Double Scaling Limit in the Random Matrix Model: The Riemann-Hilbert Approach

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Abstract

We prove the existence of the double scaling limit in the unitary matrix model with quartic interaction, and we show that the correlation functions in the double scaling limit are expressed in terms of the integrable kernel determined by the ψ function for the Hastings-McLeod solution to the Painlevé II equation. The proof is based on the Riemann-Hilbert approach, and the central point of the proof is an analysis of analytic semiclassical asymptotics for the ψ function at the critical point in the presence of four coalescing turning points.

1 Introduction

In this paper we are concerned with the double scaling limit in the unitary random matrix model with quartic interaction. The unitary random matrix model, or the unitary ensemble of random matrices, is defined by the probability distribution

$$\begin{aligned} \mu_N(dM) &= Z_N^{-1} \exp(-N \operatorname{tr} V(M)) dM \,, \\ Z_N &= \int\limits_{\mathcal{H}_N} \exp(-N \operatorname{tr} V(M)) dM \,, \end{aligned} \tag{1.1}$$

on the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices $M = (M_{ij})_{1 \le i,j \le N}$, where in general $V(M)$ is a polynomial of even degree with a positive leading coefficient, or even more generally, a real analytic function with some conditions at infinity. The basic case for the double scaling limit is the quartic matrix model when

$$V(M) = \frac{t}{2} M^2 + \frac{g}{4} M^4 \,, \quad g > 0 \,, \tag{1.2}$$

and we will consider this case only. By a change of variable one can reduce the general case to the one with $g = 1$, but we prefer to keep g because it is useful in some questions. The double scaling limit describes the asymptotics of correlation functions between eigenvalues in the limit when simultaneously $N \to \infty$ and t approaches the critical value

$$t_c = -2\sqrt{g} \,,$$

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with an appropriate relation between N and $t - t_c$ (see below). The critical value t_c is a bifurcation point: For $t \geq t_c$ the support of the limiting distribution of eigenvalues consists of one interval, while for $t < t_c$ it consists of two intervals (cf. [13]). This can be described as follows.

Let $d\nu(x)$ be the limiting eigenvalue distribution (for the proof of the existence of the limiting eigenvalue distribution and formulae for it, see [8, 17, 19], earlier physical works [4, 10], and others). The distribution $d\nu(x)$ is a unique solution to a variational problem, and for a general analytic function $V(M)$ satisfying certain conditions at infinity (see [17, 50] for more detail), the distribution $d\nu(x)$ is supported by a finite number of segments $[a_1, b_1], \dots, [a_q, b_q]$, it is absolutely continuous with respect to dx, and on $J = \bigcup_{j=1}^{q}[a_j, b_j]$ its density function is of the form

$$p(x) = \frac{1}{2\pi i} h(x) R_+^{1/2}(x)\,, \qquad R(x) = \prod_{j=1}^{q}(x - a_j)(x - b_j)\,,$$

where $h(x)$ is a polynomial and $R_+^{1/2}(x)$ means the value on the upper cut of the principal sheet of the function $R^{1/2}(z)$ with cuts on J. The variational problem for $d\nu(x)$ implies that the polynomial $h(z)$ satisfies the equation

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}\,, \tag{1.3}$$

where

$$\omega(z) \equiv \int_J \frac{p(x)dx}{z - x} = z^{-1} + O(z^{-2})\,, \qquad z \to \infty$$

(see, e.g., [19]).

Equation (1.3) enables one (see, e.g., [8]) to determine uniquely the limiting eigenvalue distribution for quartic polynomial (1.2). For $t \geq t_c$ the support of the distribution consists of one interval $[-a, a]$ and on $[-a, a]$,

$$p(x) = \frac{1}{\pi}\big(b_0 + b_2 x^2\big)\sqrt{a^2 - x^2}\,, \qquad |x| \leq a,\ t > t_c\,. \tag{1.4}$$

Here

$$a = \left(\frac{-2t + (4t^2 + 48g)^{1/2}}{3g}\right)^{1/2}, \qquad b_2 = \frac{g}{2}\,, \qquad b_0 = \frac{t + ((t^2/4) + 3g)^{1/2}}{3}\,.$$

For $t = t_c$, (1.4) reduces to

$$p(x) = \frac{gx^2\sqrt{a^2 - x^2}}{2\pi}\,, \qquad |x| \leq a\,,\ t = t_c\,, \tag{1.5}$$

where $a = 2/g^{1/4}$. Observe that $p(0) = 0$ in this case. For $t < t_c$ the support consists of two intervals, $[-a, -b]$ and $[b, a]$, and on these intervals

$$p(x) = \frac{1}{\pi} b_0 |x| \sqrt{(a^2 - x^2)(x^2 - b^2)}\,, \qquad 0 < b \leq |x| \leq a,\ t < t_c\,. \tag{1.6}$$

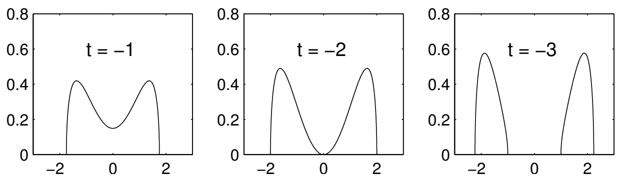

FIGURE 1.1. The density function $p(x)$ for $g = 1$ and $t = -1, -2, -3$.

Here

$$(1.7)\qquad a = \left(\frac{2\sqrt{g} - t}{g}\right)^{1/2}, \quad b = \left(\frac{-2\sqrt{g} - t}{g}\right)^{1/2}, \quad b_0 = \frac{g}{2}.$$

Figure 1.1 depicts the density function $p(x)$ for $g = 1$ and $t = -1, -2, -3$. In this case $t_c = -2$.

1.1 Correlation Functions and Orthogonal Polynomials

Our basic object of interest is the correlation between eigenvalues. The m-point correlation function $K_{Nm}(z_1, \dots, z_m)$, $m = 1, 2, \dots$, is a distribution over the space $\mathcal{D}(\mathbb{R}^m)$ of C^∞ functions $\varphi(z_1, \dots, z_m)$ on $\mathbb{R}^m$ with compact support such that for the product functions $\varphi(z_1, \dots, z_m) = \varphi_1(z_1) \cdots \varphi_m(z_m)$,

$$(1.8)\qquad K_{Nm}(\varphi_1(z_1) \cdots \varphi_m(z_m)) = \int_{\mathcal{H}_N} [\chi_{\varphi_1}(M) \cdots \chi_{\varphi_m}(M)] \mu_N(dM),$$

where

$$\chi_\varphi(M) = \sum_{j=1}^{N} \varphi(\lambda_j), \quad Me_j = \lambda_j e_j.$$

$K_{Nm}(\varphi)$ is extended from the set of product functions $\varphi_1(z_1) \cdots \varphi_m(z_m)$ to the whole space $\mathcal{D}(\mathbb{R}^m)$ by linearity.

The m-point correlation function $K_{Nm}(z_1, \dots, z_n)$ turns out to be a regular function on the set $\{z_i \neq z_j\}$, and on this set it can be expressed in terms of orthogonal polynomials. Namely, let

$$(1.9)\qquad P_n(z) = z^n + \cdots, \quad n = 0, 1, \dots,$$

be monic orthogonal polynomials on the line with respect to the weight $e^{-NV(z)}$ so that

$$(1.10)\qquad \int_{-\infty}^{\infty} P_n(z) P_m(z) e^{-NV(z)}\, dz = h_n \delta_{mn}.$$

We normalize $P_n(z)$ by the condition that the leading coefficient of $P_n(z)$ is equal to 1. The polynomials $P_n(z)$ satisfy the recursive equation [63]

$$(1.11)\qquad zP_n(z) = P_{n+1}(z) + R_n P_{n-1}(z),$$

where

$$R_n = \frac{h_n}{h_{n-1}} > 0\,. \tag{1.12}$$

We associate with $P_n(z)$ the functions

$$\psi_n(z) = \frac{1}{\sqrt{h_n}}\, P_n(z) e^{-NV(z)/2}\,, \tag{1.13}$$

which form an orthonormal basis in $L^2(\mathbb{R}^1)$,

$$\int_{-\infty}^{\infty} \psi_n(z)\psi_m(z)dz = \delta_{nm}\,. \tag{1.14}$$

From (1.11),

$$z\psi_n(z) = R_{n+1}^{1/2}\psi_{n+1}(z) + R_n^{1/2}\psi_{n-1}(z)\,. \tag{1.15}$$

The m-point correlation function $K_{Nm}(z_1, \dots, z_n)$ is expressed in terms of the functions $\psi_n(z)$ as

$$K_{Nm}(z_1, \dots, z_m) = \det(Q_N(z_i, z_j))_{i,j=1,\dots,m}\,, \quad z_i \neq z_j\,, \tag{1.16}$$

where

$$Q_N(z, w) = \sum_{k=0}^{N-1} \psi_k(z)\psi_k(w) \tag{1.17}$$

[25, 54, 64]. The Christoffel-Darboux identity reduces $Q_N(z, w)$ to

$$Q_N(z, w) = R_N^{1/2}\, \frac{\psi_N(z)\psi_{N-1}(w) - \psi_{N-1}(z)\psi_N(w)}{z - w}\,. \tag{1.18}$$

We are interested in the asymptotics of the correlation functions $K_{Nm}(z_1, \dots, z_m)$ in the limit when $N \to \infty$. Formula (1.16) reduces the problem to the asymptotics of the kernel $Q_N(z, w)$ and (1.18) further to the asymptotics of the functions $\psi_N(z)$ and $\psi_{N-1}(z)$. In the noncritical case $t < t_c$ this problem was solved in [7]. Noncritical asymptotics for the general $V(M)$ were found in [19].

1.2 Double Scaling Limit for Correlation Functions

We will consider the correlation functions $K_{Nm}(z_1, \dots, z_m; t)$ in the situation when the parameter t approaches the critical value t_c and $z_1, \dots, z_m$ are near 0. The problem is how to scale eigenvalues and $t - t_c$ to get a nontrivial limit for the correlation functions. More precisely, we want to find numbers ξ and η, critical exponents, such that the following "double scaling" limit exists:

$$\lim_{N\to\infty} \frac{1}{N^{(m-1)\eta}} K_{Nm}\left(\frac{z_1}{N^\eta}, \dots, \frac{z_m}{N^\eta}; t_c + yN^{-\xi}\right) = K_m(z_1, \dots, z_m; y)\,, \tag{1.19}$$

and it is a nonconstant function of the parameter y, $-\infty < y < \infty$. We also want to find the limiting correlation functions $K_m(z_1, \dots, z_m; y)$. We will derive the double scaling limit for correlation functions from corresponding asymptotics for orthogonal polynomials. The relation between the double scaling limit for correlation functions and the one for orthogonal polynomials can be described as follows.

1.3 Double Scaling Limit for Orthogonal Polynomials

Let us fix the parameters $t < 0$ and $g > 0$ in quartic polynomial (1.2). Since $t < 0$, $V(z)$ is a double-well function. Consider orthogonal polynomials $P_n(z)$ with respect to the weight $e^{-NV(z)}$. The bifurcation of $p(x)$ in the quartic matrix model is closely related to the bifurcation of the distribution of zeros of the orthogonal polynomials $P_n(z)$. This relation is motivated by the Heine formula for orthogonal polynomials as follows. Let $x_1, \dots, x_n$ be zeros of $P_n(z)$ that are all real. Consider the corresponding probability measure

$$d\mu_n(x) = n^{-1} \sum_{j=1}^{n} \delta(x - x_j)dx \, .$$

We can write the weight as

$$e^{-NV(x)} = e^{-nV_\lambda(x)} \, , \quad V_\lambda(x) \equiv \lambda^{-1} V(x) \, , \quad \lambda \equiv \frac{n}{N} \, .$$

By the Heine formula for orthogonal polynomials (see, e.g., [4]),

$$P_n(z) = \langle \det(z - M) \rangle_{V_\lambda} \equiv Z_n^{-1}(V_\lambda) \int \det(z - M) e^{-n \operatorname{tr} V_\lambda(M)} \, dM \, ,$$

or equivalently, in the ensemble of eigenvalues,

$$P_n(z) = \left\langle \prod_{j=1}^{n} (z - \lambda_j) \right\rangle_{V_\lambda} \, .$$

Due to the variational principle, we expect that if n is large, then the probability measure

$$d\nu(x; \{\lambda_j\}) \equiv n^{-1} \sum_{j=1}^{n} \delta(x - \lambda_j) d\lambda$$

is close to the equilibrium measure $p(x; V_\lambda)dx$ for typical $\{\lambda_j\}$. Therefore,

$$n^{-1} \log \left\langle \prod_{j=1}^{n} (z - \lambda_j) \right\rangle_{V_\lambda} \approx \int_{J_\lambda} \log(z - x) p(x; V_\lambda) dx \, ;$$

hence by Heine,

$$\begin{aligned} \int_{-\infty}^{\infty} \log(z - x) d\mu_n(x) &= n^{-1} \log P_n(z) \\ &\approx \int_{J_\lambda} \log(z - x) p(x; V_\lambda) dx \, , \quad z \in \mathbb{C} \setminus \mathbb{R} \, . \end{aligned}$$

That is, as $n \to \infty$ the logarithmic potential of $d\mu_n(x)$ converges to the logarithmic potential of $p(x; V_\lambda)dx$. Thus, we can expect that as $n, N \to \infty$ in such a way that $n/N \to \lambda$, the measure $d\mu_n(x)$ converges to $d\mu_\infty(x;\lambda) = p(x; V_\lambda)dx$. This convergence has been rigorously established for the general $V(M)$ in [19] (see also [7, 8] and works on the theory of orthogonal polynomials [52, 53, 56, 61] and references therein).

