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The Variance of the Error Function in the Shifted Circle Problem Is a Wild Function of the Shift

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Abstract: We prove that the variance of the error function in the shifted circle problem, as a function of the shift, is a continuous function which has a sharp local maximum with infinite derivatives at every rational point on a plane.

1. Introduction

Let

$$N(R;\alpha)=\#\{m\in\mathbf{Z}^2:|m-\alpha|\leqq R\},\quad \alpha=(\alpha_1,\alpha_2)\in\mathbf{R}^2\ ,$$

be the number of lattice points inside the circle of radius R with the center at α, and

$$F(R;\alpha)=\frac{N(R;\alpha)-\pi R^2}{R^{1/2}}\ .$$

As was shown in [B] and [BCDL], the limit,

$$D(\alpha)=\lim_{T\to\infty}\frac{1}{T}\int_1^T|F(R;\alpha)|^2dR$$

exists and is equal to

$$D(\alpha)=(2\pi^2)^{-1}\sum_{n=1}^{\infty}n^{-3/2}|r_\alpha(n)|^2\ , \tag{1.1}$$

where

$$r_\alpha(n)=\sum_{k^2+l^2=n}e(k\alpha_1+l\alpha_2),\quad e(t)=\exp(2\pi it)\ .$$

For $\alpha=0$ (1.1) reduces to a classical result of Cramér (see [C]). After averaging (1.1) in α we get a formula of Kendall (see [K]):

$$\int_0^1\int_0^1 D(\alpha)d\alpha=(2\pi^2)^{-1}\sum_{n=1}^{\infty}n^{-3/2}r_0(n)\ .$$

$D(\alpha)$ is the mean value of $F^2(R;\alpha)$, and it is also equal to

$$D(\alpha) = \int_{-\infty}^{\infty} x^2 \nu_\alpha(dx)\,,$$

where $\nu_\alpha(dx)$ is the limit distribution of $F(R;\alpha)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_{\{R:\, a\leq F(R;\alpha)\leq b;\, 1\leq R\leq T\}} dR = \int_a^b \nu_\alpha(dx)$$

(see [B, BCDL]). In addition,

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T F(R;\alpha)dR = \int_{-\infty}^{\infty} x\nu_\alpha(dx) = 0\,,$$

hence $D(\alpha)$ is the variance of the limit distribution of $F(R;\alpha)$. The existence of a limit distribution of $F(R;\alpha)$ for $\alpha = 0$ was proved by Heath-Brown (see [H-B]).

Since the series in the RHS of (1.1) is uniformly convergent, $D(\alpha)$ is continuous in α. Here we prove

Theorem 1.1 *For every rational* $\beta\in\mathbf{Q}^2$,

$$\lim_{\alpha\to\beta} (|\log|\alpha-\beta||\cdot|\alpha-\beta|)^{-1}(D(\alpha)-D(\beta)) = -C(\beta), \quad C(\beta)>0\,. \tag{1.2}$$

Since $D(\alpha)$ has a sharp local maximum with infinite derivatives at every rational point, it is a wild function. By wild function we mean a continuous function which is nondifferentiable on a dense set. $C(\beta)$ is defined as follows. Let Q be the integer such that

$$2Q\beta_1 = n_1, \quad 2Q\beta_2 = n_2 \tag{1.3}$$

are integers and there is no common factor dividing all three of Q, n_1, n_2. Then

$$C(\beta) = C(16/\pi^2)(Qr(Q))^{-1}\,, \tag{1.4}$$

where

$$r(Q) = \prod_{p|Q}(1+p^{-1})\,, \tag{1.5}$$

with the product taken over primes p dividing Q, and

$$C = 3\ (Q \text{ even}), \quad C = 4\ (Q \text{ odd}, (n_1+n_2) \text{ even})\,,$$
$$C = 2\ (Q \text{ odd}, (n_1+n_2) \text{ odd})\,. \tag{1.6}$$

$D(\alpha)$ achieves its global maximum at $\alpha = m$ and $\alpha = m + (1/2, 1/2)$, $m\in\mathbf{Z}^2$. Indeed,

$$|r_\alpha(n)|^2 = \left|\sum_{k^2+l^2=n} e(k\alpha_1 + l\alpha_2)\right|^2 = \left|\sum_{k^2+l^2=n} \cos(2\pi k\alpha_1)\cos(2\pi l\alpha_2)\right|^2$$

is maximum when

$$|\cos(2\pi k\alpha_1)| = |\cos(2\pi l\alpha_2)| = 1$$

and sign $(\cos(2\pi k\alpha_1)\cos(2\pi l\alpha_2))$ is the same for all k, l with $k^2+l^2=n$. This holds for all $n\in\mathbf{N}$ iff $\alpha = m$ or $\alpha = m + (1/2, 1/2)$, $m\in\mathbf{Z}^2$.

