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RIGIDITY OF THE CRITICAL PHASES ON A CAYLEY TREE

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Dedicated to Robert Minlos on the occasion of his 70th birthday

Abstract. We discuss statistical mechanics on nonamenable graphs, and we study the features of the phase transition, which are due to nonamenability. For the Ising model on the usual lattice it is known that below the critical temperature fluctuations of magnetization are much less likely in the states with nonzero magnetic field than in the pure states with zero field. We show that on the Cayley tree the corresponding fluctuations have the same order.

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1. Introduction and statement of results

This paper is a result of our attempt to understand the nature of the statistical mechanics on nonamenable graphs, and in particular the special nature of the phenomenon of the first order phase transition, whose specifics are due to nonamenability. The topic of the statistical mechanics on nonamenable graphs is a modern growing field, and for its present status the reader can consult, e.g., the papers [Ly] and [S].

Let G be an infinite, locally finite, connected graph with vertex (or site) set V; let 0 be an arbitrarily chosen vertex. We are interested in the properties of the ferromagnetic low-temperature Ising model on G under magnetic field H. So we introduce the notation

$$\mu^G_{\pm,T,H}$$

for the infinite volume Gibbs measure on the set of configurations $\Omega = \{\sigma_x = \pm 1, x \in V\}$, corresponding to the temperature T, magnetic field H and ($\pm$)-boundary conditions. They are defined as follows. We first take a finite subset $\Lambda \subset V$, and

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we define the measures $\mu^{G,\Lambda}_{\pm,T,H}$ on $\Omega_\Lambda = \{\sigma_\Lambda\colon \sigma_x = \pm 1,\, x \in \Lambda\}$ to be given by

$$\mu^{G,\Lambda}_{\pm,T,H}(\sigma_\Lambda) = Z_\Lambda^{-1}(\pm, T, H) \exp\left\{ \frac{1}{T} \left(\sum_{\substack{x,y\in\Lambda,\\ x,y \text{ n.n.}}} \sigma_x\sigma_y + H \sum_{x\in\Lambda} \sigma_x \pm \sum_{\substack{x\in\Lambda, y\in V\setminus\Lambda,\\ x,y \text{ n.n.}}} \sigma_x \right) \right\}.$$

Here the symbol n.n. means that the summation goes over all pairs of the nearest neighbour sites, and the factors $Z_\Lambda(\pm, T, H)$ (the partition function) are chosen in such a way that $\mu^{G,\Lambda}_{\pm,T,H}(\Omega_\Lambda) = 1$. It turns out that the limits $\lim_{\Lambda\to V} \mu^{G,\Lambda}_{\pm,T,H}$ exist, and those are the Gibbs measures needed. We define the spontaneous magnetization $m_\pm(T, H)$ at 0 to be the expectation $\langle\sigma_0\rangle^G_{\pm,T,H}$ of the random variable σ_0 according to the measure $\mu^G_{\pm,T,H}$. The graphs of the functions $m_\pm(T, H)$ can be seen on Fig. 1, for the case in which G is a Cayley tree. (Apart from the (+)-state and (−)-state there are other automorphism invariant states of the Ising model on Cayley trees; these states correspond to the intermediate (decaying) branch of the values of $m(T, H)$, seen on Fig. 1, but we will not be concerned with those states in the present paper. They are discussed in [BRZ1].)

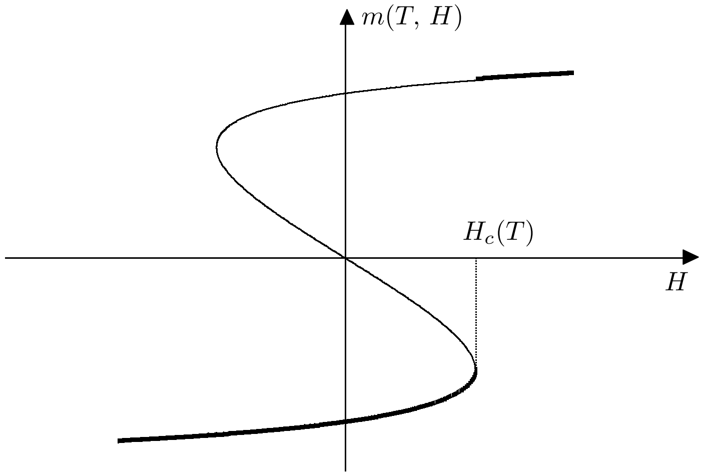

FIGURE 1. The values of the magnetization $m(T, H)$ of the Ising model on the Cayley tree $\mathcal{T}_2$ for a fixed value $T < T_c$ in various phases. The bold part of the line is the graph of the function $m_-(T, H)$ — the magnetization in the (−)-phase. This function is left-continuous.

In what follows, recall that an *automorphism* of a graph is a one-to-one onto transformation from the set of sites to itself which preserves the structure of the graph. A graph is said to be *transitive* if for any two sites x and y there is an automorphism of the graph which maps x into y (intuitively, all the sites play the same role). All the graphs discussed in this paper are assumed to be transitive. A graph is said to be *amenable* if there is a sequence of finite sets of sites, S_n, such that $|\partial_E S_n|/|S_n| \to 0$, as $n \to \infty$, where $\partial_E S$ is the edge-boundary of S, i.e., the set of bonds which have one endpoint in S and one endpoint outside S.

In case when G is the (amenable) lattice $\mathbb{Z}^d$, the following statements are well known: the measures $\mu^{\mathbb{Z}^d}_{\pm,T,H}$ do not depend on the choice of boundary conditions $+$ or $-$ if $H \neq 0$ or if $T > T_c(d)$. For $H = 0$ and $T < T_c(d)$ the two measures $\mu^{\mathbb{Z}^d}_{+,T,0}$ and $\mu^{\mathbb{Z}^d}_{-,T,0}$ are distinct. (The critical temperature $T_c(d)$ is positive iff $d > 1$.) The absence of phase transition when $H \neq 0$ was extended to all transitive amenable graphs in [JS]. Also in [JS] it was proved that the situation is different on transitive nonamenable graphs. For the latter it is proved that at low temperatures there is a phase transition even for nonzero H. Combining this result with Proposition 2 in [ST], we learn that there exists a (graph dependent) function $H_c(T)$, $0 \leq T \leq T_c \equiv T_c(G)$, such that for low T, $H_c(T) > 0$, and such that if $T < T_c$ and $|H| < H_c(T)$, then the measures $\mu^G_{+,T,H}$ and $\mu^G_{-,T,H}$ are different, while for $T \leq T_c$ and $|H| > H_c(T)$ or for $T > T_c$ they coincide. The phase transition regions for both cases are shown on Fig. 2.

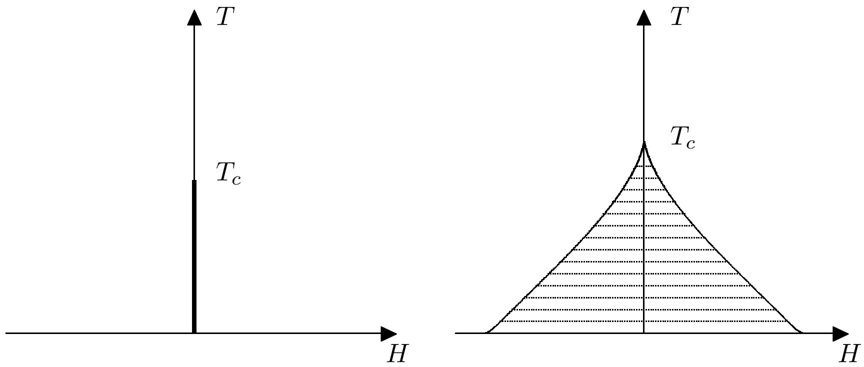

FIGURE 2. The phase diagrams for the Ising model on the lattice $\mathbb{Z}^d$ (left) and on the Cayley tree $\mathcal{T}_2$ (right). The boundary of the non-uniqueness region is made by the graphs of the functions $\pm H_c(T)$, $0 \leq T \leq T_c$.