For λ small, the zeros of $P_n(z)$ are located near the minima of the function $V(z)$ and the support of $d\mu_\infty(x;\lambda)$ consists of two intervals. For λ big, the support of $d\mu_\infty(x;\lambda)$ consists of one interval, and there exists a critical value,

$$\lambda_c = \frac{t^2}{4g}, \tag{1.20}$$

which separates the two-interval and one-interval regimes (see [7]). The problem of the double scaling limit for orthogonal polynomials is to prove the existence of a nontrivial scaling limit for the ψ function,

$$\lim_{n,N\to\infty:(n/N)=\lambda_c+yN^{-\xi}} C_n^{-1}\psi_n\left(\frac{z}{N^\eta}\right) = \psi_\infty(z;y),$$

with some critical exponents ξ and η, where $C_n \neq 0$ are some normalizing constants. Our main goal in this paper is to derive uniform asymptotics for the functions $\psi_n(z)$ on the whole complex plane as $n, N \to \infty$ in the double scaling limit regime, $(n/N) = \lambda_c + yN^{-\xi}$. When applied to the quartic matrix model, these asymptotics will give us the double scaling limit for the correlation functions at the origin as well as scaling limits in the bulk of the spectrum and at the edges. There are three basic ingredients in our approach: the string (Freud) equation, the Lax pair for the string equation, and the Riemann-Hilbert problem.

1.4 String Equation

Let $Q = (Q_{mn})_{m,n=0,1,\dots}$ be the matrix of the operator of multiplication by z in the basis $\psi_n(z)$. By (1.15),

$$Q = \begin{pmatrix} 0 & \sqrt{R_1} & 0 & \cdots \\ \sqrt{R_1} & 0 & \sqrt{R_2} & \cdots \\ 0 & \sqrt{R_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let P be the matrix of the operator d/dz. Then $P^\top = -P$, where $P^\top$ is the transposed matrix. Observe that

$$\psi_n' = -\left(\frac{NV'}{2}\right)\psi_n + \frac{P_n' e^{-NV/2}}{\sqrt{h_n}} = -\left(\frac{NV'}{2}\right)\psi_n + \frac{n}{\sqrt{R_n}}\psi_{n-1} + \cdots,$$

so that

$$\left[P + \frac{NV'(Q)}{2}\right]_{n,n-1} = \frac{n}{\sqrt{R_n}} \tag{1.21}$$

and

$$\left[P + \frac{NV'(Q)}{2}\right]_{n,n+1} = 0\,;$$

hence

$$0 = \left[P + \frac{NV'(Q)}{2}\right]_{n-1,n} = \left[-P + \frac{NV'(Q)}{2}\right]_{n,n-1}. \tag{1.22}$$

Combining (1.21) with (1.22) we obtain

$$\frac{n}{N\sqrt{R_n}} = [V'(Q)]_{n,n-1}\,, \tag{1.23}$$

the discrete string equation (see [11, 24, 34]). When V is as in (1.2), this reduces to

$$\frac{n}{N} = R_n(t + gR_{n-1} + gR_n + gR_{n+1}) \tag{1.24}$$

(cf. [4, 10, 32, 42]). Note that from (1.12) and (1.24) it follows that

$$\frac{n}{N} > tR_n + gR_n^2\,;$$

hence

$$0 < R_n < \frac{-t + \sqrt{t^2 + 4gn/N}}{2g}\,. \tag{1.25}$$

1.5 Lax Pair

Introduce the vector-valued function

$$\vec{\Psi}_n(z) = \begin{pmatrix} \psi_n(z) \\ \psi_{n-1}(z) \end{pmatrix}. \tag{1.26}$$

It satisfies the following 2×2 matrix differential equation:

$$\vec{\Psi}_n'(z) = NA_n(z)\vec{\Psi}_n(z)\,, \tag{1.27}$$

where

$$A_n(z) = \begin{pmatrix} -\left(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n\right) & \sqrt{R_n}(t + gz^2 + gR_n + gR_{n+1}) \\ -\sqrt{R_n}(t + gz^2 + gR_{n-1} + gR_n) & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix} \tag{1.28}$$

([29]; see also [7]). For the sake of completeness, let us remark that for a general even polynomial

$$V(z) = \sum_{j=1}^{k} \frac{t_{2j}z^{2j}}{2j}\,,$$

the matrix $A_n(z)$ has the form

$$A_n(z) = \begin{pmatrix} -a_n(z) & \sqrt{R_n}\,\frac{a_n(z)+a_{n+1}(z)}{z} \\ -\sqrt{R_n}\,\frac{a_{n-1}(z)+a_n(z)}{z} & a_n(z) \end{pmatrix},$$

where

$$a_n(z) = \frac{V'(z)}{2} + \sqrt{R_n}\sum_{j=2}^{k} t_{2j}\sum_{l=1}^{j-1} z^{2l-1}[Q^{2j-2l-1}]_{n,n-1}$$

[30]. This reduces to (1.28) when V is as in (1.2).

In addition, we have the recurrence equation

$$\vec{\Psi}_{n+1} = U_n(z)\vec{\Psi}_n(z)\,, \tag{1.29}$$

where

$$U_n(z) = \begin{pmatrix} \frac{z}{R_{n+1}^{1/2}} & -\frac{R_n^{1/2}}{R_{n+1}^{1/2}} \\ 1 & 0 \end{pmatrix}. \tag{1.30}$$

The compatibility condition of equations (1.27) and (1.29) is

$$U_n'(z) = NA_{n+1}(z)U_n(z) - NU_n(z)A_n(z)\,. \tag{1.31}$$

Restricting this matrix equation to the element 11, one can rederive string equation (1.24) (see [29, 30, 39]). Conversely, (1.24) implies (1.31). Thus, system (1.27) and (1.29) is the Lax pair for equation (1.24).

Remark. The Lax pair (1.27) and (1.29) can be alternatively written [29, 30, 39] as the following linear differential difference system:

$$\begin{cases} \frac{d\psi}{dz} = NV_-'(Q)\psi \\ Q\psi = z\psi \end{cases} \qquad \psi \equiv (\psi_1, \dots, \psi_n, \dots)\,.$$

Accordingly, string equation (1.42) takes the "quantum" [55, 57] commutator form

$$[Q, V_-'(Q)] = \frac{1}{N}$$

(see [39]). Here, as usual, the low triangular part M_- of a matrix M is defined by the equations $[M_-]_{nm} = [M]_{nm}$ if $n > m$ and $[M_-]_{nm} = 0$ if $n \le m$.

To describe the Riemann-Hilbert problem, consider the adjoint functions

$$\phi_n(z) = e^{NV(z)/2}\int_{-\infty}^{\infty} \frac{e^{-NV(u)/2}\psi_n(u)}{z-u}\,du\,, \quad z \in \mathbb{C}\setminus\mathbb{R}\,. \tag{1.32}$$

They share the following properties:

(1) As $z \to \infty$,

$$\phi_n(z) \sim \sqrt{h_n}z^{-n-1}e^{NV(z)/2}\left(1 + \sum_{k=1}^{\infty}\frac{\gamma_{2k}}{z^{2k}}\right).$$

(2) The function $\phi_n(z)$ has limits $\phi_{n\pm}(z)$ as z approaches the real axis from above and below, and the limits are related as

$$\phi_{n+}(z) = \phi_{n-}(z) - 2\pi i\psi_n(z)\,.$$

(3) The vector function

$$\vec{\Phi}_n(z) = \begin{pmatrix} \phi_n(z) \\ \phi_{n-1}(z) \end{pmatrix}$$

solves Lax pair equations (1.27) and (1.29).

To prove (1) expand $\frac{1}{u-z}$ into geometric series and use orthogonality. Property (2) follows from the jump condition for the Cauchy integral. Property (3) will follow from the Riemann-Hilbert problem (see Proposition 1.1 below). Define the matrix-valued function $\Psi_n(z)$ on the complex plane as

$$\Psi_n(z) = \begin{pmatrix} \psi_n(z) & \phi_n(z) \\ \psi_{n-1}(z) & \phi_{n-1}(z) \end{pmatrix}. \tag{1.33}$$

It solves the Lax pair equations

$$\begin{cases} \Psi_n'(z) = NA_n(z)\Psi_n(z) \\ \Psi_{n+1}(z) = U_n(z)\Psi_n(z) \end{cases} \tag{1.34}$$

and the Riemann-Hilbert problem for orthogonal polynomials.

1.6 Riemann-Hilbert Problem

The Riemann-Hilbert problem for orthogonal polynomials is formulated as follows ([30]; see also [7, 18]): One has to find a 2×2 matrix-valued function $\Psi_n(z)$ on the complex plane that is analytic outside of the real line and that has continuous limits from above and below the real line,

$$\Psi_{n\pm}(z) = \lim_{u \to z, \pm \operatorname{Im} u > 0} \Psi_n(u),$$

so that

(i) $\Psi_n(z)$ satisfies the jump condition on the real line,

$$\Psi_{n+}(z) = \Psi_{n-}(z)S, \quad \operatorname{Re} z = 0, \tag{1.35}$$

where

$$S = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}, \tag{1.36}$$

and

(ii) as $z \to \infty$, the function $\Psi_n(z)$ has the following uniform asymptotic expansion:

$$\Psi_n(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k} \right) e^{-(\frac{NV(z)}{2} - n \ln z + \lambda_n)\sigma_3}, \quad z \to \infty, \tag{1.37}$$

where Γ_k, $k = 0, 1, \dots$, are some constant 2×2 matrices with

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ R_n^{1/2} & 0 \end{pmatrix}, \tag{1.38}$$

λ_n is a constant, and σ_3 is the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.39}$$

To solve the Riemann-Hilbert problem, we have to find the function $\Psi_n(z)$ and real numbers λ_n and $R_n > 0$ such that (i) and (ii) hold.

PROPOSITION 1.1 *There exists a unique solution to the Riemann-Hilbert problem* (1.35)–(1.38)*, and it is given by* (1.33)*. The number R_n in this solution coincides with the recursive coefficient in* (1.11) *and*

$$\lambda_n = \frac{\ln h_n}{2}, \tag{1.40}$$

where h_n is defined in (1.10)*. The functions $\Psi_n(z)$, $n = 0, 1, \dots$, satisfy the Lax pair equations* (1.34).

For the proof of Proposition 1.1 see [7, 18, 30]. From now on, we shall "forget" about explicit equation (1.33) for the solution $\Psi_n(z)$ of the Riemann-Hilbert problem (1.35)–(1.38). It is the Riemann-Hilbert problem itself that now becomes the principal characteristic of the function $\Psi_n(z)$. In other words, we shift the focus from equation (1.33), which represents the function $\Psi_n(z)$ in terms of the orthogonal polynomials $P_n(z)$, to the equation

$$\psi_n(z) = [\Psi_n(z)]_{11},$$

which represents the orthogonal polynomials $P_n(z)$ in terms of the function $\Psi_n(z)$ (which in turn is uniquely determined as a solution of the Riemann-Hilbert problem (1.35)–(1.38)). It is also worth emphasizing that (see [7]) in the setting of the Riemann-Hilbert problem, the quantities R_n and λ_n *are not the given data*. They are evaluated via the solution $\Psi_n(z)$, which is determined by conditions (1.35)–(1.38) uniquely without any prior specification of R_n and λ_n.

Since $V(-z) = V(z)$, we obtain that

$$\psi_n(-z) = (-1)^n \psi_n(z). \tag{1.41}$$

This leads to the following equation on $\Psi_n(z)$:

$$\Psi_n(-z) = (-1)^n \sigma_3 \Psi_n(z) \sigma_3. \tag{1.42}$$

We now formulate the main result concerning the semiclassical asymptotics of the functions $\psi_n(z)$ in the double scaling limit.

1.7 Formulation of the Main Result

We will consider n such that n/N is close to the critical value $\lambda_c = t^2/(4g)$. We introduce the variable y as

$$y = c_0^{-1} N^{2/3} \left(\frac{n}{N} - \lambda_c \right), \quad c_0 = \left(\frac{t^2}{2g} \right)^{1/3}. \tag{1.43}$$

We will assume that $|y|$ is bounded; that is, we fix an arbitrary large number T_0, and we will consider such n so that

$$|y| \le T_0 \,. \tag{1.44}$$

Our results will be concerned with the asymptotics of the recurrence coefficients R_n and the functions $\psi_n(z)$. For R_n we will prove the following asymptotics:

$$\begin{aligned} R_n &= \frac{|t|}{2g} + N^{-1/3} c_1 (-1)^{n+1} u(y) + N^{-2/3} c_2 v(y) + O(N^{-1}) \,, \\ c_1 &= \left(\frac{2|t|}{g^2}\right)^{1/3}, \quad c_2 = \frac{1}{2}\left(\frac{1}{2|t|g}\right)^{1/3}, \end{aligned} \tag{1.45}$$

where $u(y)$ is the Hastings-McLeod solution to the Painlevé II equation

$$u''(y) = yu(y) + 2u^3(y) \,, \tag{1.46}$$

which is characterized by the conditions at infinity

$$\lim_{y \to -\infty} \frac{u(y)}{(-y/2)^{1/2}} = 1 \,, \quad \lim_{y \to \infty} u(y) = 0 \,, \tag{1.47}$$

and

$$v(y) = y + 2u^2(y) \,. \tag{1.48}$$

Remark. Asymptotics (1.45) has been suggested in physical papers by Douglas, Seiberg, and Shenker [23], Crnković and Moore [15], and Periwal and Shevitz [60]. The existence and uniqueness of the solution to (1.46)–(1.47) was first established in [35] (see also the later works [22, 44, 45]). It is also worth noting that the asymptotics of $u(y)$ at $+\infty$ can be specified as

$$u(y) = \mathrm{Ai}(y)(1 + O(|y|^{-1})) \,, \tag{1.47$'$}$$

where $\mathrm{Ai}(y)$ is the Airy function. Moreover, the asymptotic condition (1.47′) characterizes solution $u(x)$ uniquely, so that the first equation in (1.47) and equation (1.47′) constitute an example of the so-called connection formulae for Painlevé equations (see [28, 36, 41] for more on this matter). We should also point out that the analytic and asymptotic properties of the Hastings-McLeod solution for complex values of y can be extracted from the results of [37, 44, 45, 49, 58].