It is to be noted that the wild behavior of $D(\alpha)$ is closely related to a bumpy shape of the exponential sum

$$S_\alpha(b) = \sum_{n=1}^{\infty} |r_\alpha(n)|^2 \exp(-n/b),$$

as a function of α, when $b \to \infty$ (see [BD]). $S_\alpha(b)$ is a key tool used to study the limit distribution of $F(R;\alpha)$ in [BCDL].

As a generalization of Theorem 1.1 consider the variance

$$D_I(\alpha) = \lim_{T\to\infty} \frac{1}{T}\int_1^T |F_I(R;\alpha)|^2 dR, \tag{1.7}$$

for a general lattice-point problem, with

$$F_I(R:\alpha) = \frac{N_I(R;\alpha) - AR^2}{R^{1/2}},$$

where

$$N_I(R;\alpha) = \#\{m\in\mathbf{Z}^2 : I(m-\alpha) \leqq R^2\}, \quad \alpha\in\mathbf{R}^2; \quad A = \mathrm{Area}\{x: I(x)\leqq 1\},$$

and $I(x) > 0$ is an arbitrary C^∞ positive convex homogeneous of order 2 function on $\mathbf{R}^2\setminus\{0\}$. As was proved in [B] the limit (1.7) always exists and is equal to

$$D_I(\alpha) = (2\pi^2)^{-1}\sum_{n=1}^{\infty} |u_\alpha(n)|^2, \tag{1.8}$$

where

$$u_\alpha(n) = \sum_{k,l:\,J(k,l)=J_n} |k^2+l^2|^{-3/4}\sqrt{\rho(k,l)}\,e(k\alpha_1 + l\alpha_2),$$

$0 = J_0 < J_1 < J_2 < \cdots$ are all possible values of

$$J(k,l) = \max_{x_1,x_2\in\mathbf{R}} [2(kx_1 + lx_2) - I(x_1,x_2)], \quad k,l\in\mathbf{Z},$$

and $\rho(k,l)$ is the curvature of the curve $\Gamma = \{x: I(x) = 1\}$ at the point $x\in\Gamma$ where grad $I(x)$ is collinear to the vector (k,l). The formula (1.8) is a generalization of (1.1).

By (1.8) $D_I(\alpha)$ is independent of α if for every $n = 1, 2, \ldots$, the set of (k,l) such that $J(k,l) = J_n$ consists of one point. This can be viewed as a "generic" case of $I(x)$, so that "generically" $D_I(\alpha)$ is constant in α. On the other hand, if $I(x)$ possesses some symmetry, say, $I(-x) = I(x)$, then $D_I(\alpha)$ is, in general, nonsmooth, since the Fourier coefficients of $D_I(\alpha)$ in (1.8) decay slowly. For instance, if $I(x) = (x_1/a)^2 + (x_2/b)^2$ and a^2/b^2 is irrational then $J(k,l) = (ak)^2 + (bl)^2$ and $J(k_1,l_1) = J(k_2,l_2)$ iff $k_1 = \pm k_2, l_1 = \pm l_2$. In this case it is not difficult to prove that $D_I(\alpha)$ is nondifferentiable at half-integer points $\alpha\in(1/2)\mathbf{Z}^2$. More precisely,

$$D_I(\alpha) = -C\sqrt{\frac{\sin^2 2\pi\alpha_1}{a^2} + \frac{\sin^2 2\pi\alpha_2}{b^2}} + R_I(\alpha),$$

where $C > 0$ and $R_I(\alpha)$ is differentiable everywhere.

In addition, an "arithmetic" degeneracy of $J(k,l)$ can worsen the smoothness of $D_I(\alpha)$. For instance, if $I(x) = (x_1/a)^2 + (x_2/b)^2$ and a^2/b^2 is rational, then $\#\{(k,l): J(k,l) = J_n\}$ is unbounded as $n\to\infty$. This "arithmetic" degeneracy of $J(k,l)$ causes wild nonsmoothness of the variance $D_I(\alpha)$, namely, with the help of the same method that we use in the proof of Theorem 1.1, we can prove that for every rational $\beta \in \mathbf{Q}^2$,

$$\lim_{\alpha \to \beta} (|\log|\alpha - \beta|| \cdot |\alpha - \beta|)^{-1} (D_I(\alpha) - D_I(\beta)) = -C_I(\beta), \quad C_I(\beta) > 0 .$$

The set-up of the remainder of the paper is the following. In Sect. 2 we prove some preliminary results for Theorem 1.1. The proof of Theorem 1.1 is slightly different for $\beta = 0$ and for $\beta \neq 0$. In Sect. 3 we prove Theorem 1.1 for $\beta = 0$, and in Sect. 4 we prove Theorem 1.1 for $\beta \neq 0$.

Throughout the paper $C, C_0, C_1, \ldots$ are considered to be fixed positive constants. However they often change value from one equation to the next.