Our original interest was to analyze the structure of the phase diagram on the curves $(\pm H_c(T), T)$. For amenable transitive graphs we have $H_c(T) = 0$, so this curve is just the segment $\{0 < T < T_c,\ H = 0\}$, and on it we have at least two distinct phases, μ^G_{+,T,H_c} and μ^G_{-,T,H_c}. For the general transitive nonamenable graph G the question whether the states μ^G_{+,T,H_c} and μ^G_{-,T,H_c} are different is open. But this is known to be the case when G is a Cayley tree, $\mathcal{T}_b$, also called a homogeneous tree. Here $\mathcal{T}_b$ denotes the infinite tree of degree $b+1$, which means that every site of it has precisely $b+1$ nearest neighbors. For $\mathcal{T}_b$ the answer to the above question is known because the Ising models on homogeneous trees are exactly solvable (by means of recurrent relations), see [G] or [BRZ2].

In order to understand the behavior of our models on the transition curve $(H_c(T), T)$, let us look on the behavior of large deviation probabilities. Specifically, we have in mind the following question. Let $\Lambda \subset G$ be a ball, and

$$M_\Lambda = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x$$

be the magnetization inside Λ. A relevant quantity to look at is the probability of the fluctuation of the magnetization:

$$\mu^{G}_{-,T,H}\{M_\Lambda > m_-(T,\, H) + \delta\} \tag{1}$$

for $\delta > 0$. For example, for the Ising model on $\mathbb{Z}^d$ below the critical temperature the behavior of the probabilities (1) for large Λ is quite different depending on whether or not H is below or at its critical value, H_c, which is 0 for $G = \mathbb{Z}^d$. Indeed, for any $H < 0$ and δ small enough ($\delta < \delta(H)$)

$$\mu^{\mathbb{Z}^d}_{-,T,H}\{M_\Lambda > m_-(T,\, H) + \delta\} \sim \exp\{-c(\delta)|\Lambda|\}, \tag{2}$$

while for $H = H_c = 0$ one expects, and for $d = 2$ or for low enough T one knows, that

$$\mu^{\mathbb{Z}^d}_{-,T,H=0}\{M_\Lambda > m_-(T,\, 0) + \delta\} \sim \exp\{-c'(\delta)|\partial\Lambda|\}, \tag{3}$$

where $\partial\Lambda$ is the boundary of Λ, which for regular Λ contains only $\sim |\Lambda|^{(d-1)/d}$ sites, see [DKS], [CP], [BIV] and references therein. We call the phase *rigid*, if the probabilities (1) behave as in (2), i.e., as $\exp\{-c(\delta)|\Lambda|\}$; in case their behavior is $\exp\{-o(|\Lambda|)\}$ for Λ large, we call the phase *soft*. According to this terminology, the critical phase $\mu^{\mathbb{Z}^d}_{-,T,H=0}$ is soft: indeed, it is much easier to create a deviation in the density of $(-)$-spins in the critical phase $\mu^{\mathbb{Z}^d}_{-,T,H=0}$ than in the phases $\mu^{\mathbb{Z}^d}_{-,T,H<0}$. Here and below we use the word "critical" to denote the phase which is an "end-point" of the continuous curve of states in the space of Gibbs states. More precisely, the phase will be called *critical* if it is a point of discontinuity of the family of Gibbs states corresponding to the continuous family of interactions.

The reason for having different behaviors in (2) and (3) can be understood from the fact that the typical configurations forming the events $\{M_\Lambda > m_-(T,\, H) + \delta\}$ look quite different for $H < 0$ and for $H = 0$. To explain this, we observe first that the typical configurations of the state $\mu^{\mathbb{Z}^d}_{-,T,H}$ itself (at low T-s) look as follows: for $H \le 0$ they consist of sea of $(-)$-spins with rare $(+)$-islands. The typical configurations of the event $\{M_\Lambda > m_-(T,\, H)+\delta\}$ with $H < 0$ look qualitatively the same: they are still seas of $(-)$-spins with rare $(+)$-islands, and the only difference is that the density of the $(+)$-islands as well as their sizes are slightly bigger; this increase accounts for the behavior in (2). However, for $H = 0$ the picture is drastically different: a typical configuration of the $(-)$-state under event $\{M_\Lambda > m_-(T,\, 0) + \delta\}$ looks as follows: inside the box Λ it has a large single droplet of the size of the order of Λ, filled with the $(+)$-phase, while the rest of Λ is filled with the $(-)$-phase. (In fact, the above picture is correct only if the size of Λ exceeds a certain critical size, which depends on δ; see [SS] for details.) Because of the $\pm$-symmetry, the energy cost of having such a droplet is proportional to $|\partial\Lambda|$, which explains (3).

Another way to express the difference between the behaviors in (2) and (3) is the following: in the phase transition regime we have a competition between the "magnetic field" volume term and the boundary condition surface term. Such a competition is not "fair", of course, and the volume term, if present, always beats the surface term. The only case when the competition is won by the surface term is when the volume term vanishes, and then the surface term — the only one left! — appears in the large deviation exponent.

For nonamenable graphs the situation is very different. There both the magnetic field term and the boundary term are volume terms, and the (positive) critical value $H_c(T)$ of the magnetic field is by definition the one which balances the influence of the (negative) boundary conditions. Due to the cancellation of the volume terms, the question of the behavior of the fluctuations of the magnetization at the critical value $H_c(T)$ of the magnetic field becomes nontrivial.

Our conjecture is that for nonamenable graphs G the behavior of large deviations (1) is the same for $H \neq H_c(T)$ and for $H = H_c(T)$; in other words, we think that for every H

$$\mu^G_{-,T,H}\{M_\Lambda > m_-(T,\,H)+\delta\} \sim \exp\{-c(\delta)|\Lambda|\} \tag{4}$$

with $c(\delta) > 0$, so in particular the critical phase $\mu^G_{-,T,H=H_c(T)}$ is rigid, unlike the amenable case. (Clearly, for $H > H_c(T)$ the phase $\mu^G_{-,T,H}$ is nothing else but $\mu^G_{+,T,H}$, while $m_-(T,\,H) = m_+(T,\,H)$.) Of course, for generic values of the magnetic field H, the conjecture (4) is straightforward, since in this case either the bulk term dominates or the boundary term dominates, and both of them are volume terms for nonamenable graphs. The subtle case is when $H = H_c(T)$, and the two terms cancel. At present we are not able to prove (4) for general nonamenable graphs. What we are able to show is that such behavior indeed takes place for $G = \mathcal{T}_b$, $b \geq 2$. The main result of our paper is contained in the following

Theorem 1. *Let $\Lambda = B_R \subset \mathcal{T}_b$ be a ball of radius R centered at the origin. For every temperature T and every magnetic field H there exists a function $\phi_{H,T}(m) \geq 0$ with the following properties*:

- *$\phi_{H,T}(m)$ is convex,*
- $\displaystyle\lim_{R\to\infty} -\frac{1}{|B_R|}\ln \mu^{\mathcal{T}_b}_{-,T,H}\{c < M_{B_R} < d\} = \inf_{c\leq m\leq d}\phi_{H,T}(m),$
- *$\phi_{H,T}(m)$ vanishes at exactly one point $m = m_-(T,\,H)$. Moreover,*

$$\phi_{H,T}(m) \geq \frac{1}{4}C(T,\,H)^{-1}(m - m_-(T,\,H))^2, \tag{5}$$

with the function $C(T,\,H) > 0$ for all T and H. In particular, the critical phases $\mu^{\mathcal{T}_b}_{-,T,H_c(T)}$ are rigid.

The function $C(T,\,H)$ is defined in Proposition 5 below.

Note 1. The relation (5) is expected to fail at the critical point on Euclidean lattices.

Note 2. The $(-)$-phase is not just a Markov random field, but also a Markov chain in the sense of Chapter 12 of [G]. This fact could possibly be used for an alternative proof of the exponential decay in volume contained in Theorem 1. We thank a referee for this remark.