We will prove asymptotics (1.45) simultaneously with semiclassical asymptotics for the function $\psi_n(z)$ on the whole complex plane. To that end we will substitute asymptotics (1.45) into the coefficients of differential equation (1.27) and solve the resulting equation in the semiclassical approximation. Realization of this idea is not trivial, and in comparison with the standard semiclassical analysis, it involves a new type of special function generated by the monodromy problem associated with the second Painlevé equation. The appearance of the latter is closely related to the fact that ansatz (1.45) leads to the coalescence of four turning points in system (1.27) (cf. [48]). To see this, let us analyze (1.27).

System (1.27) can be reduced to one equation of the second order. Namely, if we denote by a_{ij}, $i, j = 1, 2$, the matrix elements of the matrix $A_n(z)$, solve ψ_{n-1} in terms of ψ_n from the first equation in (1.27),

$$\psi_{n-1} = \frac{1}{a_{12}}\left(N^{-1}\psi_n' - a_{11}\psi_n\right),$$

and substitute

$$\psi_n = \sqrt{a_{12}}\,\eta\,, \tag{1.49}$$

then we obtain from the second equation in (1.27) the Schrödinger equation on η,

$$-\eta'' + N^2 U\eta = 0\,, \tag{1.50}$$

where

$$U = -\det A_n + N^{-1}\left[(a_{11})' - a_{11}\frac{(a_{12})'}{a_{12}}\right] - N^{-2}\left[\frac{(a_{12})''}{2a_{12}} - \frac{3((a_{12})')^2}{4(a_{12})^2}\right]. \tag{1.51}$$

The zeroth-order term in the potential U is $-\det A_n$, so let us consider properties of $-\det A_n$.

From (1.28) we obtain that $-\det A_n(z)$ is a polynomial of the sixth degree,

$$-\det A_n(z) = \frac{g^2 z^4}{4}\left(z^2 - z_0^2\right) + \alpha_n z^2 + \beta_n\,, \tag{1.52}$$

where

$$z_0 = \sqrt{-\frac{2t}{g}} \tag{1.53}$$

and

$$\alpha_n = \frac{t^2}{4} - gR_n(t + gR_{n-1} + gR_n + gR_{n+1})\,, \quad \beta_n = -R_n\theta_{n-1}\theta_n\,, \tag{1.54}$$
$$\theta_n = t + gR_n + gR_{n+1}\,.$$

Due to string equation (1.24), α_n simplifies to

$$\alpha_n = \frac{t^2}{4} - \frac{gn}{N} = -g\left(\frac{n}{N} - \lambda_c\right), \tag{1.55}$$

or, according to scaling (1.43),

$$\alpha_n = -gc_0 y N^{-2/3}\,.$$

The substitution of asymptotics (1.45) into β_n gives

$$\beta_n = c_3 N^{-4/3} + c_4 N^{-5/3} + O(N^{-2})\,, \tag{1.56}$$

where

$$c_3 = -\left(\frac{g^{1/3}|t|^{1/3}}{2^{5/3}}\right)\left[v^2(y) - 4w^2(y)\right], \quad c_4 = (-1)^n\left(\frac{g^{2/3}}{2^{1/3}|t|^{1/3}}\right)w(y)\,, \tag{1.57}$$

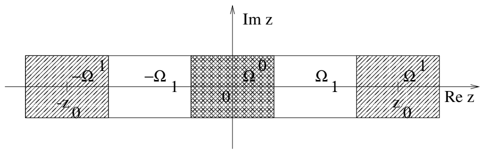

FIGURE 1.2. The region Ω.

and $w(y) \equiv u'(y)$, thus,

$$-\det A_n(z) = \frac{g^2 z^4}{4}\left(z^2 - z_0^2\right) - g\left(\frac{n}{N} - \lambda_c\right)z^2 + \left[c_3 N^{-4/3} + c_4 N^{-5/3} + O(N^{-2})\right]. \tag{1.58}$$

As $N \to \infty$, $-\det A_n(z)$ approaches the polynomial

$$a_\infty(z) = \frac{g^2 z^4}{4}\left(z^2 - z_0^2\right), \tag{1.59}$$

which has two simple roots at $\pm z_0$ and a quadruple root at the origin. Therefore, equation (1.50) has two simple turning points approaching $\pm z_0$ and four turning points coalescing at the origin.

In accord with this analysis, we divide the complex plane into several regions. The form of the semiclassical asymptotics will be different in different regions. Let d_1 and d_2 be arbitrary fixed (i.e., independent of N) numbers such that

$$0 < d_2 \le d_1 \le \frac{z_0}{4}. \tag{1.60}$$

Introduce the rectangular region Ω as

$$\Omega = \{z : |\mathrm{Re}\, z| \le z_0 + d_1,\ |\mathrm{Im}\, z| \le d_2\}. \tag{1.61}$$

Observe that the segment $[-z_0, z_0]$ lies in Ω. Introduce furthermore the rectangular subregions Ω^0 and Ω^1 of the region Ω as

$$\begin{aligned} \Omega^0 &= \{z : |\mathrm{Re}\, z| \le d_1,\ |\mathrm{Im}\, z| \le d_2\}, \\ \Omega^1 &= \{z : |\mathrm{Re}\, z - z_0| \le d_1,\ |\mathrm{Im}\, z| \le d_2\}, \end{aligned} \tag{1.62}$$

so that Ω^0 is a rectangular region centered at 0 and Ω^1 at z_0. Finally, let Ω_1 be the rectangular region between Ω^0 and Ω^1, that is,

$$\Omega_1 = \{z : d_1 \le \mathrm{Re}\, z \le z_0 - d_1,\ |\mathrm{Im}\, z| \le d_2\} \tag{1.63}$$

(see Figure 1.2). We will prove the following semiclassical asymptotics for the vector function $\vec{\Psi}_n(z)$:

(1) in $\Omega^c \equiv \overline{\mathbb{C} \setminus \Omega}$: WKB asymptotics of exponential type,
(2) in Ω_1: WKB asymptotics of cosine type,
(3) in Ω^1: turning point (TP) asymptotics (in terms of the Airy function), and
(4) in Ω^0: critical point (CP) asymptotics of Painlevé II type.

In the regions $-\Omega^1$ and $-\Omega_1$, the semiclassical asymptotics will follow by symmetry equation (1.42). As a matter of fact, both the TP and CP asymptotics will be extended to the region Ω_1, and we will use this extension to prove the WKB asymptotics of the cosine type in Ω_1.

1.8 Region $\Omega^c = \overline{\mathbb{C} \setminus \Omega}$: WKB Asymptotics

In Ω^c we will prove the following asymptotics:

$$\vec{\Psi}_n(z) = \left(1 + O\left(\frac{1}{N(1+|z|)}\right)\right)\vec{\Psi}_{\mathrm{WKB}}(z)\,, \quad z \in \Omega^c\,, \tag{1.64}$$

where

$$\vec{\Psi}_{\mathrm{WKB}}(z) \equiv \frac{1}{2\pi^{1/2}}\left(R_n^0\right)^{-1/4} T^c(z) \begin{pmatrix} e^{-N\xi^c(z)} \\ -e^{-N\xi^c(z)} \end{pmatrix}, \quad z \in \Omega^c\,, \tag{1.65}$$

with the quantities on the right defined as follows: We define

$$R_n^0 \equiv \frac{|t|}{2g} + N^{-1/3}c_1(-1)^{n+1}u(y) + N^{-2/3}c_2 v(y) \tag{1.66}$$

(cf. (1.45)), where y is defined as in (1.43), and we put

$$A_n^0(z) \equiv \begin{pmatrix} -\left(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n^0\right) & \sqrt{R_n^0}\left(t + gz^2 + gR_n^0 + gR_{n+1}^0\right) \\ -\sqrt{R_n^0}\left(t + gz^2 + gR_{n-1}^0 + gR_n^0\right) & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n^0 \end{pmatrix} \tag{1.67}$$

(cf. (1.28)). We denote the matrix elements of $A_n^0(z)$ by $a_{ij}^0(z)$. We define the function $\mu^c(z)$ as

$$\mu^c(z) \equiv \sqrt{U^0(z)}\,,$$

where $U^0(z)$ is a suitable approximation of the function $U(z)$ in (1.51), namely,

$$\begin{aligned} U^0(z) \equiv{} & \frac{g^2z^4}{4}\left(z^2 - z_0^2\right) - \left(\frac{n}{N} - \frac{t^2}{4g}\right)z^2 + \left(c_3N^{-4/3} + c_4N^{-5/3}\right) \\ & + N^{-1}\left[a_{11}^{0\,\prime}(z) - a_{11}^0(z)\frac{a_{12}^{0\,\prime}(z)}{a_{12}^0(z)}\right], \end{aligned} \tag{1.68}$$

and we put

$$\xi^c(z) \equiv \int_{z_0^N}^{z} \mu^c(u)du, \quad z \in \Omega^c\,, \tag{1.69}$$

where z_0^N is the root of $U^0(z)$ that approaches z_0 as $N \to \infty$. The contour of integration in (1.69) is taken as follows: first from z_0^N to $z_0 + d_1$ and then from

$z_0 + d_1$ to z around the region Ω in the counterclockwise direction. The matrix $T^c(z)$ is defined as

$$T^c(z) \equiv \left(\frac{a_{12}^0(z)}{\mu^c(z)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{\mu^c(z)}{a_{12}^0(z)} \end{pmatrix}. \tag{1.70}$$

Observe that $\det T^c(z) \equiv 1$. The branches for $(a_{12}^0(z))^{1/2}$, $(U^0(z))^{1/2}$, and $(\mu^c(z))^{1/2}$ are fixed by the condition that they are positive for large positive z. We will check below that the function $\vec{\Psi}_{\mathrm{WKB}}(z)$ is analytic in Ω^c. The meaning of formula (1.64) and similar formulae to follow is that there exists a 2×2 matrix-valued function $\varepsilon_N(z)$ such that

$$\vec{\Psi}_n(z) = (1 + \varepsilon_N(z))\vec{\Psi}_{\mathrm{WKB}}(z)\,, \quad z \in \Omega^c\,,$$

and

$$\sup_{z\in\Omega^c} |(1 + |z|)\varepsilon_N(z)| \le cN^{-1}\,, \quad c > 0\,.$$

For concrete calculations the function $U^0(z)$ can be simplified as follows. The function

$$U^1(z) \equiv a_{11}^{0\,\prime}(z) - a_{11}^0(z)\frac{a_{12}^{0\,\prime}(z)}{a_{12}^0(z)}$$

is analytic in Ω^c and as $N \to \infty$,

$$\begin{aligned} U^1(z) &= -\frac{gz^2}{2} - N^{-1/3}\widetilde{c}_3 - N^{-2/3}\widetilde{c}_4 + O(N^{-1})\,, \quad z \in \Omega^c\,, \\ \widetilde{c}_3 &= (-1)^n 2^{1/3} g^{1/3}|t|^{1/3}\,, \quad \widetilde{c}_4 = \frac{2^{2/3}g^{2/3}[v(y) + 4(-1)^n w(y)]}{4|t|^{1/3}}\,; \end{aligned} \tag{1.71}$$

hence

$$\begin{aligned} U^0(z) &= \frac{g^2z^4}{4}\big(z^2 - z_0^2\big) - \left(\frac{n}{N} - \lambda_c + \frac{1}{2N}\right)z^2 + \beta_N + O(N^{-2})\,, \\ \beta_N &= (c_3 - \widetilde{c}_3)N^{-4/3} + (c_4 - \widetilde{c}_4)N^{-5/3}\,, \end{aligned}$$

and the error term $O(N^{-2})$ in $U^0(z)$ can be neglected in concrete calculations, thereby reducing $U^0(z)$ to a polynomial of the sixth degree.

For the normalizing constant h_n, we will prove the asymptotics

$$h_n = e^{2N\int_{z_0^N}^{\infty} \mu^c(u)du}(1 + O(N^{-1}))\,. \tag{1.72}$$

The integral in the exponent is regularized at infinity as in formula (5.3) below.