2. Preliminaries

This section consists of identities valid for all α. The sum (1.1) may be written

$$D(\alpha) = (2\pi^2)^{-1} \sum_{n=1}^{\infty} n^{-3/2} \left| \sum_{m \in \mathbf{Z}^2 : |m|^2 = n} e(\alpha \cdot m) \right|^2$$
$$= (2\pi^2)^{-1} \sum_{mm'} |m|^{-3} e(\alpha \cdot (m - m')) , \tag{2.1}$$

summed over integer vectors m, m' with $m^2 = m'^2$. The sum (2.1) may be converted into an unrestricted sum (see Appendix B in [BCDL]),

$$D(\alpha) = 2\pi^{-2} \sum_{jhkl} e(h(l\alpha_1 - k\alpha_2))((j^2 + h^2)(k^2 + l^2))^{-3/2} , \tag{2.2}$$

summed over all $(j, h, k, l) \in \mathbf{Z}^4$ satisfying

$$j^2 + h^2 \neq 0, \quad k^2 + l^2 \neq 0 , \tag{2.3}$$

$$\text{either } j \equiv h \equiv 0, \text{ or } j \equiv h \equiv k \equiv l \equiv 1 \pmod 2 , \tag{2.4}$$

and

$$k, l \text{ are relatively prime} , \tag{2.5}$$

which means that either $|k| + |l| = 1$, or $\gcd(|k|, |l|) = 1$.

According to two possibilities allowed by (2.4), we divide $D(\alpha)$ into two parts,

$$D(\alpha) = D_e(\alpha) + D_o(\alpha) , \tag{2.6}$$

where the terms with j and h even are

$$D_e(\alpha) = \sum_{kl} U(w)(k^2 + l^2)^{-3/2} , \tag{2.7}$$

summed over integers (k, l) satisfying (2.5), and the terms with j and h odd are

$$D_o(\alpha) = \sum_{\text{odd } kl} V(w)(k^2 + l^2)^{-3/2} , \tag{2.8}$$

summed over odd integers k and l satisfying (2.5). The functions (U, V) are defined by

$$U(w) = (2\pi)^{-2} \sum_{j^2 + h^2 \neq 0} e(hw)(j^2 + h^2)^{-3/2}$$
$$= (2\pi)^{-2} \sum_{j^2 + h^2 \neq 0} \cos(2\pi hw)(j^2 + h^2)^{-3/2} ,$$

$$V(w) = (2\pi)^{-2} \sum_{\text{half-odd-integer } jh} e(hw)(j^2 + h^2)^{-3/2}$$
$$= (2\pi)^{-2} \sum_{\text{half-odd-integer } jh} \cos(2\pi hw)(j^2 + h^2)^{-3/2} . \tag{2.9}$$

In (2.7)–(2.9) we have used the abbreviation

$$w = 2(l\alpha_1 - k\alpha_2) . \tag{2.10}$$

According to (2.9),

$$U(w+1) = U(w), \quad V(w+1) = -V(w) ,$$
$$U(-w) = U(w), \quad V(-w) = V(w) . \tag{2.11}$$

Lemma 2.1. *$U(w)$ and $V(w)$ are infinitely differentiable on* $[0, 1]$, *and*

$$U'(+0) = V'(+0) = -1 . \tag{2.12}$$

Remark. (2.11), (2.12) imply that

$$U'(+0) - U'(-0) = V'(+0) - V'(-0) = -2 .$$

We shall use Lemma 2.1 with (2.11) to note that $U(w)$ and $V(w)$ are bounded absolutely; and also that $U(t) = -|t| + O(|t|^2)$ for sufficiently small t.

Proof of Lemma 2.1 is given in Appendix to the paper. We are now ready to prove Theorem 1.1. The proof is slightly different for $\beta = 0$ and $\beta \neq 0$. First we consider $\beta = 0$.

3. Proof of Theorem 1.1 for $\beta = 0$

Let $\zeta_i = |\alpha|^{-1}\alpha_i$, $i = 1, 2$, $|\alpha| = (\alpha_1^2 + \alpha_2^2)^{1/2}$, and $\zeta = (\zeta_1, \zeta_2)$, so that $\alpha = |\alpha|\zeta$, $|\zeta| = 1$. Then from (2.7),

$$|\alpha|^{-1}(D_e(\alpha) - D_e(0)) = |\alpha|^{-1} \sum_{kl} (U(2(l\alpha_1 - k\alpha_2)) - U(0))(k^2 + l^2)^{-3/2}$$
$$= \sum_{kl} (U(2(l|\alpha|\zeta_1 - k|\alpha|\zeta_2)) - U(0))((k|\alpha|)^2 + (l|\alpha|)^2)^{-3/2}|\alpha|^2$$
$$= \sum_{kl} \Phi(|\alpha|k, |\alpha|l)|\alpha|^2 \equiv I(|\alpha|) , \tag{3.1}$$

which is an approximating sum for the integral

$$I = (6/\pi^2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x_1, x_2) dx_1 dx_2 \tag{3.2}$$

with

$$\Phi(x_1, x_2) = (U(2(x_2\zeta_1 - x_1\zeta_2)) - U(0))(x_1^2 + x_2^2)^{-3/2} . \tag{3.3}$$

The summation in (3.1) goes over relatively prime k, l, and the factor $(6/\pi^2)$ in (3.2) is the density of pairs (k, l) with relatively prime k, l.