One should not think that there is no qualitative difference whatsoever between the properties of the phase $\mu^{\mathcal{T}_b}_{-,T,H}$ for $H = H_c$ and for $H < H_c$. To make this difference visible, let us pick $\delta > 0$ and consider the perturbation $\mu^{\mathcal{T}_b}_{-,T,H+\delta\chi_\Lambda}$ of the state $\mu^{\mathcal{T}_b}_{-,T,H}$, corresponding to the magnetic field $H+\delta\chi_\Lambda$, which equals H outside Λ and $H+\delta$ inside Λ. Consider the difference

$$\langle\sigma_0\rangle^{\mathcal{T}_b}_{-,T,H+\delta\chi_\Lambda} - \langle\sigma_0\rangle^{\mathcal{T}_b}_{-,T,H},$$

as Λ goes to T_b and then δ goes to zero. Then

$$\lim_{\delta\to 0}\lim_{\Lambda\to\mathcal{T}_b}\left[\langle\sigma_0\rangle^{\mathcal{T}_b}_{-,T,H+\delta\chi_\Lambda} - \langle\sigma_0\rangle^{\mathcal{T}_b}_{-,T,H}\right] = \begin{cases} m_+(T,\,H_c) - m_-(T,\,H_c), & \text{if } H = H_c,\\ 0, & \text{otherwise.}\end{cases}$$

One can interpret this fact as the formation of the droplet of the (+)-phase around the origin. However, its size is too small, so its appearance does not change the type of the large deviation behavior of our model, specified by (2).

2. Proofs

2.1. Existence of "free energy". The first step, which contains the proof of the first two statements of our theorem, is quite general.

Proposition 2. *Consider the Ising model on a tree $\mathcal{T}_b$, and let $\boldsymbol{\mu}$ be some homogeneous pure Gibbs state of it (at a certain temperature). Then there exists a function $\phi_{\boldsymbol{\mu}}(m) \geq 0$ with the following properties:*

(i) *$\phi_{\boldsymbol{\mu}}(m)$ is convex,*

(ii) $\displaystyle\lim_{\Lambda\to\mathcal{T}_b} -\frac{1}{|\Lambda|}\ln\boldsymbol{\mu}\{c < M_\Lambda < d\} = \inf_{c\leq m\leq d}\phi_{\boldsymbol{\mu}}(m).$

Proof. Though we restrict ourselves to the Ising model, the proof given below goes through verbatim for any finite range interaction. The main step consists in proving the statements for the so-called semi-infinite subtrees, the relation between measures on tree and on subtrees being given by relation (6) below.

The semi-infinite trees are obtained by cutting away a bond uv of the tree $\mathcal{T}_b$. Then $\mathcal{T}_b$ splits into two connected components, called semi-infinite trees with roots u and v, which will be denoted respectively by $\mathcal{T}(u)$ and $\mathcal{T}(v)$. If we cut away from $\mathcal{T}_b$ the origin 0 together with all $(b+1)$ bonds adjacent to it, the result is the union of $(b+1)$ semi-infinite trees $\mathcal{T}(x)$, with $x\in S_0 = \{x\in\mathcal{T}_b\colon \operatorname{dist}(0,\,x)=1\}$, so

$$\mathcal{T}_b = \bigcup_{x\in S_0}\mathcal{T}(x)\cup\{0\}.$$

The purity property of the state $\boldsymbol{\mu}$ on $\mathcal{T}_b$ is equivalent (see e.g. [P], Chapter 4 or [BG]) to the existence of $(b+1)$ pure states $\mu_{\mathcal{T}(x)}$ on semi-infinite subtrees $\mathcal{T}(x)$ and a value $h = h(\boldsymbol{\mu})$, such that

$$\boldsymbol{\mu}(d\sigma) = Z_0^{-1}e^{h\sigma_0}\prod_{x\in S_0} e^{\beta\sigma_x\sigma_0}\mu_{\mathcal{T}(x)}\,(d\sigma). \tag{6}$$

Moreover, the states $\{\mu_{\mathcal{T}(x)}\}_{x\in S_0}$ are uniquely determined by $\boldsymbol{\mu}$.

Let Λ be a ball centered at the origin: $\Lambda = \{x\colon \operatorname{dist}(x,\,0)\leq n+1\}$. Then $\Lambda = \bigcup_{x\in S_0}\Lambda_x^{(n)}\cup\{0\}$ where $\Lambda_x^{(n)} = \Lambda\cap\mathcal{T}(x)$ for $x\in S_0$. We start by showing that

$$\boldsymbol{\mu}\{c < M_\Lambda < d\} \geq A_-\prod_{x\in S_0}\mu_{\mathcal{T}(x)}\{c < M_{\Lambda_x^{(n)}} < d\}. \tag{7}$$

Picking up some $y \in S_0$, we get from equation (6) that

$$\begin{aligned}\boldsymbol{\mu}\{c < M_\Lambda < d\} &= \boldsymbol{\mu}\left\{c|\Lambda| < \sum_{t\in\Lambda} \sigma_t < d|\Lambda|\right\} \\ &\geq A_- \prod_{x\in S_0\setminus y} \mu_{\mathcal{T}(x)}\left\{c|\Lambda_x^{(n)}| < \sum_{t\in\Lambda_x^{(n)}} \sigma_t < d|\Lambda_x^{(n)}|\right\} \\ &\quad \times \sum_{\sigma_0=\pm1} \mu_{\mathcal{T}(y)}\left\{c(|\Lambda_y^{(n)}|+1) < \sigma_0 + \sum_{t\in\Lambda_y^{(n)}} \sigma_t < d(|\Lambda_y^{(n)}|+1)\right\}, \quad (8)\end{aligned}$$

where $A_- = Z_0^{-1} e^{-\beta(b+1)-|h|}$. Evidently, for any $\delta_1 \geq 0$, $\delta_2 \geq 0$, C, D

$$\mu_{\mathcal{T}(y)}\left\{C < \sum_{t\in\Lambda_y} \sigma_t < D\right\} \geq \mu_{\mathcal{T}(y)}\left\{C + \delta_1 < \sum_{t\in\Lambda_y} \sigma_t < D - \delta_2\right\},$$

so

$$\begin{aligned}\sum_{\sigma_0=\pm1} \mu_{\mathcal{T}(y)}&\left\{c(|\Lambda_y^{(n)}|+1) < \sigma_0 + \sum_{t\in\Lambda_y^{(n)}} \sigma_t < d(|\Lambda_y^{(n)}|+1)\right\} \\ &\geq \mu_{\mathcal{T}(y)}\left\{c|\Lambda_y^{(n)}| < \sum_{t\in\Lambda_y^{(n)}} \sigma_t < d|\Lambda_y^{(n)}| + d - 1\right\} \\ &\quad + \mu_{\mathcal{T}(y)}\left\{c|\Lambda_y^{(n)}| + c + 1 < \sum_{t\in\Lambda_y^{(n)}} \sigma_t < d|\Lambda_y^{(n)}|\right\} \\ &\geq \mu_{\mathcal{T}(y)}\left\{c|\Lambda_y^{(n)}| < \sum_{t\in\Lambda_y^{(n)}} \sigma_t < d|\Lambda_y^{(n)}|\right\} \quad (9)\end{aligned}$$

provided that $d - c \geq \frac{2}{|\Lambda_y^{(n)}|+1}$. Therefore for large n, (7) follows from inequalities (8) and (9).

