.23) if
\[(3.31) \quad 1/m < 2f(p).\]
The left side of (3.30) is an upper bound to the measure of the set of $\alpha$ for which (3.28) is false for a given $A$ and at least one $j$. The set of $\alpha$ for which (3.28) is false for every $A$ and some $j$ has measure less than (3.30) for every $A$, i.e. has measure zero. So we have proved that (3.28) holds for almost all $\alpha$ if (3.31) holds. We proved before that (3.22) follows from (3.28) if (3.27) holds. Thus the induction of the hypothesis (3.22) from $p+1$ to $p$ succeeds, provided that we can satisfy both (3.27) and (3.31) with the same $m$. This will be possible if and only if
\[(3.32) \quad f(p+1) < 3f(p).\]
To start the induction we use the estimate
\[(3.33) \quad |D^p R_\alpha(b)| < \sum n^p |r_0(n)|^2 \exp(-n/b) = O(b^{p+1+\epsilon}),\]
which follows from
\begin{equation}
|r_\alpha(n)| \leq r_0(n) = O(n^\varepsilon).
\end{equation}
Choose any integer \( P \). The inductive hypothesis (3.22) holds for \( p = P \) by (3.33) if
\begin{equation}
f(P) > 1/4.
\end{equation}
The induction requires only that (3.32) hold for \( p < P \), which is true if we take
\begin{equation}
f(p) = K^{p-P},
\end{equation}
with any constant \( K < 3 \). So the induction is complete and proves (3.22) with \( f(p) \) given by (3.36), for any value of \( P \). But the choice of \( P \) is arbitrary.

In particular, when \( p = 0 \), (3.22) with (3.37) implies (3.4), and Theorem 1.2 is proved.

4. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. By (2.6), \( S_\alpha(b) = S_e + S_o \). Following [BCDL] we divide \( S_e \) into two parts, \( S_{e1} + S_{e2} \), with \( \hat{w} > \delta \) and with \( \hat{w} \leq \delta \), respectively, where \( \delta > 0 \) is an arbitrary small number. Similar division is defined for \( S_o \). (B.75), (B.77) in [BCDL] prove
\begin{equation}
\lim_{\delta \to 0} \limsup_{b \to \infty} b^{-1}|S_1 - \pi| = 0,
\end{equation}
where \( S_1 = S_{e1} + S_{o1} \). Therefore Theorem 1.3 will be proved if we prove for \( S_2 = S_{e2} + S_{o2} \) the following result:

Lemma 4.1. Assume that \( \alpha = (\alpha_1, \alpha_2) \) is Diophantine, i.e., (1.5) holds. Then
\begin{equation}
\lim_{\delta \to 0} \limsup_{b \to \infty} b^{-1}|S_2| = 0.
\end{equation}

Proof. We shall estimate \( S_{e2} \); \( S_{o2} \) can be estimated in the same way. We start with the definition of \( S_{e2} \):
\[ S_{e2} = \frac{1}{2} \sum_{k,l} [F(w)F(0) - 1] \]
with the summation over \( k, l \) with \( (k,l) = 1 \) and \( \hat{w} \leq \delta \). Let us divide \( S_{e2} \) into four parts, \( S_{e2} = S_3 + S_4 + S_5 + S_6 \), where
\[ S_j = \frac{1}{2} \sum_{M_j} [F(w)F(0) - 1] \]
with
\[ M_3 = \{ k, l : (k, l) = 1; \hat{w} \leq \delta; a \leq |\log \delta|^{-2} \}; \]
\[ M_4 = \{ k, l : (k, l) = 1; \hat{w} \leq \delta; |\log \delta|^{-2} < a \leq \delta^{-1/3} \}; \]
\[ M_5 = \{ k, l : (k, l) = 1; \hat{w} \leq \delta; \delta^{-1/3} < a; \exp(-\pi^2 a\hat{w}^2) \leq a^{-1} \}; \]
\[ M_6 = \{ k, l : (k, l) = 1; \hat{w} \leq \delta; \delta^{-1/3} < a; \exp(-\pi^2 a\hat{w}^2) > a^{-1} \}. \]

Now we shall estimate in turn \( S_3, \ldots, S_6 \). Without loss of generality we may assume that the summation in \( k, l \) goes over the region \(|l| \geq |k|\), because the sum over the complementary set \(|l| < |k|\) can be estimated in the same way.