By Lemma 2.1, for small $|t|$, $U(t) - U(0) \sim -|t|$. This implies that the integral (3.2) diverges logarithmically at the origin. The approximating sum (3.1) is taken over points $(|\alpha|k, |\alpha|l) \neq 0$ which belong to the lattice with the space $|\alpha|$, so we may expect that $I(|\alpha|)$ behaves like $C|\log|\alpha||$.

We can estimate easily the part of the sum in (3.1) with $k^2 + l^2 \geqq |\alpha|^{-2}$. Lemma 2.1 implies that $U(t)$ is bounded, so this part is

$$\begin{aligned}\left|\sum_{k^2+l^2\geqq|\alpha|^{-2}} \Phi(|\alpha|k, |\alpha|l)|\alpha|^2\right| &\leqq \sum_{k^2+l^2\geqq|\alpha|^{-2}} |\Phi(|\alpha|k, |\alpha|l)|\,|\alpha|^2 \\ &\leqq C \sum_{k^2+l^2\geqq|\alpha|^{-2}} ((|\alpha|k)^2 + (|\alpha|l)^2)^{-3/2}|\alpha|^2 \\ &= C|\alpha|^{-1} \sum_{k^2+l^2\geqq|\alpha|^{-2}} (k^2 + l^2)^{-3/2} \leqq C_0 . \qquad (3.4)\end{aligned}$$

Let us fix large numbers M, N, which will be chosen later, and consider a sequence of squares

$$S_i = \{|x_1|, |x_2| \leqq M_i = M((N+1)/N)^i\}, \quad i \geqq 0 . \qquad (3.5)$$

We shall consider S_i with $i = 0, 1, \ldots, p$, where p is chosen in such a way, that

$$M_{p-1} \leqq |\alpha|^{-1} < M_p , \qquad (3.6)$$

or in other words,

$$p = [(\log(1 + N^{-1}))^{-1}||\log|\alpha|| - \log M|] + 1 . \qquad (3.7)$$

The choice of S_i ensures the following property of commensurability of S_i and S_{i+1}: We can partition S_i into $4N^2$ squares of side $N^{-1}M_i$, and S_{i+1} into $4(N+1)^2$ squares of the same side. This implies that the square annulus $S_{i+1} \backslash S_i$ is partitioned into $4(2N+1)$ squares S_{ij} of side $N^{-1}M_i$. Let m_{ij} be the center of S_{ij}.

Consider the sum

$$I_{ij}(|\alpha|) = \sum_{S_{ij}} \Phi(|\alpha|m)|\alpha|^2 ,$$

where the summation goes over relatively prime k, l with $m = (k, l) \in S_{ij}$. $I_{ij}(|\alpha|)$ is the S_{ij}-part of the sum $I(|\alpha|)$ in (3.1). We want to compare $I_{ij}(|\alpha|)$ first with

$$J_{ij}(|\alpha|) = \Phi(|\alpha|m_{ij})|\alpha|^2 \sum_{S_{ij}} 1 ,$$

and then with

$$I_{ij} = (6/\pi^2) \int_{X_{ij}} \Phi(x)dx ,$$

where

$$X_{ij} = |\alpha|S_{ij} = \{x = |\alpha|y, y \in S_{ij}\} .$$

Denote by $x_{ij} = |\alpha|m_{ij}$ the center of the square X_{ij}. The side of X_{ij} is equal to $|\alpha|N^{-1}M_i$, hence for every $x \in X_{ij}$,

$$|x - x_{ij}| \leqq |\alpha|N^{-1}M_i .$$

Let

$$U_0(x) = U(2(x_2\zeta_1 - x_1\zeta_2)) - U(0) , \qquad (3.8)$$

so that

$$\Phi(x) = U_0(x)|x|^{-3} . \qquad (3.9)$$

By Lemma 2.1, $U(w)$ is a periodic Lipshitz function, hence

$$|U_0(x) - U_0(x_{ij})| \leqq C|x - x_{ij}| \leqq C|\alpha|N^{-1}M_i , \qquad (3.10)$$

$$|U_0(x)| \leqq C|x| . \qquad (3.11)$$

Also,

$$||x|^{-3}-|x_{ij}|^{-3}|\leqq C|x-x_{ij}||x_{ij}|^{-4}\leqq C_0|\alpha|N^{-1}M_i(|\alpha|M_i)^{-4}$$
$$=C_0|\alpha|^{-3}N^{-1}(M_i)^{-3}.$$

When $x\in X_{ij}$,

$$|\Phi(x)-\Phi(x_{ij})|\leqq|U_0(x)-U_0(x_{ij})||x|^{-3}+|U_0(x_{ij})|||x|^{-3}-|x_{ij}|^{-3}|$$
$$\leqq C(|\alpha|N^{-1}M_i(|\alpha|M_i)^{-3}+|\alpha|M_i|\alpha|^{-3}N^{-1}(M_i)^{-3})$$
$$=2CN^{-1}(|\alpha|M_i)^{-2}.$$