1.9 Region Ω_1: WKB Asymptotics of Cosine Type

In the region Ω_1 we define $\vec{\Psi}_{\mathrm{WKB}}(z)$ as

$$\vec{\Psi}_{\mathrm{WKB}}(z) \equiv \frac{1}{\pi^{1/2}}\left(R_n^0\right)^{-1/4} T_1(z) \begin{pmatrix} \cos(N\xi_1(z) + \frac{\pi}{4}) \\ -\sin(N\xi_1(z) + \frac{\pi}{4}) \end{pmatrix}, \quad z \in \Omega_1, \tag{1.73}$$

where

$$\mu_1(z) \equiv (-U^0(z))^{1/2}, \quad \xi_1(z) \equiv \int_{z_0^N}^{z} \mu_1(u)du, \quad z \in \Omega_1, \tag{1.74}$$

and

$$T_1(z) = \left(\frac{a_{12}^0(z)}{\mu_1(z)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{\mu_1(z)}{a_{12}^0(z)} \end{pmatrix}, \tag{1.75}$$

so that $\det T_1(z) \equiv 1$. The functions $-U^0(z)$ and $a_{12}^0(z)$ are positive on the interval $d_1 \le z \le z_0 - d_1$, and the branches for the square roots in (1.74)–(1.75) are chosen to be positive on this interval. We will prove that

$$\vec{\Psi}_n(z) = (1 + O(N^{-1}))\vec{\Psi}_{\mathrm{WKB}}(z), \quad z \in \Omega_1. \tag{1.76}$$

1.10 Region Ω^1: TP Asymptotics

In the region Ω^1 we define the turning point vector-valued function $\vec{\Psi}_{\mathrm{TP}}(z)$ as

$$\vec{\Psi}_{\mathrm{TP}}(z) \equiv \left(R_n^0\right)^{-1/4} W(z) \begin{pmatrix} N^{1/6}\,\mathrm{Ai}(N^{2/3}w(z)) \\ N^{-1/6}\,\mathrm{Ai}'(N^{2/3}w(z)) \end{pmatrix}, \quad z \in \Omega^1, \tag{1.77}$$

where

$$w(z) = \left(\frac{3}{2}\int_{z_0^N}^{z} \sqrt{U^0(u)}\,du\right)^{2/3} \tag{1.78}$$

and

$$W(z) = \left(\frac{a_{12}^0(z)}{w'(z)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{w'(z)}{a_{12}^0(z)} \end{pmatrix}, \tag{1.79}$$

so that $\det W(z) \equiv 1$. The branches for fractional powers in (1.78)–(1.79) are chosen to be positive for $z > z_0^N$. Observe that the function $w(z)$ is analytic and has no critical points in Ω^1. We will prove that

$$\vec{\Psi}_n(z) = (1 + O(N^{-1}))\vec{\Psi}_{\mathrm{TP}}(z), \quad z \in \Omega^1. \tag{1.80}$$

1.11 Region Ω^0: CP Asymptotics

The normal form for system (1.27) at the critical point $z = 0$ is the system

$$\Psi'(z) = A(z)\Psi(z)\,, \tag{1.81}$$

where

$$A(z) = \begin{pmatrix} (-1)^n 4u(y)z & 4z^2 + (-1)^n 2w(y) + v(y) \\ -4z^2 + (-1)^n 2w(y) - v(y)) & -(-1)^n 4u(y)z \end{pmatrix}, \tag{1.82}$$

and

$$v(y) = y + 2u^2(y)\,, \quad w(y) = u'(y)\,, \tag{1.83}$$

where $u(y)$ is the Hastings-McLeod solution of the Painlevé II equation (1.46) (see Section 3). We will consider a special solution to (1.81),

$$\vec{\Phi}(z) = \begin{pmatrix} \Phi^1(z) \\ \Phi^2(z) \end{pmatrix}, \tag{1.84}$$

which is characterized by the following properties:

(1) $\vec{\Phi}(z)$ is real, i.e.,

$$\vec{\Phi}(\bar{z}) = \overline{\vec{\Phi}(z)}\,. \tag{1.85}$$

(2) It satisfies the parity equation

$$\vec{\Phi}(-z) = (-1)^n \sigma_3 \vec{\Phi}(z)\,. \tag{1.86}$$

(3) On the real axis the functions $\Phi^j(z)$ have the asymptotics

$$\begin{aligned} \Phi^1(z) &= \cos\left(\frac{4z^3}{3} + yz - \frac{\pi n}{2}\right) + O(z^{-1})\,, \\ \Phi^2(z) &= -\sin\left(\frac{4z^3}{3} + yz - \frac{\pi n}{2}\right) + O(z^{-1})\,, \quad z \to \pm\infty\,. \end{aligned} \tag{1.87}$$

The existence of the solution $\vec{\Phi}(z)$ is a nontrivial fact in the modern theory of Painlevé equations (see, e.g., [22, 41]). It should also be noted that, in addition to properties (1.85)–(1.87), the function $\vec{\Phi}(z)$ is an entire function on the complex z-plane, and its asymptotic behavior is known in the whole neighborhood of $z = \infty$ (see Proposition 3.2 below).

Remark. As a function of the parameter y, the vector $\vec{\Phi}(z)$ is a meromorphic function that satisfies the linear differential equation (cf. (1.81))

$$\frac{\partial \Psi(z)}{\partial y} = B(z)\Psi(z)\,, \tag{1.81$'$}$$

where

$$B(z) = \begin{pmatrix} (-1)^n u(y) & z \\ -z & -(-1)^n u(y) \end{pmatrix}. \tag{1.82$'$}$$

Moreover, the asymptotic distributions of the complex poles (which coincide with the poles of the Painlevé function $u(y)$) and the large $|y|$ asymptotics of $\vec{\Phi}(z)$ can be extracted from the general asymptotic results concerning the oscillatory Riemann-Hilbert problem associated with the second Painlevé equation; see [22, 37, 41, 44, 49].

In the region Ω^0 we define the critical point function as

$$\vec{\Psi}_{\mathrm{CP}}(z) \equiv \frac{1}{2\pi^{1/2}} \left(R_n^0\right)^{-1/4} V(z)\vec{\Phi}(N^{1/3}\zeta(z)) \,, \tag{1.88}$$

where the matrix-valued function $V(z)$ and the function $\zeta(z)$ will be defined below in Section 4 (see (4.65) and (4.82)). Both $V(z)$ and $\zeta(z)$ are analytic in Ω^0, $\det V(z) \equiv 1$, and $\zeta(z)$ has no critical points in Ω^0. It is worth noting that the functions $V(z)$, $\vec{\Phi}(z)$, and $\zeta(z)$ depend on the parameter y. We will prove that in Ω^0,

$$\vec{\Psi}_n(z) = (1 + O(N^{-1}))\vec{\Psi}_{\mathrm{CP}}(z) \,, \quad z \in \Omega^0 \,. \tag{1.89}$$

We can now formulate the main result concerning the double scaling limit for orthogonal polynomials.

THEOREM 1.2 *There exists $d_2^0 > 0$ such that for all d_2 in the interval $0 < d_2 \le d_2^0$ and all d_1 satisfying inequalities* (1.60)*, the following holds*: *Let Ω, Ω^0, Ω^1, and Ω_1 be the regions defined in* (1.61)–(1.63) (*see Figure* 1.2)*. Let $T_0 > 0$ be an arbitrary number and the variable y, defined in* (1.43)*, satisfy bound* (1.44)*. Then the recurrence coefficients R_n obey asymptotic formula* (1.45)*, and for the vector-valued function $\vec{\Psi}_n(z)$, asymptotic relations* (1.64)*,* (1.72)*,* (1.76)*,* (1.80)*, and* (1.89) *hold.*

Remark. It is interesting to compare Theorem 1.2 with the results of Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou [19] and of Baik, Deift, and Johansson [2], which are both based on the Riemann-Hilbert approach as well. The general theorem of [19] can be applied to our problem at the critical point $y = 0$, and for this case it gives asymptotics formulae for the function $\vec{\Psi}_n(z)$ with an error term of the order of $N^{-1/3}$. In the work of Baik, Deift, and Johansson [2], asymptotics similar to the ones of Theorem 1.2 (with the error term of the order of $N^{-2/3}$) have been obtained for the double scaling limit of a Riemann-Hilbert problem on the circle. We provide some comments on the relation of our approach and the one of [2] in Appendix F.

A different type of the double scaling limit in discrete string equation (1.23), which leads to the appearance of the Painlevé I equation, was discovered in [11, 24, 34] in connection with the matrix model of two-dimensional quantum gravity. This limit, on the level of solutions of the string equation, was analyzed in [29, 30, 39, 40, 38, 47] in the framework of the Riemann-Hilbert isomonodromy approach, and our method is very close to the scheme used in the works cited. An important difference, however, is that we construct an approximation of the orthogonal polynomials of the order of N^{-1} and in a finite neighborhood of the critical point. The original approach of [29, 30, 39, 40, 38, 47] gives only estimates of the order of $N^{-1/3}$ and in a neighborhood of the size of $N^{-1/3}$.

Finally, it is interesting to notice that the Hastings-McLeod solution to Painlevé II also appears in the Tracy-Widom distribution function at the edge of the spectrum; see [65].

1.12 Double Scaling Limit for Correlation Functions

From Theorem 1.2 we will derive the following results concerning the double scaling limit for correlation functions of the quartic matrix model:

THEOREM 1.3 *Let*

$$\vec{\Phi}(z;y) = \begin{pmatrix} \Phi^1(z;y) \\ \Phi^2(z;y) \end{pmatrix}$$

be the solution for $n = 0$ *to system* (1.81) *and relations* (1.85) *through* (1.87). *Then the following double scaling limit holds*:

$$\lim_{N\to\infty} \frac{1}{(cN^{1/3})^{m-1}} K_{Nm}\left(\frac{u_1}{cN^{1/3}}, \dots, \frac{u_m}{cN^{1/3}}; t_c + c_0 y N^{-2/3}\right) = \det(Q_c(u_i, u_j; y))_{i,j=1,\dots,m}, \tag{1.90}$$

where $c = \zeta'(0) > 0$, *and*

$$Q_c(u,v;y) = \frac{\Phi^1(u;y)\Phi^2(v;y) - \Phi^1(v;y)\Phi^2(u;y)}{\pi(u-v)}. \tag{1.91}$$

Furthermore, if z *is in the bulk of the spectrum, i.e.,* $0 < |z| < z_0$, *then*

$$\lim_{N\to\infty} \frac{1}{(p(z)N)^{m-1}} K_{Nm}\left(z + \frac{u_1}{p(z)N}, \dots, z + \frac{u_m}{p(z)N}; t_c + c_0 y N^{-2/3}\right) = \det(Q_b(u_i, u_j))_{i,j=1,\dots,m}, \tag{1.92}$$

where

$$Q_b(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}, \tag{1.93}$$

the sine kernel. At the edge of the spectrum,

$$\lim_{N\to\infty} \frac{1}{(cN^{2/3})^{m-1}} K_{Nm}\left(z_0 + \frac{u_1}{cN^{2/3}}, \dots, z_0 + \frac{u_m}{cN^{2/3}}; t_c + c_0 y N^{-2/3}\right) = \det(Q_e(u_i, u_j))_{i,j=1,\dots,m}, \tag{1.94}$$

where $c = w'(z_0) > 0$ *and*

$$Q_e(u,v) = \frac{\mathrm{Ai}(u)\,\mathrm{Ai}'(v) - \mathrm{Ai}(v)\,\mathrm{Ai}'(u)}{u-v}, \tag{1.95}$$

the Airy kernel.

Remark. The sine kernel in the bulk of the spectrum is established for a general (fixed) $V(M)$ by Pastur and Shcherbina [59] in a very different approach (see also a nonrigorous derivation of the sine kernel in the physical work of Brézin and Zee [12]). In the Riemann-Hilbert approach the sine kernel in the bulk of the spectrum is obtained for a general (fixed) $V(M)$ by Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou in [19] (see also the earlier paper [7] for the quartic noncritical case). The Airy kernel at the edge is established for the Gaussian model by Bowick and Brézin [9], Forrester [31], Moore [55], and Tracy and Widom [65], and for the quartic model by Bleher and Its [7].

1.13 Plan for the Rest of the Paper

In Section 2 we will give a formal (perturbative) derivation of asymptotics (1.45) for R_n. It will be justified in subsequent sections. In Section 3 we will discuss a normal form for the system of differential equations on the ψ function at the critical point. In Section 4 we will develop a three-step construction of the approximate solution in a neighborhood of the critical point. At the critical point, four turning points coalesce, making the asymptotic analysis rather nonstandard. We will be looking for the solution in the form $V(z)\Phi(N^{1/3}\zeta(z))$, where $\Phi(z)$ is a ψ function for the Hastings-McLeod solution to Painlevé II. First we will construct $\zeta(z)$ in the zeroth-order approximation; then we will construct $V(z)$ in the zeroth-order approximation, and after that we will correct $\zeta(z)$ to include terms of the first order. The basic role in this construction will be played by the equation of the equality of periods [6, 46]; see (4.64).

In Section 5 we will construct the WKB approximate solution in Ω^c and prove that it matches the critical point approximate solution up to terms of the order of N^{-1}. In Section 6 we will construct a turning points approximate solution in the region Ω^1, and we will show that it matches the WKB approximate solution. In Section 7 we will construct the WKB solution of cosine type in the intermediate region Ω_1, and we will prove that it matches the CP and TP solutions. In addition, we will prove that it matches the WKB solution on the top and on the bottom of Ω_1.

The analysis of Sections 5 through 7 provides us with an *explicit* matrix-valued function $\Psi_n^0(z)$ that solves asymptotically, as $N \to \infty$, the basic Riemann-Hilbert problem (1.35)–(1.38). In Section 8 we will prove that the quotient $\Psi_n(z)[\Psi_n^0(z)]^{-1}$ is equal to $I + O(N^{-1}(1+|z|)^{-1})$, and this will conclude the proof of Theorem 1.2. We emphasize that we only use differential equation (1.27) to motivate our choice of the function $\Psi_n^0(z)$. The uniform estimate for the difference $\Psi_n(z)[\Psi_n^0(z)]^{-1} - I$ is proven by means independent of the WKB theory of differential equations. Indeed, the proof of the estimate is based on the analysis of the Riemann-Hilbert problem, which is solved by the quotient $\Psi_n(z)[\Psi_n^0(z)]^{-1}$.