In \( M_3 \), \( a \) is small, so by (2.13),
\[ |F(w)F(0) - 1| \leq C \exp(-a^{-1}) = C \exp(-(k^2 + l^2)/b), \]
hence
\[ |S_3| \leq C \sum_{k^2 + l^2 \geq b|\log \delta|^2} \exp(-(k^2 + l^2)/b) \leq C_0 b \exp(-|\log \delta|^2), \]
which satisfies (4.2).

From (2.13), (2.14), \( |F(w)F(0) - 1| \leq Ca \), hence
\[ |S_4| \leq C \delta^{-1/3} \sum_{M_4} 1. \]

By (B.48), (B.49) in [BCDL], for every fixed \( k \) the fraction of \( l \) with \( \hat{w} < \delta \) does not exceed \( 2\delta + 4/N \), hence
\[ \sum_{M_4} 1 \leq (2\delta + 4/N) \sum_{a \leq \delta^{-1/3}} 1 = (2\delta + 4/N) \sum_{k^2 + l^2 \leq b\delta^{-1/3}} 1 \leq C(2\delta + 4/N)b\delta^{-1/3}. \]

Hence
\[ |S_4| \leq C_0 b(2\delta + 4/N)\delta^{-2/3}. \]
Since we can take \( N \to \infty \) as \( b \to \infty \), \( S_4 \) also satisfies (4.2).

In \( M_5 \), by (2.14), \( |F(w)| \leq Ca^{1/2} \exp(-\pi^2 a\hat{w}^2) \leq C_0 a^{-1/2} \), hence
\[ |F(w)F(0) - 1| \leq C_1, \]
and
\[ |S_5| \leq C_1 \sum_{a \geq \delta^{-1/3}} 1 = C_1 \sum_{k^2 + l^2 \leq b\delta^{1/3}} 1 \leq C_2 b\delta^{1/3}. \]
Thus \( S_5 \) satisfies (4.2).
In $M_6$, $\exp(-\pi^2 a \hat{w}^2) > a^{-1}$, hence $\pi^2 a \hat{w}^2 < \log a$, and
\begin{equation}
\hat{w} < \pi^{-1}(a^{-1} \log a)^{1/2}.
\end{equation}
Therefore $\hat{w}$ small for large $a$. Due to the Diophantine condition this implies that for some $\zeta > 0$ in the circle
\begin{equation}
k^2 + l^2 \leq b^\zeta
\end{equation}
there is no point from $M_6$. Indeed, in $M_6$, due to (1.5) and (4.3),
\begin{equation}
C(k^2 + l^2)^{-D} \leq \hat{w} \leq \pi^{-1}((k^2 + l^2)/b)^{1/2}\left|\log((k^2 + l^2)/b)\right|^{1/2}.
\end{equation}
This implies that for large $b$,
\begin{equation}
k^2 + l^2 > b^\zeta
\end{equation}
with $\zeta = (2D + 1)^{-1} + \varepsilon$, $\varepsilon > 0$, hence in the circle (4.4) there is no point from $M_6$.

Let us divide $M_6$ into annular parts $M_{6j} = M_6 \cap A_j$ with
\begin{equation}
A_j = \{2^{j-1} \delta^{-1/3} < a \leq 2^j \delta^{-1/3}\} = \{2^{-j} \delta^{1/3} b \leq k^2 + l^2 < 2^{-j+1} \delta^{1/3} b\},
\end{equation}
j = 1, \ldots, J, where $J$ is the least integer number with $2^{-J} \delta^{1/3} b < b^\zeta$. Let us fix some $j$, $1 \leq j \leq J$, and estimate
\begin{equation}
S_{6j} = \sum_{M_{6j}} |F(w)F(0) - 1| \leq C a |M_{6j}|
\end{equation}
where $a = b/(k^2 + l^2)$ refers to an arbitrary point inside $M_{6j}$.