Therefore,

$$|I_{ij}(|\alpha|)-J_{ij}(|\alpha|)|=\left|\sum_{S_{ij}}(\Phi(|\alpha|m)-\Phi(|\alpha|m_{ij}))|\alpha|^2\right|$$
$$\leqq\sum_{S_{ij}}CN^{-1}(|\alpha|M_i)^{-2}|\alpha|^2$$
$$=CN^{-1}(M_i)^{-2}\sum_{S_{ij}}1\leqq CN^{-1}(M_i)^{-2}(N^{-1}M_i)^2$$
$$=CN^{-3}. \tag{3.12}$$

Similarly, since Area $X_{ij}=(|\alpha|N^{-1}M_i)^2$, and by (3.9), (3.11),

$$|\Phi(x)|\leqq C|x|^{-2}\leqq C(|\alpha|M_i)^{-2}, \tag{3.13}$$

we obtain

$$|I_{ij}-J_{ij}(|\alpha|)|=\left|(6/\pi^2)\int_{X_{ij}}(\Phi(x)-\Phi(x_{ij}))dx\right.$$
$$\left.+\Phi(x_{ij})\left((6/\pi^2)(|\alpha|N^{-1}M_i)^2-|\alpha|^2\sum_{S_{ij}}1\right)\right|$$
$$\leqq C(N^{-1}(|\alpha|M_i)^{-2}(N^{-1}|M_i||\alpha|)^2$$
$$+(|\alpha|M_i)^{-2}|\alpha|^2(N^{-1}M_i)^2\varepsilon_{MN})$$
$$=2C(N^{-3}+N^{-2}\varepsilon_{MN}), \tag{3.14}$$

where

$$\varepsilon_{MN}=\sup_{ij}\left|(6/\pi^2)-(N^{-1}M_i)^{-2}\sum_{S_{ij}}1\right|. \tag{3.15}$$

Since $M_{i+1}\geqq M_0=M$,

$$\lim_{M\to\infty}\varepsilon_{MN}=0. \tag{3.16}$$

To prove (3.16) let us assume that S is an arbitrary square of side $a>0$ on the plane such that the origin is outside of S and $\max_{x\in S}|x|\leqq La$, where $L>0$ is a fixed number. Then by the Möbius inversion formula,

$$\sum_{(k,l)\in S:\gcd(k,l)=1}1=\sum_{d=1}^{\infty}\mu(d)\sum_{(dk,dl)\in S}1=\sum_{d=1}^{La}\mu(d)\left[\frac{a^2}{d^2}+O\left(\frac{a}{d}\right)\right]$$
$$=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}a^2+O(a\log a)=\frac{6}{\pi^2}a^2+O(a\log a),\quad a\to\infty,$$

where $\mu(d)$ is the Möbius function. This proves (3.16).

From (3.12), (3.14) we obtain

$$|I_{ij}(|\alpha|) - I_{ij}| \leqq C(N^{-3} + N^{-2}\varepsilon_{MN}) . \tag{3.17}$$

Due to (3.7),

$$p \leqq CN|\log|\alpha|| , \tag{3.18}$$

hence (3.17) implies

$$\left|\sum_{i=0}^{p}\sum_{j=1}^{4(2N+1)} I_{ij}(|\alpha|) - \sum_{i=0}^{p}\sum_{j=1}^{4(2N+1)} I_{ij}\right| \leqq Cp(N^{-2} + N^{-1}\varepsilon_{MN})$$

$$\leqq C_0(N^{-1} + \varepsilon_{MN})|\log|\alpha|| , \tag{3.19}$$

or

$$\left|\sum_{S_p\setminus S_0} \Phi(|\alpha|m)|\alpha|^2 - (6/\pi^2)\int_{X_p\setminus X_0} \Phi(x)dx\right| \leqq C(N^{-1} + \varepsilon_{MN})|\log|\alpha|| , \tag{3.20}$$

where

$$X_i = \{|x_1|, |x_2| \leqq |\alpha|M_i\} = |\alpha|S_i . \tag{3.21}$$

Notice that by (3.6),

$$|\alpha|M_p \geqq 1 ,$$

hence, when $m = (k, l)\in \mathbf{Z}^2\setminus S_p$, either $|k| \geqq M_p \geqq |\alpha|^{-1}$, or $|l| \geqq |\alpha|^{-1}$, hence by (3.4),

$$\left|\sum_{\mathbf{Z}^2\setminus S_p} \Phi(|\alpha|m)|\alpha|^2\right| \leqq \sum_{|m|\geqq|\alpha|^{-1}} |\Phi(|\alpha|, m)||\alpha|^2 \leqq C . \tag{3.22}$$

Similarly,

$$\left|\int_{\mathbf{R}^2\setminus X_p} \Phi(x)dx\right| \leqq C \int_{\{|x|\geqq 1\}} |x|^{-3}dx \leqq C_0 . \tag{3.23}$$