We shall now prove that the measures $\mu_{\mathcal{T}(x)}$ on the semi-infinite trees satisfy statements (i) and (ii) of the proposition. For a semi-infinite tree $\mathcal{T}(x)$ with root at x, let us introduce the set $S_x = \{z \in \mathcal{T}(x)\colon \operatorname{dist}(z, x) = 1\}$. Similarly to above, a pure state $\mu_{\mathcal{T}(x)}$ has the following canonical representation

$$\mu_{\mathcal{T}(x)} = Z_x^{-1} e^{h\sigma_x} \prod_{z\in S_x} e^{\beta\sigma_z\sigma_x} \mu_{\mathcal{T}(z)},$$

where $\{\mu_{\mathcal{T}(z)}\}_{z\in S_x}$ are in turn some pure states. Of course, the partition functions Z_x are the same for all $x \in S_0$. Since the initial state $\boldsymbol{\mu}$ is homogeneous, all subtree states $\mu_{\mathcal{T}(u)}$ coincide. By the same reasoning as used above for the proof of inequality (7), we get that the measures on semi-infinite subtrees satisfy the subadditivity property:

$$\mu_{\mathcal{T}(x)}\big\{c < M_{\Lambda_x^{(n)}} < d\big\} \geq B_- \prod_{z \in S_x} \mu_{\mathcal{T}(z)}\big\{c < M_{\Lambda_z^{(n-1)}} < d\big\},$$

where $B_- = Z_x^{-1} e^{-\beta b - |h|}$. In fact, a straightforward generalization is immediate:

$$\mu_{\mathcal{T}(x)}\big\{c < M_{\Lambda_x^{(n)}} < d\big\} \geq B_- \prod_{z \in S_x} \mu_{\mathcal{T}(z)}\big\{c_z < M_{\Lambda_z^{(n-1)}} < d_z\big\}, \tag{10}$$

provided $\sum_{z \in S_x} c_z = |S_x| c$ and $\sum_{z \in S_x} d_z = |S_x| d$. Let us introduce the functions

$$\phi_n(J) = -\frac{1}{|\Lambda_x^{(n)}|} \ln \mu_{\mathcal{T}(x)}\{M_{\Lambda_x^{(n)}} \in J\},$$

defined for all open intervals $J \subset (-1, 1)$. Since $|\Lambda_x^{(n)}| = b|\Lambda_z^{(n-1)}| + 1$ and $|S_x| = b$, they satisfy

$$0 \leq \phi_n(J) \leq -\frac{1}{|\Lambda_x^{(n)}|} \ln B_- + \phi_{n-1}(J).$$

Therefore the limits $\phi(J) = \lim_{n\to\infty} \phi_n(J)$ exist.

By standard arguments (see, e.g., [L], [LP], [LS]) one can check that this limiting function satisfies:

(a) the principle of the smallest term: for $J' \cap J'' \neq \varnothing$

$$\phi(J' \cup J'') = \phi(J') \wedge \phi(J');$$

(b) the continuity property: $\phi(J) = \inf\{\phi(I) \colon (\text{closure of } I) \subset J\}$.

These properties imply that if we let

$$\phi_{\boldsymbol{\mu}}(m) = \sup\{\phi(J) \colon m \in J\},$$

then

$$\phi(J) = \inf\{\phi_{\boldsymbol{\mu}}(m) \colon m \in J\}. \tag{11}$$

By virtue of inequality (10) one gets ([R], [L]), that $\phi_{\boldsymbol{\mu}}(m)$ is a convex function of m. Thus the measures on semi-infinite subtrees $\mathcal{T}(x)$ satisfy the statements of our proposition. Moreover, by estimate (7), we get also that

$$\limsup_{\Lambda \to \mathcal{T}_b} -\frac{1}{|\Lambda|} \ln \boldsymbol{\mu}\{c < M_\Lambda < d\} \leq \inf_{c<m<d} \phi_{\boldsymbol{\mu}}(m). \tag{12}$$

Let us now prove that equality actually holds in (12). To this end we first notice that by (6)

$$\begin{aligned}\boldsymbol{\mu}\{c < M_\Lambda < d\} &= \sum_{\{\sigma:\, c|\Lambda| < \sum_{t\in\Lambda}\sigma_t < d|\Lambda|\}} Z_0^{-1} e^{h\sigma_0} \prod_{x\in S_0} \mu_{\mathcal{T}(x)}(\sigma|_{\mathcal{T}(x)}) e^{\beta\sigma_x\sigma_0} \\ &\le A_+ \sum_{\substack{\{\sigma:\, c(b+1)+(c-1)/|\Lambda_x^{(n)}| \\ <\sum_{x\in S_0} M_{\Lambda_x^{(n)}} \\ <d(b+1)+(d+1)/|\Lambda_x^{(n)}|\}}} \prod_{x\in S_0} \mu_{\mathcal{T}(x)}(\sigma|_{\mathcal{T}(x)}) \\ &= A_+ \sum_{\substack{\{m_x:\, c(b+1)+(c-1)/|\Lambda_x^{(n)}| \\ <\sum_{x\in S_0} m_x \\ <d(b+1)+(d+1)/|\Lambda_x^{(n)}|\}}} \prod_{x\in S_0} \mu_{\mathcal{T}(x)}(M_{\Lambda_x^{(n)}} = m_x) \end{aligned} \tag{13}$$

where $A_+ = Z_0^{-1} e^{\beta(b+1)+|h|}$, and the summation in the last line is restricted to the rational values of m_{x_i} of the form $\frac{M}{|\Lambda_x^{(n)}|}$ with integer M. Next, from relation (11) and the continuity of $\phi_{\boldsymbol{\mu}}(m)$ we find that for any $\delta > 0$ there exits $N_\delta > 0$ such that for all m_x

$$\mu_{\mathcal{T}(x)}(M_{\Lambda_x^{(n)}} = m_x) \le \exp\{-|\Lambda_x^{(n)}|\,(\phi_{\boldsymbol{\mu}}(m_x) - \delta)\}$$

for all $n \ge N_\delta$. Inserting this estimate into (13) one obtains:

$$\begin{aligned} A_+ &\sum_{\substack{\{m_x:\, c(b+1)+(c-1)/|\Lambda_x^{(n)}| \\ <\sum_{x\in S_0} m_x \\ <d(b+1)+(d+1)/|\Lambda_x^{(n)}|\}}} \prod_{x\in S_0} \mu_{\mathcal{T}(x)}(M_{\Lambda_x^{(n)}} = m_x) \\ &\le A_+ \sum_{\{m_x:\, c-\varepsilon<(b+1)^{-1}\sum_{x\in S_0} m_x < d+\varepsilon\}} \prod_{x\in S_0} \exp\{-|\Lambda_x^{(n)}|\,(\phi_{\boldsymbol{\mu}}(m_x) - \delta)\} \end{aligned}$$

for any $\varepsilon > 0$. Since the number of terms in the last sum is less than $2(b+1)|\Lambda_x^{(n)}|$, by convexity of $\phi_{\boldsymbol{\mu}}$ we finally get the estimate of the probability $\boldsymbol{\mu}\{c < M_\Lambda < d\}$ from above:

$$\begin{aligned} &\boldsymbol{\mu}\{c < M_\Lambda < d\} \\ &\le A_+ \sum_{\{m_x: c-\varepsilon<(b+1)^{-1}\sum_{x\in S_0} m_x<d+\varepsilon\}} \exp\Big\{-|\Lambda_x^{(n)}|(b+1)\Big\{\phi_{\boldsymbol{\mu}}\Big((b+1)^{-1}\sum_{x\in S_0} m_x\Big) - \delta\Big\}\Big\} \\ &\qquad \le A_+\, 2(b+1)|\Lambda_x^{(n)}| \exp\Big\{-|\Lambda_x^{(n)}|(b+1)\Big(\inf_{c-\varepsilon<m<d+\varepsilon} \phi_{\boldsymbol{\mu}}(m) - \delta\Big)\Big\}. \end{aligned}$$

This estimate implies that

$$\liminf_{\Lambda\to\mathcal{T}_b} -\frac{1}{|\Lambda|} \ln \boldsymbol{\mu}\{c < M_\Lambda < d\} \ge \inf_{c-\varepsilon<m<d+\varepsilon} \phi_{\boldsymbol{\mu}}(m) - \delta$$

for any ε, $\delta > 0$, which together with estimate (12) and continuity of $\phi_{\boldsymbol{\mu}}(m)$ proves the equality in (12) and thus our proposition. □

We introduce the specific logarithmic moment generating function

$$\rho_\Lambda(\delta) = \frac{1}{|\Lambda|} \ln\langle \exp[\delta M_\Lambda |\Lambda|]\rangle_{\boldsymbol{\mu}} \tag{14}$$

where $\langle \cdot \rangle_{\boldsymbol{\mu}}$ denotes the expectation with respect to $\boldsymbol{\mu}$. As a consequence of the proof above, we get the statement, similar to one in [GMS]:

Corollary 3. *The limit* $\lim_{n\to\infty} \rho_\Lambda(\delta)$ *exists and defines a convex function* $\rho(\delta)$. *Moreover* $\rho(\delta)$ *and* $\phi_{\boldsymbol{\mu}}(m)$ *are the Legendre transform of each other*: $\phi_{\boldsymbol{\mu}}(m) = \sup_\delta[m\delta - \rho(\delta)]$ *and* $\rho(\delta) = \sup_m[m\delta - \phi_{\boldsymbol{\mu}}(m)]$.