Let $s$ be the width of the annulus $A_j$. For $(k, l) \in A_j$,
\begin{equation}
C_0 s < (k^2 + l^2)^{1/2} < C_1 s.
\end{equation}
For a fixed $k$, the number of $l$ with $\hat{w} < \lambda = \pi^{-1}(a^{-1} \log a)^{1/2}$ is estimated by (see (B.47) of [BCDL])
\begin{equation}
(s/N + 1)(\lambda N + 2) = \lambda s + \lambda N + s/N + 2,
\end{equation}
where $N < s$ is the denominator of an approximant $M/N$ of $2\alpha_2$. So
\begin{equation}
|M_{6j}|/|A_j| \leq C(\lambda + 1/N),
\end{equation}
and
\begin{equation}
|S_{6j}| \leq C \sum_{A_j} a(\lambda + 1/N).
\end{equation}
Let $N_i \leq s < N_{i+1}$, where $N_i$ are the denominators of subsequent approximants. The Diophantine condition implies
\begin{equation}
CN_i^{-D} \leq |M_i/N_i - \alpha_2| \leq |M_i/N_i - M_{i+1}/N_{i+1}| = (N_i N_{i+1})^{-1},
\end{equation}
hence \( N_i \geq (CN_i+1)^{(D-1)^{-1}} \geq (Cs)^{(D-1)^{-1}} \) and \( N_i^{-1} \leq C_0s^{-(D-1)^{-1}} \). Therefore from (4.7),
\[
|S_{6j}| \leq C \sum_{A_j} a(\lambda + s^{-\gamma})
\]
with \( \gamma = (D - 1)^{-1} \). Hence
\[
|S_{6j}| \leq C_0 \sum_{A_j} a((a^{-1} \log a)^{1/2} + (k^2 + l^2)^{-\gamma/2})
\]
or
\[
|S_6| \leq C_0 \sum_{(1/2)b^{-1} \leq k^2 + l^2 \leq b \delta^{1/3}} a((a^{-1} \log a)^{1/2} + (k^2 + l^2)^{-\gamma/2}).
\]

Now,
\[
\sum_{k^2 + l^2 \leq b \delta^{1/3}} (a \log a)^{1/2} = \sum_{k^2 + l^2 \leq b \delta^{1/3}} (b/(k^2 + l^2))^{1/2} \log^{1/2}(b/(k^2 + l^2)) \\
\leq C\delta^{-1/6} \log^{1/2} \delta^{-1/3} b \delta^{1/3} = (C/3) b \delta^{1/6} |\log \delta|^{1/2},
\]
and
\[
\sum_{(1/2)b^{-1} \leq k^2 + l^2} a(k^2 + l^2)^{-\gamma/2} = \sum_{(1/2)b^{-1} \leq k^2 + l^2} b(k^2 + l^2)^{-1-\gamma/2} \leq Cb^{1-\zeta\gamma},
\]
which implies
\[
|S_6| \leq Cb(\delta^{1/6} |\log \delta|^{1/2} + b^{-\zeta\gamma}).
\]
Therefore \( S_6 \) satisfies (4.2), and Lemma 4.1 is proved.

**Proof of Theorem 1.4.** In virtue of (4.1), Theorem 1.4 will be proved if we prove the following lemma:

**Lemma 4.2.** Assume that \( \alpha = (\alpha_1, \alpha_2) \) satisfies (1.6), (1.7). Then
\[
(4.8) \quad \lim_{\delta \to 0} \limsup_{b \to \infty} |b^{-1} S_2 - \varepsilon(pq)(p^2 + q^2)| = 0,
\]
with \( \varepsilon(n) = (n \mod 2) + 1 \).