The ideas and techniques used in Section 8 are close to the Deift-Zhou nonlinear steepest descent method [20], although there is an essential difference as well. We give more details on this matter in Appendix B. Finally, in Section 9 we will prove Theorem 1.3.

For what follows it will be convenient to make the substitution

$$\Psi_n(z) = \widetilde{\Psi}_n(z) \begin{pmatrix} (2\pi)^{-1/2} & 0 \\ 0 & (2\pi)^{1/2} \end{pmatrix}. \tag{1.96}$$

This does not change the Lax pair equations (1.34) and simplifies (1.35) to

$$\widetilde{\Psi}_{n+}(z) = \widetilde{\Psi}_{n-}(z) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}. \tag{1.97}$$

For the sake of brevity we will again denote $\widetilde{\Psi}_n(z)$ by $\Psi_n(z)$, and we will take into account substitution (1.96) at the very end.

2 Formal Painlevé II Asymptotics near the Critical Point

String equation (1.24) is supplemented by the initial conditions

$$R_0 = 0, \quad R_1 = \frac{\int_{-\infty}^{\infty} z^2 e^{-NV(z)} dz}{\int_{-\infty}^{\infty} e^{-NV(z)} dz} \tag{2.1}$$

(see (1.12)). The critical point $\lambda_c = t^2/(4g)$ is a bifurcation point for R_n. Namely, for $\lambda < \lambda_c$ the numbers R_n are attracted to two different branches for odd and even n, so that

$$\lim_{n,N\to\infty : n/N\to\lambda} R_n = \begin{cases} R(\lambda) & \text{if } n = 2k+1 \\ L(\lambda) & \text{if } n = 2k, \end{cases} \tag{2.2}$$

where

$$R, L = \frac{-t \pm \sqrt{t^2 - 4\lambda g}}{2g}, \quad \lambda < \lambda_c ; \tag{2.3}$$

see [7]. Contrariwise, for $\lambda > \lambda_c$, the numbers R_n are attracted to one branch,

$$\lim_{n,N\to\infty : n/N\to\lambda} R_n = R(\lambda), \tag{2.4}$$

where

$$R = \frac{-t + \sqrt{t^2 + 12g\lambda}}{6g}, \quad \lambda > \lambda_c . \tag{2.5}$$

At the critical point,

$$R = L = -\frac{t}{2g}, \quad \lambda = \lambda_c .$$

Figure 2.1 presents results of numerical integration of string equation (1.24) with the parameters $t = -1$, $g = 1$, and $N = 400$. In this case $\lambda_c = \frac{1}{4}$, so that $n_c = 100$. The authors thank Bobby Ramsey for his help in carrying out the numerical integration. It is worth noting that the numerical integration has been done by minimizing a global variational functional on trajectories $\{R_n,\ n = 0, 1, \dots, n_0\}$ which proved to be an efficient method of integration.

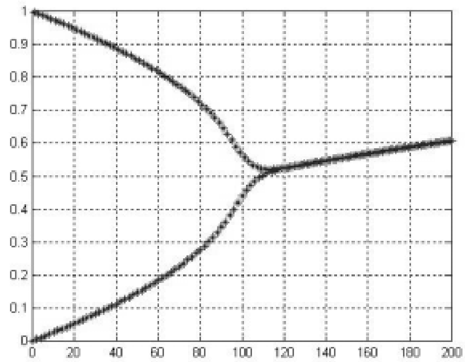

FIGURE 2.1. Computer integration of the string equation: R_n versus n. The solution shown corresponds to $t = -1$, $g = 1$, and $N = 400$.

The problem of the double scaling limit is to find the asymptotic behavior of R_n near λ_c. First we will calculate the double scaling formally (by perturbation theory) and later we will prove it rigorously. Let us assume that

$$R_n = \begin{cases} R(y) \equiv -\frac{t}{2g} + N^{-\beta}u(y) + N^{-\gamma}v(y) & \text{if } n = 2k+1 \\ L(y) \equiv -\frac{t}{2g} - N^{-\beta}u(y) + N^{-\gamma}v(y) & \text{if } n = 2k, \end{cases} \tag{2.6}$$

where y is determined by the equation

$$\frac{n}{N} = \lambda_c + N^{-\alpha}y\,,$$

with some exponents $\alpha, \beta, \gamma > 0$ to be determined. Our assumption here is that $u(y)$ and $v(y)$ are smooth functions of $y \in \mathbb{R}$. Now we substitute ansatz (2.6) into (1.24), first for odd n and then for even,

$$\begin{aligned} \lambda_c + N^{-\alpha}y &= R(y)\big[t + g(2L(y) + R(y)) + gN^{2\alpha-2}\Delta L(y)\big]\,, \\ \lambda_c + N^{-\alpha}y &= L(y)\big[t + g(2R(y) + L(y)) + gN^{2\alpha-2}\Delta R(y)\big]\,, \end{aligned} \tag{2.7}$$

where Δ is defined as

$$\Delta f(y) = \frac{f(y - N^{\alpha-1}) - 2f(y) + f(y + N^{\alpha-1})}{N^{2\alpha-2}}\,.$$

For our calculations we can replace Δ by the operator of the second derivative. Subtracting the second equation in (2.7) from the first one, we obtain that

$$(R-L)(t + gR + gL) + gN^{2\alpha-2}(RL'' - LR'') = 0\,.$$

Substituting (2.6) and neglecting higher-order terms in N^{-1}, we obtain that

$$4guv + N^{2\alpha-2+\gamma}tu'' = 0\,. \tag{2.8}$$

To get a nontrivial scaling we set

$$2\alpha - 2 + \gamma = 0\,; \tag{2.9}$$

then (2.8) reduces to the equation

$$v = -\frac{tu''}{4gu}\,. \tag{2.10}$$

The first equation in (2.7) gives that

$$N^{-\alpha}y = -N^{-\gamma}2tv - N^{-2\beta}gu^2 \tag{2.11}$$

(modulo smaller terms). To get a nontrivial scaling we put

$$\alpha = \gamma = 2\beta\,.$$

Combining these relations with (2.9), we obtain that

$$\alpha = \gamma = \frac{2}{3}\,, \qquad \beta = \frac{1}{3}\,, \tag{2.12}$$

so that (2.6) is written as

$$R_n = -\frac{t}{2g} + (-1)^{n+1}N^{-1/3}u(y) + N^{-2/3}v(y)\,, \quad y = N^{2/3}\left(\frac{n}{N} - \lambda_c\right), \tag{2.13}$$

and (2.11) reduces to the equation

$$y = -2tv - gu^2\,. \tag{2.14}$$

Substituting v from (2.10), we obtain that

$$y = \frac{t^2}{2g}\,\frac{u''(y)}{u(y)} - gu^2(y)\,, \tag{2.15}$$

which is the Painlevé II equation. From (2.10),

$$v(y) = -\frac{t}{4g}\,\frac{u''(y)}{u(y)} = \frac{y + gu^2(y)}{(-2t)}\,. \tag{2.16}$$

To bring (2.15) and (2.16) to a standard form of the Painlevé II equation, we make a rescaling of u, v, and y. To that end we rewrite (2.13) as

$$\begin{aligned} R_n &= -\frac{t}{2g} + c_1(-1)^{n+1}N^{-1/3}u(y) + c_2N^{-2/3}v(y)\,, \\ y &= c_0^{-1}N^{2/3}\left(\frac{n}{N} - \frac{t^2}{4g}\right), \end{aligned} \tag{2.17}$$

where

$$c_0 = \left(\frac{t^2}{2g}\right)^{1/3}, \quad c_1 = \left(\frac{2|t|}{g^2}\right)^{1/3}, \quad c_2 = \frac{1}{2}\left(\frac{1}{2|t|g}\right)^{1/3}. \tag{2.18}$$

Then (2.15) reduces to

$$u'' = uy + 2u^3\,, \tag{2.19}$$

which is a standard form of the Painlevé II equation (in fact, a particular case of the Painlevé II). As we have already indicated in the introduction, ansatz (2.17) had been suggested in physical papers by Douglas, Seiberg, and Shenker [23], Crnković and Moore [15], and Periwal and Shevitz [60].

Equations (2.3) and (2.5) give the boundary conditions

$$u(y)\begin{cases}\sim\sqrt{-y/2} & \text{if } y\to-\infty\\ \to 0 & \text{if } y\to\infty.\end{cases} \tag{2.20}$$

This selects a special solution to the Painlevé equation, the Hastings-McLeod solution [35] (cf. Section 1). It has the following asymptotics:

$$\begin{aligned} u(y) &= \left(-\frac{y}{2}\right)^{1/2}\left(1+\frac{1}{4y^3}+\cdots\right),\\ v(y) &= -\frac{1}{4y^2}+O(y^{-4}), \quad y\to-\infty,\\ u(y) &= \mathrm{Ai}(y)\big(1+O(e^{-(2/3)y^{3/2}})\big),\\ v(y) &= y+O\big(e^{-(4/3)y^{3/2}}\big), \quad y\to\infty, \end{aligned} \tag{2.21}$$

where $\mathrm{Ai}(y)$ is the Airy function. Of course, the numbers R_n in (2.17) do not satisfy the Freud equation (1.24) exactly, and we have for them the Freud equation with an error term,

$$\frac{n}{N} = R_n(t+gR_{n-1}+gR_n+gR_{n+1})+O(N^{-4/3}). \tag{2.22}$$

More precisely, substitution of (2.17) into (1.24) gives that

$$\begin{aligned} &R_n(t+gR_{n-1}+gR_n+gR_{n+1})-\frac{n}{N} = \\ &N^{-2/3}c_0[v(y)-y-2u^2(y)]+N^{-1}(-1)^n[u''(y)-u(y)v(y)]+O(N^{-4/3}), \end{aligned} \tag{2.23}$$

which implies (2.22).

To get an asymptotic expansion for R_n we use (2.17) with

$$u(y)=\sum_{j=0}^{\infty}N^{-(2/3)j}u_j(y), \qquad v(y)=\sum_{j=0}^{\infty}N^{-(2/3)j}v_j(y). \tag{2.24}$$

Then from the terms of the order of $N^{-2/3}$ and N^{-1} we obtain Painlevé equations (2.19) on u_0 and v_0. After that from the terms of the order of $N^{-4/3}$ and $N^{-5/3}$ we obtain a linear system on u_1 and v_1; then from the terms of the order of N^{-2} and $N^{-7/3}$ we obtain a linear system on u_2, v_2, etc.

To illustrate this process, assume that $g = 1$ and $t = -2$ (the general case can be reduced to this one). Then

$$
\begin{aligned}
&R_n(t + gR_{n-1} + gR_n + gR_{n+1}) - \frac{n}{N} \\
&\quad = N^{-2/3}2^{1/3}\big(v_0 - 2u_0^2 - y\big) + N^{-1}\big(u_0'' - u_0v_0\big) \\
&\qquad + N^{-4/3}2^{1/3}\left(-4u_0u_1 + v_1 + \frac{2^{1/3}}{16}\big(2v_0'' - 16u_0u_0'' + 3v_0^2\big)\right) \\
&\qquad - N^{-5/3}\left(v_0u_1 + u_0v_1 + \frac{2^{1/3}}{4}\big(u_0v_0'' - v_0u_0''\big)\right) + \cdots .
\end{aligned}
\tag{2.25}
$$

Equating coefficients on the right to zero, we obtain the equations

$$v_0 = y + 2u_0^2\,, \quad u_0'' = u_0v_0\,, \tag{2.26}$$

and

$$
\begin{aligned}
4u_0u_1 - v_1 &= \frac{2^{1/3}}{16}\big(2v_0'' - 16u_0u_0'' + 3v_0^2\big)\,, \\
v_0u_1 + u_0v_1 &= -\frac{2^{1/3}}{4}\big(u_0v_0'' - v_0u_0''\big)\,,
\end{aligned}
\tag{2.27}
$$

etc., from which we subsequently determine the functions $u_j(y)$ and $v_j(y)$.

3 The Ψ Functions for Painlevé II

Our next step is to derive a model (normal form) equation for the matrix differential equation

$$\Psi_n'(z) = NA_n(z)\Psi_n(z)$$

at the critical point $z = 0$. To that end we substitute (2.17) into the matrix elements of $A_n(z)$, rescale z as

$$z = CN^{-1/3}s\,, \quad C = \left(\frac{32}{|t|g}\right)^{1/6}, \tag{3.1}$$

and keep the leading terms. This gives the equation

$$\Psi'(s) = A(s)\Psi(s) \tag{3.2}$$

with

$$A(s) = \begin{pmatrix} (-1)^n 4u(y)s & 4s^2 + (-1)^n 2w(y) + v(y) \\ -4s^2 + (-1)^n 2w(y) - v(y) & -(-1)^n 4u(y)s \end{pmatrix}, \tag{3.3}$$

where

$$v(y) = y + 2u^2(y)\,, \qquad w(y) = u'(y)\,,$$

and $u(y)$ is the Hastings-McLeod solution of the Painlevé II equation (2.19). For another form of the model equation, see the physical paper by Akemann, Damgaard, Magnea, and Nishigaki [1].