Remark 4. Proposition 5 below implies, for the $(-)$-state, that the derivatives of ρ satisfy

$$\rho'_\Lambda(\delta) - \rho'_\Lambda(0) \le C\delta,$$

uniformly in Λ.

Note that in general the function $\phi_{\boldsymbol{\mu}}(m)$ depends not only on the interaction and the temperature, but also on the phase $\boldsymbol{\mu}$, if there are more than one phase at this temperature. This is very different from the lattice $\mathbb{Z}^d$, where the free energy function depends only on the interaction and the temperature and does not depend on the state.

2.2. A device for getting an upper estimate of large deviation probabilities. The most important statement of our main result is the claim that $\phi_{H,T}(m) > 0$ once $m \neq m_-(T, H)$. Its proof is based on the following computation and the subsequent application of the novel large deviation technique, contained in Lemma 6 below.

Proposition 5. *Let* $\Lambda = B_R \subset \mathcal{T}_b$ *be a ball of radius* R *centered at the origin. Let the infinite volume Gibbs state* $\mu_{-,T,H+\delta\chi_\Lambda}$ *correspond to the n.n. Ising model at the temperature* T, *with* $(-)$-*boundary conditions at infinity and external magnetic field equal to* H *outside* Λ *and to* $H+\delta$ *inside it, with* $\delta > 0$. *Then there exists* $C = C(T, H)$, $0 < C(T, H) < \infty$, *such that for all* R

$$\langle M_\Lambda\rangle^{\mathcal{T}_b}_{-,T,H+\delta\chi_\Lambda} = m_-(T, H) + \gamma_{T,H}(\delta, R), \tag{15}$$

where

$$0 \le \gamma_{T,H}(\delta, R) \le C\delta. \tag{16}$$

The proof of Proposition 5 is technically the most involved part of our paper. Note also, that unless $T \le T_c$ and $H = H_c(T)$, the statements (15) and (16) are immediate. Indeed, by FKG inequality

$$m_-(T, H) \le \langle M_\Lambda\rangle^{\mathcal{T}_b}_{-,T,H+\delta\chi_\Lambda} \le m_-(T, H+\delta),$$

so the relations (15) and (16) hold for all points (H, T), where the function $m_-(T, H)$ has finite derivative in H. Note finally that in the case of the lattice $\mathbb{Z}^d$, $d \ge 2$, the relations (15), (16) are violated at (the critical value of) $H = 0$: namely, for every $\delta > 0$ the average $\langle M_\Lambda\rangle^{\mathbb{Z}^d}_{-,T,\delta\chi_\Lambda}$ eventually reaches $m_+(T, 0)$, as Λ grows, and moreover

$$\langle M_\Lambda\rangle^{\mathbb{Z}^d}_{-,T,\delta\chi_\Lambda} \to m_+(T, \delta)$$

as $R \to \infty$.

Before proving Proposition 5 let us explain how it is used in the derivation of our main result.

Lemma 6. *Let ξ_N, $N = 1, 2, \ldots,$ be random variables, such that $\infty > A > \xi_N \geq a > -\infty$ and $\mathsf{E}(\xi_N) = d$ for all N, with $d > a$. Let $P_N(x)$ be their probability distribution functions. For every N, $h > 0$ define ξ_N^h to be the "tilted" random variable, corresponding to the distribution*

$$P_N^h(dx) = \frac{\exp\{Nhx\}\, P_N(dx)}{\int \exp\{Nhx\}\, P_N(dx)}.$$

Suppose that

$$\mathsf{E}(\xi_N^h) = d + \gamma_N(h),$$

and for $\gamma(h) = \limsup_{N\to\infty} \gamma_N(h)$ we have

$$\gamma(h) \to 0 \quad \text{as} \quad h \to 0.$$

Then for every $g > 0$,

$$\mathsf{P}\{\xi_N - d > g\} \leq C(g) \exp\{-\varphi(g)N\},$$

where

$$\varphi(g) = \frac{g}{2}\,\gamma^{-1}\Big(\frac{g}{2}\Big) > 0 \quad \text{and} \quad C(g) = \frac{2(d-a)+g}{g}.$$

Here the "inverse" function γ^{-1} is given by

$$\gamma^{-1}(g) = \sup\{h \colon \gamma(h) \leq g\}, \qquad g > 0.$$

Note that $\gamma^{-1}(g) > 0$.

This lemma, together with Proposition 5 implies that the function $\phi_{H,T}(m)$ satisfies

$$\phi_{H,T}(m) \geq \frac{1}{4} C(T,\, H)^{-1}(m - m_-(T,\, H))^2,$$

which proves Theorem 1. Here the function $C(T,\, H)$ is the one from Proposition 5.

Proof of Lemma 6. Note that for any h and $g_1 < g_2$

$$\begin{aligned}\frac{\mathsf{P}\{\xi_N^h \geq d+g_2\}}{\mathsf{P}\{\xi_N^h \leq d+g_1\}} &\geq \frac{\exp\{Nh(d+g_2)\}\, \mathsf{P}\{\xi_N \geq d+g_2\}}{\exp\{Nh(d+g_1)\}\, \mathsf{P}\{\xi_N \leq d+g_1\}} \\ &\geq \exp\{Nh(g_2-g_1)\}\, \mathsf{P}\{\xi_N \geq d+g_2\},\end{aligned}$$

so

$$\mathsf{P}\{\xi_N^h \geq d+g_2\} \geq \exp\{Nh(g_2-g_1)\}\, \mathsf{P}\{\xi_N \geq d+g_2\}\, \mathsf{P}\{\xi_N^h \leq d+g_1\}. \tag{17}$$

On the other hand,

$$\begin{aligned}\mathsf{E}(\xi_N^h) &\geq a\, \mathsf{P}\{\xi_N^h \leq d+g_1\} + (d+g_2)\, \mathsf{P}\{\xi_N^h \geq d+g_2\} \\ &\qquad + (d+g_1)\big[1 - \mathsf{P}\{\xi_N^h \leq d+g_1\} - \mathsf{P}\{\xi_N^h \geq d+g_2\}\big] \\ &= (d+g_1) + (a-d-g_1)\, \mathsf{P}\{\xi_N^h \leq d+g_1\} + (g_2-g_1)\, \mathsf{P}\{\xi_N^h \geq d+g_2\} \\ &\geq (d+g_1) + \big[(a-d-g_1) + (g_2-g_1)\exp\{Nh(g_2-g_1)\}\, \mathsf{P}\{\xi_N \geq d+g_2\}\big] \\ &\qquad \times \mathsf{P}\{\xi_N^h \leq d+g_1\},\end{aligned}$$

where we have used (17) in the last step. Therefore, for every h such that $\mathsf{E}(\xi_N^h) \le d + g_1$ we necessarily have that the last expression in the square brackets is non-positive, so

$$\mathsf{P}\{\xi_N \ge d + g_2\} \le \frac{d + g_1 - a}{g_2 - g_1} \exp\{-Nh(g_2 - g_1)\}.$$

Taking now $g_1 = g_2/2$, we arrive for large N at

$$\mathsf{P}\{\xi_N \ge d + g\} \le \frac{2(d-a) + g}{g} \exp\Big\{-\frac{g}{2}\,\gamma^{-1}\Big(\frac{g}{2}\Big)N\Big\}. \qquad \square$$

2.3. Recursions on the tree.

Proof of Proposition 5. Assume that $H = H_c$. Denote for the sake of brevity $\langle\cdot\rangle = \langle\cdot\rangle^{\mathcal{T}_b}_{-,T,H+\delta\chi_\Lambda}$. We will deduce Proposition 5 from the following lemma.