By (3.13),

$$\left|\sum_{S_0} \Phi(|\alpha|m)|\alpha|^2\right| \leqq C\sum_{S_0} |m|^{-2} \leqq C \sum_{0<|m|\leqq 2M} |m|^{-2} \leqq C_0 \log M , \tag{3.24}$$

and similarly,

$$\left|\int_{\{X_0\setminus\{|x|\leqq|\alpha|\}\}} \Phi(x)dx\right| \leqq C \int_{\{|\alpha|\leqq|x|\leqq 2M|\alpha|\}} |x|^{-2}dx \leqq C_0 \log M . \tag{3.25}$$

Equations (3.20), (3.22)–(3.25) imply

$$\left|\sum_{m} \Phi(|\alpha|m)|\alpha|^2 - (6/\pi^2)\int_{|x|\geqq|\alpha|} \Phi(x)dx\right| \leqq C((N^{-1} + \varepsilon_{MN})|\log|\alpha|| + \log M) , \tag{3.26}$$

hence by (3.1),

$$\left||\alpha|^{-1}(D_e(\alpha) - D_e(0)) - (6/\pi^2)\int_{|x|\geqq|\alpha|} \Phi(x)dx\right| \leqq C((N^{-1} + \varepsilon_{MN})|\log|\alpha|| + \log M) . \tag{3.27}$$

For every $\delta > 0$ we can choose N such that $CN^{-1} < \delta$, and then $M > N$ such that $C\varepsilon_{MN} < \delta$, so that the RHS of (3.27) is less than $2\delta|\log|\alpha|| + C\log M$. Now we can choose $\varepsilon = 1/M^N$ so that $C\log M < \delta|\log|\alpha||$, when $|\alpha| < \varepsilon$. Hence by (3.27),

$$\left| |\alpha|^{-1}(D_e(\alpha) - D_e(0)) - (6/\pi^2) \int_{|x| \geqq |\alpha|} \Phi(x)dx \right| < 3\delta|\log|\alpha||\,,$$

when $|\alpha| < \varepsilon$. This proves that

$$|\alpha|^{-1}(D_e(\alpha) - D_e(0)) - (6/\pi^2) \int_{|x| \geqq |\alpha|} \Phi(x)dx = o(|\log|\alpha||), \quad |\alpha| \to 0\,. \tag{3.28}$$

By (3.3),

$$\int_{|x| \geqq |\alpha|} \Phi(x)dx = \int_{|x| \geqq |\alpha|} (U(2(x_2\zeta_1 - x_1\zeta_2)) - U(0))|x|^{-3}dx$$
$$= \int_{|y| \geqq |\alpha|} (U(2y_1) - U(0))|y|^{-3}dy\,,$$

where $y = (y_1, y_2)$ with

$$y_1 = x_2\zeta_1 - x_1\zeta_2\,,$$
$$y_2 = x_2\zeta_2 + x_1\zeta_1\,.$$

For small $|y_1|$,

$$U(2y_1) - U(0) = -2|y_1| + O(y_1^2)\,.$$

A straightforward evaluation gives

$$\int_{1 \geqq |y| \geqq |\alpha|} |y_1||y|^{-3}dy = \int_0^{2\pi} d\varphi \int_{|\alpha|}^{1} r\,dr\,r|\cos\varphi|r^{-3} = 4|\log|\alpha||\,,$$

hence

$$\int_{|y| \geqq |\alpha|} (U(2y_1) - U(0))|y|^{-3}dy = -8|\log|\alpha|| + O(1)\,,$$

hence using the fact that $U(\cdot)$ is bounded,

$$\int_{|x| \geqq |\alpha|} \Phi(x)dx = -8|\log|\alpha|| + O(1)\,. \tag{3.29}$$

Therefore from (3.28),

$$|\alpha|^{-1}(D_e(\alpha) - D_e(0)) = -(48/\pi^2)|\log|\alpha|| + o(|\log|\alpha||)\,,$$

or in other words,

$$\lim_{\alpha \to 0} (|\log|\alpha|| \cdot |\alpha|)^{-1}(D_e(\alpha) - D_e(0)) = -48/\pi^2\,. \tag{3.30}$$

The same considerations give

$$\lim_{\alpha \to 0} (|\log|\alpha|| \cdot |\alpha|)^{-1}(D_o(\alpha) - D_o(0)) = -16/\pi^2\,. \tag{3.31}$$

The result for the odd part is three times less because the density of relatively prime odd pairs k, l is $2/\pi^2$, and not $6/\pi^2$. From (3.30), (3.31),

$$\lim_{\alpha \to 0} (|\log|\alpha|| \cdot |\alpha|)^{-1}(D(\alpha) - D(0)) = -64/\pi^2\,.$$

For $\beta = 0$ Theorem 1.1 is proved.

4. Proof of Theorem 1.1 for $\beta \neq 0$

Let us partition all relatively prime pairs k, l into subsets

$S_r = \{k, l$ are relatively prime and $ln_1 - kn_2 \equiv r \bmod 2Q\}$, $r = 0, 1, \ldots, 2Q-1$.