3.1 Turning Points of the Model Equation

The turning points of (3.2) are the zeros of $\det A(s)$ on the complex plane. Observe that

$$\det A(s) = 16s^4 + 8ys^2 + v^2(y) - 4w^2(y)\,; \tag{3.4}$$

hence the turning points are solutions of the biquadratic equation

$$s^4 + \frac{y}{2}s^2 + \frac{v^2(y) - 4w^2(y)}{16} = 0\,. \tag{3.5}$$

Denote for this equation,

$$\begin{aligned} p = p(y) &\equiv \frac{y}{2}\,, \quad q = q(y) \equiv \frac{v^2(y) - 4w^2(y)}{16}\,, \\ D = D(y) &\equiv p^2(y) - 4q(y) = \frac{y^2 - v^2(y)}{4} + w^2(y)\,. \end{aligned} \tag{3.6}$$

PROPOSITION 3.1 (i) *The discriminant* $D(y)$ *is positive for all* $y \in \mathbb{R}$, *so that equation* (3.5) *has four roots,* $\pm s_{1,2} = \pm s_{1,2}(y)$, *such that*

$$-\infty < s_1^2(y) < s_2^2(y) < \infty\,.$$

(ii) *For all* $y \in \mathbb{R}$, $s_1^2(y) < 0$. *There exists some* $y_0 > 0$ *such that*

$$s_2^2(y) > 0\,, \quad y < y_0\,, \quad s_2^2(y) < 0\,, \quad y > y_0\,.$$

(iii) *The following asymptotics hold*:

$$\begin{aligned} s_1^2 \sim -\frac{1}{16y^2}\,, \quad s_2^2 \sim -\frac{y}{2}\,, \quad y \to -\infty\,, \\ s_{1,2}^2 = -\frac{y}{4} \mp \frac{1}{4(2\pi)^{1/2}y^{1/2}}e^{-(2/3)y^{3/2}} + \cdots\,, \quad y \to \infty\,. \end{aligned} \tag{3.7}$$

PROOF: We have

$$D'(y) = -u^2(y) - 2u'(y)\big(yu(y) + 2u^3(y) - u''(y)\big) = -u^2(y)$$

and $D(\infty) = 0$; hence

$$D(y) = \int_y^\infty u^2(x)dx > 0\,, \tag{3.8}$$

which proves (i). To prove (ii) and (iii), observe that by (3.8),

$$q'(y) = \frac{y}{8} - \frac{D'(y)}{4} = \frac{y + 2u^2(y)}{8} = \frac{v(y)}{8}\,, \quad q(0) = -\frac{D(0)}{4} < 0\,. \tag{3.9}$$

Since $q(-\infty) = 0$, we obtain that

$$q(y) = \frac{1}{8}\int_{-\infty}^y v(y)dy = \frac{1}{32y} + O(|y|^{-3})\,, \quad y \to -\infty\,.$$

This gives the first line in (3.7). From (3.8) and (2.21),

$$D(y) = \frac{1}{8\pi y} e^{-(4/3)y^{3/2}} (1 + O(y^{-3/2}))\,, \quad y \to \infty\,.$$

This gives the second line in (3.7). Thus, (iii) is proved.

To prove (ii) we will show that $q(y)$ has a unique zero y_0 on the real axis. Then (ii) follows. Since

$$v(y) = y + 2u^2(y) > 0\,, \quad y > 0\,,$$

we obtain from (3.9) that $q(y)$ is increasing for $y \geq 0$ and it has one zero $y_0 > 0$ on the positive half-axis (observe that $q(\infty) = \infty$). We claim that $q(y)$ has no zeros on the negative half-axis. Indeed,

$$v''(y) = 4(u'(y))^2 + 4u''(y)u(y) = 4(u'(y))^2 + 4u^2(y)v(y)\,;$$

hence if $v(y) > 0$ then $v''(y) > 0$. From (2.21) we have that $v(y)$ is negative for negative y sufficiently large in absolute value. On the other hand, $v(0) = 2u^2(0) \geq 0$; hence there is $y_1 \leq 0$ such that $v(y_1) = 0$. We claim that $v(y)$ can have only one zero. Assume that there are two. Then there exists $y_2 < 0$ such that $v(y_2) = 0$ and $v(t) > 0$ for $y_2 - \delta < t < y_2$ for some $\delta > 0$. But then $v'(y_2) \leq 0$ and $v''(t) > 0$ for $y_2 - \delta < t < y_2$. Now take any $y < y_2$. By the Taylor expansion,

$$v(y) = v(y_2) + v'(y_2)(y - y_2) + \frac{1}{2}\int_y^{y_2} (t - y)v''(t)dt\,,$$

hence $v(y) > 0$ as long as $v(t) \geq 0$ on $[y, y_2]$, which implies that $v(y) > 0$ for all $y \leq y_2$, which is not true. The contradiction proves the unicity of the zero of $v(y)$. This in turn proves that $q(y)$ has no zero for negative y's. Indeed, $q(-\infty) = q(y_0) = 0$; hence if $q(y_3) = 0$ for some $y_3 \leq 0$, then $v(y) = 8q'(y)$ would have at least two zeros, which is not true. This finishes the proof of Proposition 3.1. □

3.2 Standard Form of the Model Equation

The matrix (3.3) differs from the standard Flaschka-Newell form of the Ψ equation for Painlevé II [27]. To put it in standard form we make the gauge transformation

$$\Psi(s) = U\widetilde{\Psi}(s)\,, \quad U = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n i \end{pmatrix} = \begin{pmatrix} 1 & (-1)^n \\ -i & (-1)^n i \end{pmatrix}. \tag{3.10}$$

Then

$$\widetilde{\Psi}'(s) = \widetilde{A}(s)\widetilde{\Psi}(s)\,, \tag{3.11}$$

where

$$\widetilde{A}(s) = U^{-1}A(s)U = \begin{pmatrix} -4is^2 - iv & 4us + 2iw \\ 4us - 2iw & 4is^2 + iv \end{pmatrix}, \quad v = y + 2u^2\,. \tag{3.12}$$

Solutions to equation (3.11) are entire functions of the complex variable s. Their behavior at $s = \infty$ is governed by six Stokes matrices (the infinity is an irregular singular point of (3.11) of Poincare index 3). A key fact for our analysis is that the Stokes matrices of (3.11), with u, v, and w determined via the Hastings-McLeod Painlevé function, are known. More precisely (see [22, 41]), there exist six Stokes solutions $\widetilde{\Psi}_j(s)$ to equation (3.11) such that

$$\lim_{|s|\to\infty} \widetilde{\Psi}_j(s) e^{i((4/3)s^3+ys)\sigma_3} = I \quad \text{if} \left| \arg s - \frac{(j-1)\pi}{3} \right| \le \frac{\pi}{3} - \varepsilon\,. \tag{3.13}$$

The Stokes solutions are uniquely determined by (3.13), and they are related as follows:

$$\begin{aligned} &\widetilde{\Psi}_1(s) = \widetilde{\Psi}_6(s) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{\Psi}_2(s) = \widetilde{\Psi}_1(s) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \widetilde{\Psi}_3(s) = \widetilde{\Psi}_2(s), \\ &\widetilde{\Psi}_4(s) = \widetilde{\Psi}_3(s) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \widetilde{\Psi}_5(s) = \widetilde{\Psi}_4(s) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{\Psi}_6(s) = \widetilde{\Psi}_5(s)\,. \end{aligned} \tag{3.14}$$

Remark. The existence of the Stokes solutions $\widetilde{\Psi}_j(s)$ is a fact of the general theory of systems of linear ODEs with rational coefficients (see, e.g., [62]), and it has nothing to do with the particular choice of the parameters u, y, and w in (3.11). For any triple u, y, and w, the solutions $\widetilde{\Psi}_j(s)$ exist, they are holomorphic with respect to u, y, and w, and the asymptotics (3.13) is uniform with respect to u, y, and w varying in a compact set. Also, in the general case, the relations (3.14) are replaced by the general equation

$$\widetilde{\Psi}_{j+1}(s) = \widetilde{\Psi}_j(s) S_j\,,$$

where the Stokes matrices S_j are some *transcendental* functions of parameters u, y, and w. A remarkable fact ([27, 43]; see also [41] and indeed classical works of R. Garnier [33]) is that the Stokes matrices S_j form a complete set of the first integrals for the Painlevé equation (2.19). The particular choice of Stokes matrices indicated in (3.14) corresponds to a selection of the Hastings-McLeod solution of the Painlevé II equation (2.19).

Equations (3.14) lead in turn to three special vector solutions to (3.2).

PROPOSITION 3.2 *Assume that $A(s)$ is defined as in* (3.3). *Then for every real y, there exist vector solutions $\vec{\Phi}(s) = \vec{\Phi}(s; y)$ and $\vec{\Phi}_j(s) = \vec{\Phi}_j(s; y)$, $j = 1, 2$, of the equation $\vec{\Phi}'(s) = A(s)\vec{\Phi}(s)$ on the complex plane such that*

(i) $\vec{\Phi}(s)$ *is real, i.e.,*

$$\vec{\Phi}(\bar{s}) = \overline{\vec{\Phi}(s)}\,, \tag{3.15}$$

and it satisfies the parity equation

$$\vec{\Phi}(-s) = (-1)^n \sigma_3 \vec{\Phi}(s)\,. \tag{3.16}$$

As $|s| \to \infty$,

$$\vec{\Phi}(s) \sim \left(\sum_{k=0}^{\infty} \frac{\vec{\Gamma}_k^0}{s^k} \right) e^{-i((4/3)s^3+ys)} \quad \textit{if}\ \varepsilon < \arg s < \pi - \varepsilon \ \ \forall \varepsilon > 0 \tag{3.17}$$

with $\vec{\Gamma}_0^0 = \binom{i^n}{i^{n-1}}$, *and*

$$\vec{\Phi}(s) \sim \left(I + \sum_{k=1}^{\infty} \frac{\Gamma_{2k}}{s^{2k}}\right) \begin{pmatrix} \cos\left(\frac{4s^3}{3} + ys - \frac{\pi n}{2}\right) \\ -\sin\left(\frac{4s^3}{3} + ys - \frac{\pi n}{2}\right) \end{pmatrix} \tag{3.17'}$$
$$\textit{if } |\arg s| < \frac{\pi}{3} - \varepsilon \quad \textit{or} \quad |\arg s - \pi| < \frac{\pi}{3} - \varepsilon\,,$$

where Γ_{2k} *are some matrix-valued coefficients.*

(ii) $\vec{\Phi}_1(s)$ *has the asymptotics as* $|s| \to \infty$,

$$\vec{\Phi}_1(s) \sim \left(\sum_{k=0}^{\infty} \frac{\vec{\Gamma}_k^1}{s^k}\right) e^{i((4/3)s^3 + ys)} \quad \textit{if } -\frac{\pi}{3} + \varepsilon < \arg s < \frac{4\pi}{3} - \varepsilon \tag{3.18}$$

with $\vec{\Gamma}_0^1 = \binom{(-i)^{n+1}}{(-i)^n}$.

(iii) $\vec{\Phi}_2(s) = \overline{\vec{\Phi}_1(\overline{s})}$ *and*

$$\vec{\Phi}_2(-s) = \begin{pmatrix} (-1)^{n+1} & 0 \\ 0 & (-1)^n \end{pmatrix} \vec{\Phi}_1(s)\,, \tag{3.19}$$

$$\vec{\Phi}_1(s) - \vec{\Phi}_2(s) = -i\vec{\Phi}(s)\,. \tag{3.20}$$

In what follows we will use the matrix-valued functions

$$\Phi^u(s) = (\vec{\Phi}(s), \vec{\Phi}_1(s))\,, \quad \Phi^d(s) = (\vec{\Phi}(s), \vec{\Phi}_2(s))\,. \tag{3.21}$$

As follows from Proposition 3.2, they satisfy the relations

$$\Phi^u(s) = \Phi^d(s) S\,, \quad S = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \tag{3.22}$$
$$\Phi^{u,d}(-s) = (-1)^n \sigma_3 \Phi^{d,u}(s) \sigma_3\,, \quad \overline{\Phi^u(\overline{s})} = \Phi^d(s)\,.$$

At infinity for all $\varepsilon > 0$, the function $\Phi^u(s)$ has the asymptotics

$$\Phi^u(s) = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left(\sum_{j=0}^{\infty} \frac{m_j}{s^j}\right) e^{-(\frac{4}{3} i s^3 + iys + \gamma)\sigma_3} \tag{3.23}$$
$$\text{as } s \to \infty,\ \varepsilon \le \arg s \le \pi - \varepsilon\,,$$

where

$$\gamma = -i\frac{n\pi}{2}\,, \quad m_0 = I\,,$$

and the rest of the matrix coefficients m_j can be found recursively by substituting series (3.23) into equation (3.2). The first two coefficients are given by the following equations:

$$m_1 = -\frac{iD}{2}\sigma_3 - \frac{(-1)^n u}{2}\sigma_1\,, \quad m_2 = \frac{u^2 - D^2}{8} I + (-1)^n \frac{w + uD}{4}\sigma_2\,, \tag{3.24}$$
$$D = w^2 - u^4 - yu^2\,,$$

where σ_j are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.25}$$

The proof of Proposition 3.2 is given in Appendix A below.

4 Semiclassical Approximation near the Critical Point

In this section, which plays the central role in the whole paper, we will construct a semiclassical approximation to the equation

$$\Psi'(z) = NA_n^0(z)\Psi(z) \tag{4.1}$$

in a fixed neighborhood of the critical point $z = 0$. It is worth emphasizing from the very beginning that we will not prove any results concerning properties and asymptotics of solutions to equation (4.1). It will be used only as a tool in constructing an approximate solution to the RH problem.