Lemma 7. *Let $0 \le R_2 \le R_1 < R$ be two (integer) radii such that*

$$R_1 = \max\{0,\, R - l_1(\delta)\}, \qquad R_2 = \max\{0,\, R - l_2(\delta)\}, \tag{18}$$

where $l_1(\delta)$, $l_2(\delta)$ are independent of R and

$$\lim_{\delta\to 0} \frac{l_1(\delta)}{\delta^{-1/2}} = 0, \qquad \lim_{\delta\to 0} \frac{l_2(\delta)}{\delta^{-1/2}} = 1. \tag{19}$$

Then there exists some $\delta_0 > 0$ and constant $C = C(T)$ such that for every $0 < \delta \le \delta_0$ and every x in the shell $B_{R_1,R} = \{x\colon R_1 \le \operatorname{dist}(0,\, x) \le R\}$,

$$\langle\sigma_x\rangle = m_-(T,\, H) + \varepsilon_x(T,\, H,\, \delta,\, R), \qquad x \in B_{R_1,R}, \tag{20}$$

where

$$0 \le \varepsilon_x(T,\, H,\, \delta,\, R) \le C\delta \operatorname{dist}(x,\, B_R^c), \qquad \textit{with} \quad B_R^c = \mathcal{T}_b \setminus B_R. \tag{21}$$

In particular, for

$$\varepsilon(T,\, H,\, \delta) \equiv \limsup_{R\to\infty} \max_{x\in B_{R_1,R}} \varepsilon_x(T,\, H,\, \delta,\, R)$$

we have

$$\lim_{\delta\to 0} \varepsilon(T,\, H,\, \delta) = 0. \tag{22}$$

On the other hand, for every x in the ball B_{R_2},

$$\langle\sigma_x\rangle = m_+(T,\, H) - \hat\varepsilon_x(T,\, H,\, \delta,\, R), \tag{23}$$

where $\hat\varepsilon_x(T,\, H,\, \delta,\, R) \ge 0$ and

$$\lim_{\delta\to 0} \hat\varepsilon(T,\, H,\, \delta) = 0, \qquad \textit{where} \quad \hat\varepsilon(T,\, H,\, \delta) \equiv \limsup_{R\to\infty} \max_{x\in B_{R_2}} \hat\varepsilon_x(T,\, H,\, \delta,\, R). \tag{24}$$

Derivation of Proposition 5 from Lemma 7. Observe that

$$\langle M_\Lambda\rangle = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\sigma_x\rangle. \tag{25}$$

By (20),

$$\langle\sigma_x\rangle = m_- + \varepsilon_x, \qquad |\varepsilon_x| \le C\delta k, \qquad k = \operatorname{dist}(x,\, \Lambda^c), \tag{26}$$

hence

$$\frac{1}{|\Lambda|}\sum_{x\in B_{R_1,R}}\langle\sigma_x\rangle = \frac{|B_{R_1,R}|}{|B_R|} m_- + \varepsilon \tag{27}$$

where

$$\varepsilon \le C\delta \sum_{k=0}^{\infty} b^{-k}(k+1) = C_1\delta. \tag{28}$$

Obviously, $|\langle\sigma_x\rangle| \le 1$ for all x, hence

$$\frac{1}{|\Lambda|} \sum_{x\in B_{R_1-1}} \langle\sigma_x\rangle \le \frac{|B_{R_1-1}|}{|B_R|} \le C_2 b^{-l_1(\delta)}. \tag{29}$$

We can take $l_1(\delta) = \delta^{-1/3}$. Then $b^{-l_1(\delta)} \le C_3\delta$. Combining (27) with (29) we obtain that

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle\sigma_x\rangle = m_-(T, H) + O(\delta), \quad \delta \to 0. \tag{30}$$

So for $H = H_c$ Proposition 5 follows. □

Proof of Lemma 7. We use the exact solution for the Gibbs state $\mu_{-,\delta} \equiv \mu_{-,T,H+\delta\chi_\Lambda}$ in terms of the "effective magnetic field" $\{h_x,\, x \in \mathcal{T}_b\}$, see [BRZ2]. Namely, for every connected finite set $V \subset \mathcal{T}_b$ such that $0 \in V$,

$$\mu_{-,\delta}(\sigma_V) = Z_V^{-1} \exp\left[\beta \sum_{x,y\in V,\ \mathrm{dist}(x,y)=1} \sigma_x\sigma_y + \beta \sum_{x\in V} H_x\sigma_x + \beta \sum_{x\in\partial V} h_x\sigma_x\right], \tag{31}$$

where $H_x = H + \delta\chi_\Lambda(x)$ and

$$\partial V = \{x \in V\colon\ \mathrm{dist}(x, V^c) = 1\}. \tag{32}$$

The effective magnetic field h_x satisfies the recurrence equation,

$$h_x = \sum_{y\in S_x} f_\beta(h_y + H_y), \quad x \in \mathcal{T}_b, \tag{33}$$

where

$$f_\beta(h) = \frac{1}{\beta}\operatorname{artanh}[\tanh(\beta)\tanh(\beta h)], \tag{34}$$

and

$$S_x = \{y\colon\ \mathrm{dist}(x, y) = 1,\, \mathrm{dist}(0, y) = \mathrm{dist}(0, x) + 1\}. \tag{35}$$

When $V = \{0\}$, (31) reduces to the one-point distribution

$$\mu_{-,\delta}(\sigma_0) = Z_0^{-1} \exp[\beta(H_0 + h_0)\sigma_0], \tag{36}$$

which gives that

$$\langle\sigma_0\rangle = \tanh[\beta(H_0 + h_0)]. \tag{37}$$

Observe that (33) differs somewhat for $x = 0$ and $x \ne 0$: for $x = 0$ the set S_x consists of $(b+1)$ points while for $x \ne 0$ it consists of b points. We will call the field (33) the effective magnetic field with the origin at 0.

Recurrence relation (33) is supplemented by the "initial data",

$$h_x = h_-, \quad \forall x,\, \mathrm{dist}(0, x) \ge R, \tag{38}$$

where $h_- = h_-(T, H)$ is the smallest solution of the fixed point equation,

$$h = b f_\beta(h + H). \tag{39}$$

Recall that this equation has three solutions if $0 \le H < H_c(T)$, two solutions if $H = H_c(T)$ and $T < T_c$, and one solution in all other cases. We denote by $h_+(T, H)$ the biggest solution. (Clearly, in the third case we have $h_-(T, H) = h_+(T, H)$.) For x in the ball B_R, (33) reduces to the equation

$$h_{k-1} = bf_\beta(h_k + H + \delta), \quad k \le R; \qquad h_R = h_-, \tag{40}$$

where $k = \mathrm{dist}(0, x)$.

The proof of Lemma 7 consists of two parts. In the first part we estimate the effective magnetic field h_x in the ball B_R. Namely, we will prove the following lemma.