We can rewrite (2.17) as

$$D_e(\alpha) = \sum_r D_{er}(\alpha) \tag{4.1}$$

with

$$D_{er}(\alpha) = \sum_{S_r} U(w)(k^2 + l^2)^{-3/2} . \tag{4.2}$$

Our aim is to estimate

$$D_{er}(\alpha) - D_{er}(\beta) = \sum_{S_r} (U(w) - U(v))(k^2 + l^2)^{-3/2} ,$$

$$w = 2(l\alpha_1 - k\alpha_2), \quad v = 2(l\beta_1 - k\beta_2) = Q^{-1}(ln_1 - kn_2) \equiv Q^{-1} r \bmod 2 . \tag{4.3}$$

Denote

$$\delta = \alpha - \beta, \quad \zeta = |\delta|^{-1}\delta = (\zeta_1, \zeta_2), \quad \eta = (-\zeta_2, \zeta_1), \quad m = (k, l) \in \mathbf{Z}^2.$$

Then (4.3) reduces to

$$|\delta|^{-1}(D_{er}(\alpha) - D_{er}(\beta)) = \sum_{S_r} (U(Q^{-1}r + 2|\delta| m \cdot \eta) - U(Q^{-1}r))|m|\delta||^{-3}|\delta|^2$$

$$= \sum_{S_r} \Phi_r(|\delta| m)|\delta|^2 , \tag{4.4}$$

where

$$\Phi_r(x) = (U(Q^{-1}r + 2x \cdot \eta) - U(Q^{-1}r))|x|^{-3}, \quad x \cdot \eta = x_1\eta_1 + x_2\eta_2 .$$

As in the proof of (3.28) we obtain now

$$|\delta|^{-1}(D_{er}(\alpha) - D_{er}(\beta)) = d_r \int_{|x| \geqq |\delta|} \Phi_r(x)dx + o(|\log|\delta||) , \tag{4.5}$$

where d_r is the density of S_r.

By Lemma 2.1 $U(w) \in C^\infty([0, 1])$, so when $r \neq 0, Q$,

$$\Phi_r(x) = Cx \cdot \eta |x|^{-3} + O(|x|^{-1}), \quad |x| \to 0 . \tag{4.6}$$

Since

$$\int_{1 \geqq |x| \geqq |\delta|} x \cdot \eta |x|^{-3} dx = 0 ,$$

(4.5) implies

$$|\delta|^{-1}(D_{er}(\alpha) - D_{er}(\beta)) = o(|\log|\delta||), \quad r \neq 0, Q . \tag{4.7}$$

In the case when $r = 0, Q$, $\Phi_r(x) = \Phi(x)$ and so by (3.29),

$$\int_{|x| \geqq |\delta|} \Phi_r(x)dx = \int_{|x| \geqq |\delta|} \Phi(x)dx = -8|\log|\delta|| + O(1) ,$$

hence (4.5) implies

$$|\delta|^{-1}(D_{er}(\alpha) - D_{er}(\beta)) = -8d_r|\log|\delta|| + o(|\log|\delta||), \quad r = 0, Q . \tag{4.8}$$

From (4.7), (4.8),

$$|\delta|^{-1}(D_e(\alpha) - D_e(\beta)) = -8\Delta|\log|\delta|| + o(|\log|\delta||) , \tag{4.9}$$

where $\Delta = d_0 + d_Q$ is the density of k, l such that

$$\gcd(k, l) = 1 \tag{4.10}$$

and

$$ln_1 - kn_2 \equiv 0 \bmod Q . \tag{4.11}$$

Similar considerations applying to $D_o(\alpha) - D_o(\beta)$ leads us to

$$|\delta|^{-1}(D_o(\alpha) - D_o(\beta)) = -8(\Delta_1 - \Delta_2)|\log|\delta|| + o(|\log|\delta||) , \tag{4.12}$$

where $\Delta_{1,2}$ are the densities of odd relatively prime pairs k, l such that

$$ln_1 - kn_2 \equiv 0, Q \bmod 2Q , \tag{4.13}$$

respectively.

From (4.9), (4.12) (remember $\delta = \alpha - \beta$),

$$|\alpha - \beta|^{-1}(D(\alpha) - D(\beta)) = -8(\Delta + \Delta_1 - \Delta_2)|\log|\alpha - \beta|| + o(|\log|\alpha - \beta||) . \tag{4.14}$$

which implies

$$\lim_{\alpha \to \beta} (|\alpha - \beta||\log|\alpha - \beta||)^{-1}(D(\alpha) - D(\beta)) = -8(\Delta + \Delta_1 - \Delta_2) . \tag{4.15}$$

It remains to calculate Δ and the other densities in (4.15). Δ is the density of pairs (k, l) satisfying (4.10) and (4.11). Since the highest common factor of n_1 and n_2 is prime to Q, the pairs (k, l) satisfying (4.11) form an integer lattice of density Q^{-1}. Within this lattice, the condition (4.10) eliminates a fraction $f(p)$ of pairs, independently for every prime p. Therefore

$$\Delta = (Q^{-1})\prod_p (1 - f(p)) , \tag{4.16}$$

with

$$f(p) = p^{-1} \text{ if } p \text{ divides } Q, \quad f(p) = p^{-2} \text{ otherwise} . \tag{4.17}$$