**Proof.** The proof of Lemma 4.2 repeats word for word the one of Lemma 4.1 excepting one point: we proved in Lemma 4.1 that if \( \alpha \) is Diophantine then in the circle (4.4) there is no point from \( M_6 \); now we state that if \( \alpha \) satisfies (1.6), (1.7) then in the circle (4.4) there are exactly two points from \( M_6 \),
\[
(4.9) \quad (k, l) = \pm(-q, p).
\]
Notice that due to (1.6), if (4.9) holds then
\[
w = 2(l\alpha_1 - k\alpha_2) = \pm 2(pa_1 + qa_2) = \pm 2r,
\]
which implies
\[
\sum_{(1/2)b^{-1} \leq k^2 + l^2} a(k^2 + l^2)^{-\gamma/2} = \sum_{(1/2)b^{-1} \leq k^2 + l^2} b(k^2 + l^2)^{-1-\gamma/2} \leq Cb^{1-\zeta\gamma},
\]
and
\[
|S_6| \leq Cb(\delta^{1/6} |\log \delta|^{1/2} + b^{-\zeta\gamma}).
\]
Therefore \( S_6 \) satisfies (4.2), and Lemma 4.1 is proved.
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hence \( \hat{w} = 0 \), so that these two points contribute to \( S_e \) the term
\[
F^2(0) - 1 = \pi b(k^2 + l^2)^{-1} + O(1) = \pi b(p^2 + q^2)^{-1} + O(1).
\]

If \( pq \) is odd, then these two points contribute a similar term to \( S_o \). Therefore totally they contribute to \( S_n(b) \) the term \( \pi b\varepsilon(pq)(p^2 + q^2)^{-1} + O(1) \).

These considerations show that (4.8) will be proved if we prove that (4.9) are the only points from \( M_6 \) in the circle (4.4). Without loss of generality we may assume \( p \neq 0 \). Assume
\[
(k, l) \neq \pm (-q, p).
\]

We have
\[
\alpha_1 \sim \alpha_2 = (l/p)(\alpha_1 p + \alpha_2 q) - \alpha_2(k + lq/p) = (l/p)r - \alpha_2(k + lq/p).
\]

Note that
\[
k + lq/p \neq 0.
\]

Indeed, otherwise \( lq = -kp \), and since the pairs \( (k, l) \) and \( (p, q) \) are coprime, \( (k, l) = \pm (-q, p) \), which contradicts (4.10).

(4.11), (4.12) and (1.7) imply that for every integer \( n \),
\[
|2p(\alpha_1 - \alpha_2) - n| = |2\alpha_2(kp + lq) + 2lr - n| \geq C(2|kp + lq|)^{-2D},
\]

hence if \( m \) is the closest integer to \( w \), then
\[
p\hat{w} = p|2(\alpha_1 - \alpha_2) - m| \geq C(2|kp + lq|)^{-2D}.
\]

Hence
\[
\hat{w} \geq C_{pq}(k^2 + l^2)^{-D}.
\]

This proves that (1.6), (1.7) and (4.10) imply (4.13). If we assume in addition that \( (k, l) \in M_6 \), then (4.5) holds. Since (4.5) implies (4.6), the point \( (k, l) \) lies outside of the circle (4.4). This means that (4.9) are the only points from \( M_6 \) in this circle. Lemma 4.2 is proved.

Appendix. Proof of Corollary 2 of Theorem 1.5. The proof of Corollary 2 is the same as the proof of Theorem 1.5 = Theorem B.3 in [BCDL], except that \( w = 2(\alpha_1 - \alpha_2) \) is now an approximate integer instead of an exact integer when
\[
(lP_{j1} - kP_{j2}) \equiv 0 \mod Q_j.
\]

From (2.14),
\[
F(w)F(0) - 1 > \pi a \exp(-1),
\]

with the particular choice of \( b \) given by
\[
b = b_j = \pi|\varepsilon_j|^{-2(Q_j)^2}.
\]
Instead of (1.11) we now have
\begin{equation}
(A.4) \quad S_n(b_j) > C \exp(-1)b_j(Q_j r(Q_j))^{-1} \log(b_j/Q_j) + O(b_j).
\end{equation}
From (1.15), (A.3) and (A.4), (1.16) follows immediately.

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**References**