Lemma 8. *For every x in the shell $B_{R_1,R}$,*

$$h_x = h_-(T, H) + \bar\varepsilon_x(T, H, \delta, R), \quad x \in B_{R_1,R}, \tag{41}$$

where

$$|\bar\varepsilon_x(T, H, \delta, R)| \le C\delta\, \mathrm{dist}(x, B_R^c). \tag{42}$$

In particular,

$$\lim_{\delta\to 0} \bar\varepsilon(T, H, \delta) = 0, \qquad \bar\varepsilon(T, H, \delta) \equiv \limsup_{R\to\infty} \max_{x\in B_{R_1,R}} |\bar\varepsilon_x(T, H, \delta, R)|. \tag{43}$$

On the other hand, for every x in the ball B_{R_2},

$$h_x = h_+(T, H) + \tilde\varepsilon_x(T, H, \delta, R), \tag{44}$$

where

$$\lim_{\delta\to 0} \tilde\varepsilon(T, H, \delta) = 0, \qquad \tilde\varepsilon(T, H, \delta) \equiv \limsup_{R\to\infty} \max_{x\in B_{R_2}} |\tilde\varepsilon_x(T, H, \delta, R)|. \tag{45}$$

Proof of Lemma 8. Using (39) we rewrite (40) as

$$h_{k-1} - h_- = bf_\beta(h_k + H + \delta) - bf_\beta(h_- + H). \tag{46}$$

For the notational convenience we will use, together with the coordinate k, which measures the distance from the origin, the coordinate l, which measures the distance from B_R^c, so that $k + l = R$. Let

$$g_l = h_{R-l} - h_-. \tag{47}$$

Then (46) reduces to

$$g_{l+1} = bf_\beta(h_- + H + g_l + \delta) - bf_\beta(h_- + H), \qquad g_0 = 0. \tag{48}$$

Observe that the definition of h_- as a minimal solution to (39) implies that for $H = H_c$

$$bf'_\beta(h)|_{h=h_-+H} = 1, \tag{49}$$

hence (48) is well approximated by the equation

$$\bar g_{l+1} = \bar g_l + \delta + a(\bar g_l + \delta)^2, \quad \text{with} \quad a = \frac{1}{2} bf''_\beta(h)|_{h=h_-+H} > 0. \tag{50}$$

In a more quantitative way, the solution to the equation

$$\hat g_{l+1} = \hat g_l + \delta + 2a(\hat g_l + \delta)^2 \tag{51}$$

(with bigger quadratic part) provides an upper estimate for the solution of (48): $g_l \le \hat g_l$, once $\hat g_l$ is small enough: $|\hat g_l| \le c(\beta)$ for some function c. Consider $\hat g_l$ with

$0 \le l+1 \le l_1(\delta)$ (see (19)). For these l's the quadratic term in (51) is negligible; we can compare its solution with the solution of the equation $\tilde{g}_{l+1} = \tilde{g}_l + \delta$, i.e., with $\tilde{g}_l = l\delta$. The estimate of the quadratic term is easy and in this way we obtain that for small δ,

$$l\delta \le \hat{g}_l \le 2l\delta, \qquad 0 \le l \le l_1(\delta). \tag{52}$$

[Indeed, by induction we estimate $2a(g_l+\delta)^2 \le 2a(2l+1)^2\delta^2 \le 8a(l_1(\delta))^2\delta^2 = o(\delta)$]. This proves the first part of Lemma 8, relation (42). Observe that the lower bound, $l\delta \le \hat{g}_l$, holds for all $l \ge 0$.

Consider now g_l in the interval $l \ge l_2(\delta) = \delta^{-1/2}$. In this interval we can estimate g_l from below by the solution of another truncated version of (50):

$$\hat{g}_{l+1} = \hat{g}_l + \frac{1}{2}a\hat{g}_l^2, \qquad \hat{g}_{l_0} = \delta^{1/2}, \qquad l_0 = \delta^{-1/2}, \tag{53}$$

provided that again $\hat{g}_l \le c(\beta)$. (We assume in (53) that $\delta^{-1/2}$ is integer.) The latter equation is well approximated by the differential equation

$$\hat{g}' = \frac{1}{2}a\hat{g}^2, \qquad \hat{g}(\delta^{-1/2}) = \delta^{1/2}, \tag{54}$$

which has the solution

$$\hat{g}(t) = \frac{\delta^{1/2}}{1 - \frac{1}{2}a\delta^{1/2}(t - \delta^{-1/2})}. \tag{55}$$

Note that $\hat{g}(t) \to \infty$ as $t \to (1+2a^{-1})\delta^{-1/2}$. This divergence of $\hat{g}(t)$ implies that inside the interval $\delta^{-1/2} \le l \le (1+2a^{-1})\delta^{-1/2}$ the proviso $\hat{g}_l \le c(\beta)$ starts to be violated, and so for these l-s the numbers g_l become of the order of 1. But after the sequence g_l approaches the values of the order of 1, in a finite number of steps it approaches the fixed point $h_+(T, H, \delta) - h_-$ of equation (46) and this proves the second part of Lemma 8, (45). Lemma 8 is thus proved. □

Lemma 8 allows us to estimate $\langle\sigma_0\rangle$. Indeed, according to (44), $h_0 = h_+ + o(1)$ as $\delta \to 0$, and from (37) we obtain that $\langle\sigma_0\rangle = m_+(T, H) + o(1)$. To estimate $\langle\sigma_{x_0}\rangle$ for other sites $x_0 \in B_R$ we can use the effective magnetic field with the origin at x_0. This change of the origin will change h_x to a new value, $h_x^{\upharpoonleft}$, provided x belongs to the path $\pi(0, x_0)$ connecting 0 and x_0. We will denote by $h_k^{\upharpoonleft}$ the value $h_x^{\upharpoonleft}$ at the (unique) site $x \in \pi(0, x_0)$ satisfying $\mathrm{dist}(0, x) = k$. The new effective field $h_k^{\upharpoonleft}$ is determined by the recurrence equations

$$h_k^{\upharpoonleft} = \begin{cases} f_\beta(h_{k-1}^{\upharpoonleft} + H + \delta) + (b-1)f_\beta(h_{k+1} + H + \delta), & \text{if } 1 \le k < \mathrm{dist}(0, x_0), \\ f_\beta(h_{k-1}^{\upharpoonleft} + H + \delta) + bf_\beta(h_{k+1} + H + \delta), & \text{if } k = \mathrm{dist}(0, x_0), \end{cases}$$
$$h_0^{\upharpoonleft} = h_0, \tag{56}$$

see [BRZ2]. The magnetization at x_0 is then found as

$$\langle\sigma_{x_0}\rangle = \tanh[\beta(H + \delta + h_k^{\upharpoonleft})], \qquad k = \mathrm{dist}(0, x_0). \tag{57}$$

Of course, recursion (56) does not depend on x_0, except for the last equation; in what follows we denote by $h_x^{\upharpoonleft}$, where $x \in B_R$, the value of the solution $h_k^{\upharpoonleft}$ to (56) at $k = \mathrm{dist}(0, x)$. In the following lemma we evaluate $h_x^{\upharpoonleft}$.

Lemma 9. *For every x in the shell $B_{R_1,R}$,*

$$h_x^{\uparrow} = h_-(T,\, H) + \varepsilon_x'(T,\, H,\, \delta,\, R), \qquad x \in B_{R_1,R}, \tag{58}$$

where

$$|\varepsilon_x'(T,\, H,\, \delta,\, R)| \le C\delta[\mathrm{dist}(x,\, B_R^c) + 1]. \tag{59}$$

In particular,

$$\lim_{\delta\to 0} \varepsilon'(T,\, H,\, \delta) = 0, \qquad \varepsilon'(T,\, H,\, \delta) \equiv \limsup_{R\to\infty} \max_{x\in B_{R_1,R}} |\varepsilon_x'(T,\, H,\, \delta,\, R)|. \tag{60}$$

On the other hand, for every x in the ball B_{R_2},

$$h_x^{\uparrow} = h_+(T,\, H) + \varepsilon_x''(T,\, H,\, \delta,\, R), \tag{61}$$

where

$$\lim_{\delta\to 0} \varepsilon''(T,\, H,\, \delta) = 0, \qquad \varepsilon''(T,\, H,\, \delta) \equiv \limsup_{R\to\infty} \max_{x\in B_{R_2}} |\varepsilon_x''(T,\, H,\, \delta,\, R)|. \tag{62}$$

Proof. We use the contracting property of the map $h \to f_\beta(h+H)$:

$$\left|\frac{d}{dh} f_\beta(h)\right| \le \tanh\beta \equiv a < 1. \tag{63}$$

Let us first show that $h_k^{\uparrow}$ is close to $h_+ = h_+(T,\, H)$ when $0 \le k \le R_2$ and R is large. We have from (56) that

$$h_k^{\uparrow} - h_+ = f_\beta(h_{k-1}^{\uparrow} + H + \delta) - f_\beta(h_+ + H) + (b-1)[f_\beta(h_{k+1} + H + \delta) - f_\beta(h_+ + H)], \tag{64}$$

hence (63) gives that

$$|h_k^{\uparrow} - h_+| \le a|h_{k-1}^{\uparrow} - h_+| + \tilde\varepsilon_k, \tag{65}$$

where

$$\tilde\varepsilon_k = b\delta + (b-1)|h_{k+1} - h_+|. \tag{66}$$

By Lemma 8,

$$\tilde\varepsilon \equiv \max_{1\le k\le R_2-1} \tilde\varepsilon_k \le b\delta + (b-1)\tilde\varepsilon(T,\, H,\, \delta) \to 0 \quad \text{as} \quad \delta \to 0. \tag{67}$$

Iterating (65) we obtain that

$$|h_k^{\uparrow} - h_+| \le a^k|h_0^{\uparrow} - h_+| + \frac{1}{1-a}\tilde\varepsilon. \tag{68}$$

Since $h_0^{\uparrow} = h_0$ is close to h_+ for R large (see (44)), this estimate proves the second part of Lemma 9, (62).