Denote

$$\theta_n^0 \equiv t + gR_n^0 + gR_{n+1}^0 \,. \tag{4.2}$$

From (1.66) we get that

$$\begin{aligned}
\theta_n^0 &= N^{-2/3}c_5 + N^{-1}c_6 + O(N^{-4/3})\,,\\
c_5 &= \left(\frac{g^2}{2|t|}\right)^{1/3}[v(y) + 2(-1)^n w(y)]\,,\\
c_6 &= \left(\frac{g}{2|t|}\right)[2(-1)^n u(y)v(y) + v'(y)]\,,\\
\theta_{n-1}^0 &= N^{-2/3}c_7 + N^{-1}c_8 + O(N^{-4/3})\,,\\
c_7 &= \left(\frac{g^2}{2|t|}\right)^{1/3}[v(y) - 2(-1)^n w(y)]\,,\\
c_8 &= \left(\frac{g}{2|t|}\right)[2(-1)^n u(y)v(y) - v'(y)]\,.
\end{aligned} \tag{4.3}$$

This gives the matrix elements of $A_n^0(z)$ as

$$\begin{aligned}
a_{11}^0(z) &= -\frac{gz^3}{2} - \left[c_1 g(-1)^{n+1}N^{-1/3}u(y) + c_2 g N^{-2/3}v(y)\right]z\,,\\
a_{12}^0(z) &= \left(R_n^0\right)^{1/2}\left[gz^2 + N^{-2/3}c_5 + N^{-1}c_6 + O(N^{-4/3})\right],\\
a_{21}^0(z) &= -\left(R_n^0\right)^{1/2}\left[gz^2 + N^{-2/3}c_7 + N^{-1}c_8 + O(N^{-4/3})\right],
\end{aligned} \tag{4.4}$$

where the constants c_1 and c_2 are defined in (1.45). In the semiclassical approximation the function $\det A_n^0(z)$ will be important. We will use the function

$$d(z) \equiv -\frac{g^2 z^4}{4}(z^2 - z_0^2) + g\left(\frac{n}{N} - \lambda_c\right)z^2 - \left(c_3 N^{-4/3} + c_4 N^{-5/3}\right) \tag{4.5}$$

as a suitable approximation to $\det A_n^0(z)$ (cf. (1.58)). From (1.67) we have that

$$\det A_n^0(z) = d(z) + \alpha_n^0 z^2 + \beta_n^0\,, \quad \alpha_n^0 = O(N^{-4/3})\,, \quad \beta_n^0 = O(N^{-2})\,. \tag{4.5'}$$

4.1 Three-Step Critical Point Solution

In a neighborhood of $z = 0$ we are looking for a *critical point approximate solution* to equation (4.1) in the following form:

$$\Psi_{\mathrm{CP}}(z) = V(z)\Phi(N^{1/3}\zeta(z))\,, \tag{4.6}$$

where $V(z)$ is a gauge matrix-valued function, $\Phi(z)$ is a matrix-valued solution to the model equation

$$\Phi'(z) = A(z)\Phi(z) \tag{4.7}$$

with $A(z)$ defined as in (3.3), and $\zeta(z)$ is an analytic change of variable with $\zeta'(0) \neq 0$. We will choose $\Phi(z)$ differently for $\operatorname{Im} z \geq 0$ and $\operatorname{Im} z \leq 0$ to secure the necessary multiplicative jump. Here we will carry out a general analysis of (4.1), and we will assume that $\Phi(z)$ is any solution to (4.7). Substituting (4.6) into (4.1), we obtain the equation

$$V(z)\left[\zeta'(z)N^{-2/3}A(N^{1/3}\zeta(z))\right]V^{-1}(z) = A_n^0(z) - N^{-1}V'(z)V^{-1}(z)\,. \tag{4.8}$$

From this equation we determine iteratively $\zeta(z)$ and $V(z)$ in three steps: (1) the zeroth-order approximation for $\zeta(z)$, (2) the zeroth-order approximation for $V(z)$, and (3) the first-order approximation for $\zeta(z)$.

4.2 Zeroth-Order Approximation for $\zeta(z)$

Taking the determinant of both sides in (4.8), we obtain that

$$[\zeta'(z)]^2 N^{-4/3} \det A(N^{1/3}\zeta(z)) = \det\left[A_n^0(z) - N^{-1}V'(z)V^{-1}(z)\right]. \tag{4.9}$$

In the zeroth-order approximation we will neglect terms of the order of N^{-1}. So we drop the term $N^{-1}V'(z)V^{-1}(z)$ on the right,

$$[\zeta'(z)]^2 N^{-4/3} \det A(N^{1/3}\zeta(z)) = \det A_n^0(z)\,, \tag{4.10}$$

which is an equation on $\zeta(z)$ alone. By (3.4),

$$\begin{aligned} N^{-4/3}\det A(N^{1/3}\zeta) = f(\zeta) \equiv {} & 16\zeta^4(z) + 8N^{-2/3}y\zeta^2(z) \\ & + N^{-4/3}[v^2(y) - 4w^2(y)]\,. \end{aligned} \tag{4.11}$$

We replace $\det A_n^0(z)$ by $d(z)$ (see (4.5)), reducing (4.10) to the equation

$$(\zeta')^2 f(\zeta) = d(z)\,. \tag{4.12}$$

Now we drop the $O(N^{-1})$ terms in f and d, and we introduce the functions

$$f^0(\zeta) = 16\zeta^4 + 8N^{-2/3}y\zeta^2 \tag{4.13}$$

and

$$d^0(z) = \frac{g^2z^4}{4}(z_0^2 - z^2) + g\left(\frac{n}{N} - \lambda_c\right)z^2 . \tag{4.14}$$

Equation (4.12) then reduces to

$$(\zeta')^2 f^0(\zeta) = d^0(z) \tag{4.15}$$

or

$$\zeta'\sqrt{f^0(\zeta)} = \sqrt{d^0(z)} . \tag{4.16}$$

To fix a branch for the square roots, observe that both $f^0(z)$ and $d^0(z)$ are positive at $z = z_0/2$ for large N. We will assume that both square roots are also positive at $z_0/2$. Equation (4.16) is separable and is easy to solve. The problem is to find an analytic solution.

Consider this problem more carefully. Let us make the change of variables

$$z = CN^{-1/3}s , \qquad \zeta = N^{-1/3}\sigma , \tag{4.17}$$

where C is the same as in (3.1). We will call (s, σ) local coordinates (at the critical point) and (z, ζ) global ones. In the local coordinates, equation (4.16) reduces to

$$\sigma'\sqrt{\varphi^0(\sigma)} = \sqrt{\delta^0(s)} , \tag{4.18}$$

where

$$\begin{aligned} \varphi^0(\sigma) &\equiv N^{4/3} f^0(N^{-1/3}\sigma) = 16\sigma^4 + 8y\sigma^2 , \\ \delta^0(s) &\equiv C^2N^{4/3}d^0(CN^{-1/3}s) \\ &= 16s^4 + 8c_0^{-1}N^{2/3}\left(\frac{n}{N} - \lambda_c\right)s^2 - N^{-2/3}c_9s^6 , \end{aligned} \tag{4.19}$$

where c_0 is defined in (1.43) and

$$c_9 = \frac{2^{14/3}g^{2/3}}{|t|^{4/3}} . \tag{4.20}$$

Using the definition of y in (1.43), we reduce (4.19) to

$$\begin{aligned} \varphi^0(\sigma) &= 16\sigma^4 + 8y\sigma^2 , \\ \delta^0(s) &= 16s^4 + 8ys^2 - N^{-2/3}c_9s^6 . \end{aligned} \tag{4.21}$$

Observe that φ^0 and δ^0 differ only by a term of the order of $N^{-2/3}$. Therefore, we are looking for an analytic solution to (4.18) in the form

$$\sigma(s) = s + N^{-2/3}\sigma_1(s) + N^{-4/3}\sigma_2(s) + \cdots . \tag{4.22}$$

To secure the analyticity of $\sigma(s)$ at $s = \pm s_0$ we need the condition

$$\int_0^{\sigma_0} \sqrt{\varphi^0(\tau)}\, d\tau = \int_0^{s_0} \sqrt{\delta^0(\tau)}\, d\tau\,. \tag{4.23}$$

Indeed, $\sigma = \sigma(s)$, as a solution to equation (4.18) with initial condition $\sigma(0) = 0$, is determined by the implicit equation

$$\int_0^{\sigma} \sqrt{\varphi^0(\tau)}\, d\tau = \int_0^{s} \sqrt{\delta^0(\tau)}\, d\tau\,. \tag{4.24}$$

The integrals on both sides have the same $s^{3/2}$ singularity at $\sigma = \sigma_0$ and $s = s_0$, respectively. Therefore, to have an analytic solution $\sigma(s)$ at $s = s_0$, we should be able to rewrite (4.24) as

$$\int_{\sigma_0}^{\sigma} \sqrt{\varphi^0(\tau)}\, d\tau = \int_{s_0}^{s} \sqrt{\delta^0(\tau)}\, d\tau\,, \tag{4.25}$$

which implies (4.23). We will call (4.23) the equation of periods.

From (4.21) it is not difficult to evaluate the periods by perturbation theory in $1/N^{2/3}$, and they are certainly different. Therefore we need a parameter to adjust the periods. To that end we change the relation between n and y from (1.43) to

$$y = c_0^{-1} N^{2/3}\left(\frac{n}{N} - \lambda_c\right) + \alpha N^{-2/3}\,, \tag{4.26}$$

where α is a parameter. Then (4.21) changes as follows:

$$\begin{aligned} \varphi^0(\sigma) &= 16\sigma^4 + 8y\sigma^2\,, \\ \delta^0(s) &= 16s^4 + 8(y - \alpha N^{-2/3})s^2 - c_9 N^{-2/3} s^6\,, \end{aligned} \tag{4.27}$$

and we find the value of the parameter $\alpha = \alpha(y)$ from equation (4.23). Let us analyze this procedure in more detail.

When $y = 0$ we take $\alpha = 0$. Then both φ^0 and δ^0 have a quadruple zero at the origin, and equation (4.23) holds. For $y \neq 0$, let us scale the functions $\varphi^0(\sigma)$ and $\delta^0(s)$ as follows:

$$\begin{aligned} \hat\varphi(\sigma) &\equiv -\frac{1}{16y^2}\varphi^0\big(\sqrt{-2y}\,\sigma\big) = -\sigma^4 + \sigma^2\,, \\ \hat\delta(s) &\equiv -\frac{1}{16y^2}\delta^0\big(\sqrt{-2y}\,s\big) \\ &= -s^4 + (1 - \hat\alpha N^{-2/3})s^2 - \frac{N^{-2/3}c_9 y}{2}s^6\,, \quad \alpha = y\hat\alpha\,, \end{aligned} \tag{4.28}$$

where $\sqrt{-2y} > 0$ for $y < 0$ and $\operatorname{Im}\sqrt{-2y} > 0$ for $y > 0$. Then (4.23) reduces to

$$\int_0^{1} \sqrt{\hat\varphi(\tau)}\, d\tau = \int_0^{\hat s} \sqrt{\hat\delta(\tau)}\, d\tau\,, \tag{4.29}$$

where $\hat s$ is the zero of $\hat\delta(s)$ that is closest to 1.

$$\hat{\delta}(\hat{s}) = 0\,, \quad \hat{s} = 1 + O(N^{-2/3})\,. \tag{4.30}$$

To analyze (4.29) we use the implicit function theorem. To that end, introduce, for small $|x_1|$ and $|x_2|$, the function

$$I(x_1, x_2) = \int_0^{\hat{s}(x_1,x_2)} \sqrt{-s^4 + (1-x_1)s^2 - x_2 s^6}\, ds\,, \tag{4.31}$$

where $\hat{s}(x_1, x_2)$ is the zero of the function under the radical closest to 1. We claim that $I(x_1, x_2)$ is an analytic function at $x_1 = x_2 = 0$ and

$$\left.\frac{\partial I(x_1, x_2)}{\partial x_1}\right|_{x_1=x_2=0} \neq 0\,, \quad \left.\frac{\partial I(x_1, x_2)}{\partial x_2}\right|_{x_1=x_2=0} \neq 0\,. \tag{4.32}$$

To prove the analyticity, apply the Cauchy theorem and rewrite $I(x_1, x_2)$ as

$$I(x_1, x_2) = \frac{1}{2} \oint_{\Gamma_0} \sqrt{-s^4 + (1-x_1)s^2 - x_2 s^6}\, ds\,, \tag{4.31$'$}$$

where Γ_0 is a circle on the complex plane of radius 1 centered at $s = 1$. In this form one can differentiate $I(x_1, x_2)$ with respect to x_1 and x_2 and prove the analyticity at $x_1 = x_2 = 0$. To prove (4.32), differentiate (4.31$'$), set $x_1 = x_2 = 0$, and return to the real integral

$$\left.\frac{\partial I(x_1, x_2)}{\partial x_1}\right|_{x_1=x_2=0} = -\int_0^1 \frac{s^2}{2\sqrt{-s^4 + s^2}}\, ds \neq 0\,,$$

and similarly for $(\partial I(x_1, x_2))/\partial x_2$. Equation (4.29) is equivalent to

$$I(x_1, x_2) = I(0, 0)\,, \quad x_1 = \hat{\alpha} N^{-2/3}\,, \quad x_2 = \frac{N^{-2/3} c_9 y}{2}\,.$$

The implicit function theorem ensures the existence of an analytic solution $x_1 = x_1(x_2)$ with $dx_1/dx_2(0) \neq 0$. This in turn gives an analytic solution $\alpha = \alpha(y) \sim cy^2$, $c \neq 0$, to (4.23).