Since

$$\prod_p (1 - p^{-2}) = (6/\pi^2) , \tag{4.18}$$

Eq. (4.16) with (1.5) gives

$$\Delta = (Qr(Q))^{-1}(6/\pi^2) . \tag{4.19}$$

If we ignore the condition (4.13), the density of pairs of odd (k, l) satisfying (4.11) and (4.10) is (4.16) with the factor $(1 - f(2))$ arising from the prime $p = 2$ replaced by (1/4). Therefore

$$\Delta_1 + \Delta_2 = (1/3)\Delta, \quad (Q \text{ odd}); \; \Delta_1 + \Delta_2 = (1/2)\Delta, (Q \text{ even}) . \tag{4.20}$$

On the other hand, if Q is odd, both k and l being odd, the condition (4.13) reduces to

$$(n_1 + n_2) \equiv (0, 1) (\text{mod } 2) , \tag{4.21}$$

which implies

$$\Delta_1 = (1/3)\Delta, \quad \Delta_2 = 0 \quad ((n_1 + n_2) \text{ even});$$

$$\Delta_1 = 0, \quad \Delta_2 = (1/3)\Delta \quad ((n_1 + n_2) \text{ odd}). \tag{4.22}$$

Finally, if Q is even, at least one of n_1 and n_2 must be odd, and the two cases (4.13) will be satisfied equally often, so that

$$\Delta_1 = \Delta_2 = (1/4)\Delta, \quad (Q \text{ even}) . \tag{4.23}$$

Putting together (4.20), (4.22) and (4.23), we obtain

$$\Delta + \Delta_1 - \Delta_2 = (C/3)\Delta , \tag{4.24}$$

with C given by (1.6). Putting together (4.24) with (4.15) and (4.19), we obtain (1.2) with (1.4). Theorem 1.1 is proved.

Appendix. Proof of Lemma 2.1

We have

$$U(w) = \frac{1}{2\pi^{5/2}} \int_0^\infty a^{-5/2}(F_a(w)F_a(0) - 1)da ,$$

$$V(w) = \frac{1}{2\pi^{5/2}} \int_0^\infty a^{-5/2} G_a(w)G_a(0)da ,$$

with

$$\sum_x \exp(-x^2/a)e(xw) = F_a(w) \text{ or } G_a(w) , \tag{A.1}$$

where the sum is over integer x for F_a and over half-odd-integer x for G_a. By the Poisson summation formula, (A.1) gives

$$F_a(w) = (\pi a)^{1/2} \sum_{p=-\infty}^{\infty} \exp(-\pi^2 a(p+w)^2) ,$$

$$G_a(w) = (\pi a)^{1/2} \sum_{p=-\infty}^{\infty} (-1)^p \exp(-\pi^2 a(p+w)^2) . \tag{A.2}$$

Divide integrals into $a < 1$ and $a > 1$. Integrals $a < 1$ are analytic in w by (A.1). Integrals $a > 1$ are analytic in w by (A.2) when w is real and not integer. So U and V are analytic on (0, 1). If w is close to 0, we have by (A.2),

$$F_a(w)F_a(0),\, G_a(w)G_a(0) = \pi a \exp(-\pi^2 a w^2) + R_a(w) ,$$

where $\int_1^\infty a^{-5/2} R_a(w)da$ is even and analytic in w. Only the first term contributes to $U'(w)$, $V'(w)$ as $w \to 0$, and gives

$$U'(w),\, V'(w) \sim -\pi^{1/2} w \int_1^\infty a^{-1/2} \exp(-\pi^2 a w^2)da$$

$$= -2\pi^{-1/2} \frac{w}{|w|} \int_{\pi w}^\infty \exp(-b^2)db ,$$

which is analytic at $w = +0$ with $U'(+0) = V'(+0) = -1$.

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References

[B] Bleher, P.M.: On the distribution of the number of lattice points inside a family of convex ovals. Duke Math. J. **67**, 461–481 (1992)

[BCDL] Bleher, P.M., Cheng, Zh., Dyson, F.J., Lebowitz, J.L.: Distribution of the error term for the number of lattice points inside a shifted circle. Commun. Math. Phys. **154**, 433–469 (1993)

[BD] Bleher, P.M., Dyson, F.J.: Mean square value of exponential sums related to representation of integers as sum of two squares. Preprint IASSNS-HEP-92/84, Institute for Advanced Study, Princeton, 1992

[C] Cramér, H.: Über zwei Sätze von Herrn G.H. Hardy. Math. Zeit. **15**, 201–210 (1922)

[H-B] Heath-Brown, D.R.: The distribution and moments of the error term in the Dirichlet divisor problem. Acta Arithmetica **60**, 389–415 (1992)

[K] Kendall, D.G.: On the number of lattice points inside a random oval. Quart. J. Math. (Oxford) **19**, 1–26 (1948)

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