To prove the first part of Lemma 9 let us notice that the same consideration as above (with the use of the first part of Lemma 8) gives us that in the shell $B_{R_1,R}$ we have an estimate similar to (68),

$$|h_{R_1+m}^{\uparrow} - h_-| \le a^m|h_{R_1}^{\uparrow} - h_-| + \frac{1}{1-a}\bar\varepsilon, \tag{69}$$

where

$$\bar\varepsilon = \bar\varepsilon(T,\, H,\, \delta) \to 0 \quad \text{as} \quad \delta \to 0. \tag{70}$$

Observe that (56) implies that for all k,

$$|h_k^{\uparrow}| \le b+1 \tag{71}$$

(because $|f_\beta(t)| < 1$). Similarly, $|h_-| \le b+1$, hence (69) gives that

$$|h_{R_1+m}^{\uparrow} - h_-| \le 2(b+1)a^m + \frac{1}{1-a}\bar{\varepsilon}. \tag{72}$$

If we take here $m > |\ln\delta|$ and denote $\widetilde{R}_1 = R_1 + |\ln\delta|$, then we will obtain that for $k \ge \widetilde{R}_1$

$$|h_k^{\uparrow} - h_-| \le \varepsilon, \tag{73}$$

where $\varepsilon = \varepsilon(T, H, \delta) \to 0$ as $\delta \to 0$. This proves (60). It does not matter that we proved it for $k \ge \widetilde{R}_1$, and not for $k \ge R_1$, because we can redefine R_1 to $R_1 - |\ln\delta|$ and it will still satisfy assumption (19).

To prove (59) we have to slightly refine the above estimates. Namely, similar to (65), (66) we have the estimate

$$|h_k^{\uparrow} - h_-| \le a\,|h_{k-1}^{\uparrow} - h_-| + A\delta + B\,|h_{k+1} - h_-|, \tag{74}$$

with some $A = A(T) > 0$ and $B = B(T) > 0$. If we iterate this inequality we obtain that

$$|h_k^{\uparrow} - h_-| \le a^m|h_{R_1} - h_-| + \frac{A\delta}{1-a} + B\sum_{j=0}^{m} a^j|h_{k+1-j} - h_-|, \quad m = k - R_1. \tag{75}$$

By (42),

$$|h_{k+1-j} - h_-| \le C\delta(R - k - 1 + j), \tag{76}$$

hence

$$\sum_{j=0}^{m} a^j|h_{k+1-j} - h_-| \le C\delta \sum_{j=0}^{m} a^j(R - k - 1 - j) \le C_1\delta(R - k + 1), \tag{77}$$

and thus (75) gives that

$$|h_k^{\uparrow} - h_-| \le a^m|h_{R_1} - h_-| + \frac{A\delta}{1-a} + C_2\delta(R - k + 1), \quad m = k - R_1, \tag{78}$$

where $C_2 = BC_1$. Taking $m \ge \log_a \delta$, so that $a^m \le \delta$, we obtain that

$$|h_k^{\uparrow} - h_-| \le C_3\delta + C_2\delta(R - k). \tag{79}$$

This proves (59). Lemma 9 is proved. □

Lemma 7 follows immediately from Lemmas 8 and 9 by formula (57). □

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References

[BG] P. M. Blekher, N. N. Ganikhodzhaev, *Pure phases of the Ising model on Bethe lattices*, Teor. Veroyatnost. i Primenen. **35** (1990), no. 2, 220–230. (Russian) MR **92c:**82039. English translation: Theory Probab. Appl. **35** (1990), no. 2, 216–227 (1991)

[BIV] T. Bodineau, D. Ioffe, Y. Velenik, *Rigorous probabilistic analysis of equilibrium crystal shapes*, J. Math. Phys. **41** (2000), no. 3, 1033–1098. MR **2001f:**82015

[BRZ1] P. M. Bleher, J. Ruiz, V. A. Zagrebnov, *On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice*, J. Statist. Phys. **79** (1995), no. 1-2, 473–482. MR **96d:**82009

[BRZ2] P. M. Bleher, J. Ruiz, V. A. Zagrebnov, *On the phase diagram of the random field Ising model on the Bethe lattice*, J. Statist. Phys. **93** (1998), no. 1-2, 33–78. MR **99h:**82043

[CP] R. Cerf, Á. Pisztora, *On the Wulff crystal in the Ising model*, Ann. Probab. **28** (2000), no. 3, 947–1017. MR **2001i:**82017

[DKS] R. Dobrushin, R. Kotecký, S. Shlosman, *Wulff construction. A global shape from local interaction.*, Translations of Mathematical Monographs, vol. 104, American Mathematical Society, Providence, RI, 1992. MR **93k:**82002

[GMS] L. Galgani, L. Manzoni, Scotti A., *Asymptotic equivalence of equilibrium ensembles of classical statistical mechanics*, J. Math. Phys. **12** (1971), 933–935.

[G] H.-O. Georgii, *Gibbs measures and phase transitions*, Walter de Gruyter & Co., Berlin, 1988. MR **89k:**82010

[JS] J. Jonasson, J. E. Steif, *Amenability and phase transition in the Ising model*, J. Theoret. Probab. **12** (1999), no. 2, 549–559. MR **2000b:**60238

[L] O. Lanford, *Entropy and equilibrium states in classical statistical mechanics*, Statistical mechanics and mathematical problems (Lenard A., ed.), Lecture Notes in Physics, vol. 20, Springer-Verlag, Berlin, 1973. MR 52 #16416

[LS] J. L. Lebowitz, R. H. Schonmann, *Pseudo-free energies and large deviations for non-Gibbsian FKG measures*, Probab. Theory Related Fields **77** (1988), no. 1, 49–64. MR **89g:**60306

[LP] J. T. Lewis and C.-E. Pfister, *Thermodynamic probability theory: some aspects of large deviations*, Uspekhi Mat. Nauk **50** (1995), no. 2(302), 47–88 (Russian), English translation: Russian Math. Surveys **50** (1995), no. 2, 279–317. MR **96k:**60067

[Ly] R. Lyons, *Phase transitions on nonamenable graphs*, J. Math. Phys. **41** (2000), no. 3, 1099–1126. MR **2001c:**82028

[P] C. J. Preston, *Gibbs states on countable sets*, Cambridge Tracts in Mathematics, vol. 68, Cambridge University Press, London, 1974. MR 57 #14194a

[R] D. Ruelle, *Statistical mechanics: Rigorous results*, W. A. Benjamin, Inc., New York–Amsterdam, 1969. MR 44 #6279

[S] R. H. Schonmann, *Multiplicity of phase transitions and mean-field criticality on highly nonamenable graphs*, preprint, 2000.

[SS] R. H. Schonmann, S. B. Shlosman, *Constrained variational problem with applications to the Ising model*, J. Statist. Phys. **83** (1996), no. 5–6, 867–905. MR **97h:**82030

[ST] R. H. Schonmann, N. I. Tanaka, *Lack of monotonicity in ferromagnetic Ising model phase diagrams*, Ann. Appl. Probab. **8** (1998), no. 1, 234–245. MR **99b:**60171

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