Let us summarize our calculations. In the zeroth-order approximation, the change-of-variable function $\zeta_0(z)$ is determined from the initial value problem

$$\zeta_0' \sqrt{f^0(\zeta_0)} = \sqrt{d^0(z)}\,, \quad \zeta_0(0) = 0\,, \tag{4.33}$$

where

$$\begin{aligned} f^0(z) &= 16z^4 + 8N^{-2/3} y z^2\,, \quad \frac{n}{N} = \lambda_c + c_0 N^{-2/3}(y - \alpha N^{-2/3})\,, \\ d^0(z) &= \frac{g^2 z^4}{4}\left(z_0^2 - z^2\right) + g\left(\frac{n}{N} - \lambda_c\right) z^2\,, \end{aligned} \tag{4.34}$$

and α is determined from (4.23). We consider the solution to (4.33) in the region

$$(-\Omega_1) \cup \Omega^0 \cup \Omega_1 = \{z : -z_0 + d_1 \leq \operatorname{Re} z \leq z_0 - d_1,\ |\operatorname{Im} z| \leq d_2\}\,;$$

see Figure 1.2. We also notice that, because of the uniqueness of the solution of the initial value problem (4.33), the following symmetry relation takes place:

$$\zeta_0(-z) = -\zeta_0(z)\,. \tag{4.35}$$

When z is separated from 0, i.e., $|z| \geq \omega_0 > 0$ where $\omega_0 < d_1$ does not depend on N, (4.34) gives

$$\sqrt{f^0(z)} = 4z^2 + N^{-2/3}yz + O(N^{-4/3})\,, \quad \sqrt{d^0(z)} = \sqrt{d(z)} + O(N^{-4/3})$$

(cf. (4.5) and (4.9)); hence equation (4.33) implies that

$$4\zeta_0'(z)\zeta_0^2(z) + N^{-2/3}y\zeta_0'(z) = \sqrt{d(z)} + O(N^{-4/3})\,, \quad |z| \geq \omega_0\,.$$

In what follows, we use the function

$$\mu_0(z) \equiv \sqrt{-d(z)} \tag{4.36}$$

rather than $\sqrt{d(z)}$, so we rewrite the last equation as

$$\begin{aligned} 4i\zeta_0'(z)\zeta_0^2(z) + N^{-2/3}iy\zeta_0'(z) &= \mu_0(z) + O(N^{-4/3})\,, \\ & z \in \big((-\Omega_1) \cup \Omega^0 \cup \Omega_1\big) \setminus \{|z| < \omega_0\}\,. \end{aligned} \tag{4.37}$$

Observe that the function $-d(z)$ is positive for large N if $z \geq z_0 + d_1$ (cf. (4.5) and (4.9)), and we take the branch of the square root for $\mu_0(z)$ that is positive for $z \geq z_0 + d_1$. In (4.37) we continue $\mu_0(z)$ analytically going around z_0 from above. We note that, for sufficiently large N, $\mu_0(z)$ is holomorphic in the domain

$$\{|z| > \omega_0\} \setminus \big((-\infty, -z^N] \cup [z^N, +\infty)\big)\,,$$

where z^N denotes the zero of $d(z)$ that approaches z_0 as $N \to \infty$. In particular, $\mu_0(z)$ is holomorphic in the domain $((-\Omega_1) \cup \Omega^0 \cup \Omega_1) \setminus \{|z| < \omega_0\}$.

Equation (4.37) can be used to describe, within an error term of the order of $N^{-4/3}$, the function $\zeta_0(z)$ by an elementary explicit formula. Indeed, as long as z is separated from 0, the function $\mu_0(z)$ admits the asymptotic representation (cf. (4.5) and (4.36)),

$$\begin{gathered} \mu_0(z) = \frac{igz^2}{2}\sqrt{z_0^2 - z^2} + \frac{ic_0 y}{\sqrt{z_0^2 - z^2}} N^{-2/3} + O(N^{-4/3})\,, \\ z \in \Lambda(\omega_0) \equiv \big((-\Omega_1) \cup \Omega^0 \cup \Omega_1\big) \setminus \{|z| < \omega_0\}\,, \end{gathered} \tag{4.38}$$

where $\sqrt{z_0^2 - z^2}$ denotes the branch of the root that is analytic in $\mathbb{C} \setminus ((-\infty, z_0] \cup [z_0, \infty))$ and positive for $-z_0 < z < z_0$. This estimate allows us to rewrite (4.37) as

$$\begin{aligned} & 4\zeta_0'(z)\zeta_0^2(z) + N^{-2/3}y\zeta_0'(z) = \\ & \qquad \frac{gz^2}{2}\sqrt{z_0^2 - z^2} + \frac{c_0 y}{\sqrt{z_0^2 - z^2}} N^{-2/3} + O(N^{-4/3})\,, \quad z \in \Lambda(\omega_0)\,. \end{aligned} \tag{4.39}$$

By integrating the last equation and taking into account the symmetry $z \to -z$, we arrive at the relation

$$\frac{4}{3}\zeta_0^3(z) + N^{-2/3}y\zeta_0(z) = D_\infty(z) + N^{-2/3}yD_1(z) + O(N^{-4/3}), \quad z \in \Lambda(\omega_0), \tag{4.40}$$

where we have introduced the notation

$$D_\infty(z) := \int_0^z \frac{gu^2}{2}\sqrt{z_0^2 - u^2}\, du \tag{4.41}$$

and

$$D_1(z) := c_0 \int_0^z \frac{du}{\sqrt{z_0^2 - u^2}}. \tag{4.42}$$

Notice that both the functions $D_\infty(z)$ and $D_1(z)$ are analytic and odd in $\mathbb{C} \setminus ((-\infty, z_0] \cup [z_0, \infty))$. (Of course, $D_\infty(z)$ and $D_1(z)$ can be expressed in terms of elementary functions, but we will not need these expressions; the integral representations are already elementary enough and quite convenient for any further analysis.) Equation (4.40), in its turn, implies the estimate

$$\zeta_0(z) = \zeta_\infty(z) + N^{-2/3}y\zeta_1(z) + O(N^{-4/3}), \quad z \in \Lambda(\omega_0), \tag{4.43}$$

where the functions $\zeta_\infty(z)$ and $\zeta_1(z)$ are defined by the equations

$$\zeta_\infty(z) = \left[\frac{3}{4}D_\infty(z)\right]^{1/3} \tag{4.44}$$

and

$$\zeta_1(z) = \frac{D_1(z) - \zeta_\infty(z)}{4\zeta_\infty^2(z)}, \tag{4.45}$$

respectively. By a straightforward calculation one can see that both $\zeta_\infty(z)$ and $\zeta_1(z)$ are analytic at $z = 0$. In fact, the first terms of the relevant Taylor series are

$$\zeta_\infty(z) = C^{-1}z - \frac{1}{10Cz_0^2}z^3 + \cdots, \quad D_1(z) = C^{-1}z - \frac{1}{6Cz_0^2}z^3 + \cdots, \tag{4.46}$$

and

$$\zeta_1(z) = \frac{1}{60c_0^2}z + \cdots \tag{4.47}$$

(note the absence of the singularity of $\zeta_1(z)$ at $z = 0$). Therefore, the functions $\zeta_\infty(z)$ and $\zeta_1(z)$ are analytic in the full rectangle $(-\Omega_1) \cup \Omega^0 \cup \Omega_1$. This, by virtue of the maximum principle, allows us to extend the asymptotic formulae (4.43) to the full rectangle,

$$\zeta_0(z) = \zeta_\infty(z) + N^{-2/3}y\zeta_1(z) + O(N^{-4/3}), \quad z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1. \tag{4.48}$$

Initial value problem (4.33), which we have used to define the function $\zeta_0(z)$, was obtained by neglecting terms of the order of N^{-1} in basic equation (4.9). Therefore, in view of uniform asymptotics (4.46), it makes perfect sense to redefine $\zeta_0(z)$ as

$$\zeta_0(z) \equiv \zeta_\infty(z) + N^{-2/3} y \zeta_1(z)\,, \quad y = c_0^{-1} N^{2/3}\left(\frac{n}{N} - \lambda_c\right) \tag{4.49}$$

(note that we dropped the parameter α in the definition of y). With this new definition we preserve all three basic properties of the function $\zeta_0(z)$, which we will use in the next step and later in matching the critical point and WKB asymptotics (see Section 5 and Appendix A). These properties are

(i) analyticity and $\zeta_0'(z) \neq 0$ in the rectangle $(-\Omega_1) \cup \Omega^0 \cup \Omega_1$,
(ii) estimate (4.37),
(iii) estimate (4.22) for the function $\sigma(s) \equiv N^{1/3}\zeta_0(CN^{-1/3}s)$, and
(iv) symmetry relation (4.35).

Our next step is to construct the gauge matrix $V(z)$ in the zeroth-order approximation.

4.3 Gauge Matrix in the Zeroth-Order Approximation

In the zeroth-order approximation, equation (4.8) reduces to

$$V(z)\big[\zeta_0'(z) N^{-2/3} A(N^{1/3}\zeta_0(z))\big] V^{-1}(z) = A_n^0(z)\,, \tag{4.50}$$

which can be viewed as a generalized eigenvector problem, because by step 1 the matrices

$$\zeta_0'(z) N^{-2/3} A(N^{1/3}\zeta_0(z))$$

and $A_n^0(z)$ have the same (up to terms of the order of $N^{-4/3}$) eigenvalues. We will be looking for an approximate solution $V(z)$ to (4.50), with a possible error term in the equation of the order of $O(N^{-1})$, and we will replace the matrix elements of $A_n^0(z)$ by their suitable approximations (cf. (4.4)). We put

$$\begin{aligned} a_{11}^0(z) = -a_{22}^0(z) = &-\frac{gz^3}{2} \\ &- \big[c_1 g(-1)^{n+1} N^{-1/3} u(y) + c_2 g N^{-2/3} v(y)\big] z\,, \\ a_{12}^0(z) = &\big(R_n^0\big)^{1/2}\big(gz^2 + N^{-2/3}c_5\big)\,, \\ a_{21}^0(z) = &-\big(R_n^0\big)^{1/2}\big(gz^2 + N^{-2/3}c_7\big)\,. \end{aligned} \tag{4.51}$$

LEMMA 4.1 *Let $B = (b_{ij})$ and $D = (d_{ij})$ be two 2×2 matrices such that*

$$\operatorname{tr} B = \operatorname{tr} D = 0\,, \quad \det B = \det D\,.$$

Then the equation $VB = DV$ *has the following two explicit solutions*:

$$V_1 = \begin{pmatrix} d_{12} & 0 \\ b_{11} - d_{11} & b_{12} \end{pmatrix}, \quad V_2 = \begin{pmatrix} b_{21} & d_{11} - b_{11} \\ 0 & d_{21} \end{pmatrix}. \tag{4.52}$$

The proof of Lemma 4.1 is given in Appendix B.

We apply Lemma 4.1 to solve equation (4.50) with

$$B = \zeta_0'(z) N^{-2/3} A(N^{1/3}\zeta_0(z)), \quad D = A_n^0(z).$$

The problem is that in (4.50) we need an analytic matrix-valued function $V(z)$ that is invertible in some fixed neighborhood of the origin. Neither V_1 nor V_2 in (4.52) are invertible. Nevertheless, we will find a linear combination of V_1 and V_2 (plus some negligibly small terms) that is analytic and invertible.

Let us rewrite (4.50) in the local coordinates (s, σ) defined in (4.17):

$$\begin{aligned} V_0(s)\big[\sigma'(s) A(\sigma(s))\big] V_0^{-1}(s) &= C N^{2/3} A_n^0(C N^{-1/3} s), \\ V_0(s) &= V(C N^{-2/3} s). \end{aligned} \tag{4.53}$$

Define

$$\begin{aligned} B_0(s) &= \sigma'(s) A(\sigma(s)) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \\ b_{11} &= -b_{22} = (-1)^n 4u\sigma'(s)\sigma(s), \\ b_{12} &= \sigma'(s)(4\sigma^2(s) + (-1)^n 2w + v), \\ b_{21} &= \sigma'(s)(-4\sigma^2(s) + (-1)^n 2w - v), \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} D_0(s) &= C N^{2/3} A_n^0(C N^{-1/3} s) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \\ d_{11} &= -d_{22} = (-1)^n 4us - N^{-1/3} c_0^{-1}(vs + 4s^3), \\ d_{12} &= \left(\frac{R_n^0 2g}{|t|}\right)^{1/2} (4s^2 + (-1)^n 2w + v), \\ d_{21} &= \left(\frac{R_n^0 2g}{|t|}\right)^{1/2} (-4s^2 + (-1)^n 2w - v). \end{aligned} \tag{4.55}$$

To solve (within the error $O(N^{-2/3})$) the equation

$$V_0(s) B_0(s) V_0^{-1}(s) = D_0(s), \tag{4.56}$$

we can use any linear combination of V_1 and V_2 in (4.52). To determine an appropriate linear combination, consider the matrix element $b_{11} - d_{11}$ that appears in V_1

and with minus sign in V_2. By (4.54) and (4.55),

$$b_{11} - d_{11} = (-1)^n 4u\sigma'(s)\sigma(s) - (-1)^n 4us + N^{-1/3} c_0^{-1} s(4s^2 + v)\,. \tag{4.57}$$

By (4.22), $\sigma(s) = s + N^{-2/3}\sigma_1(s) + \cdots$; hence

$$b_{11} - d_{11} = N^{-1/3} c_0^{-1} s(4s^2 + v) + O(N^{-2/3})\,. \tag{4.58}$$

Thus the function $b_{11}(s) - d_{11}(s)$ has three zeros: $s = 0$ and two other zeros that are determined, in the zeroth-order approximation in $N^{-1/3}$, by the quadratic equation

$$4s^2 + v = 0\,. \tag{4.59}$$

(The function $b_{11}(s) - d_{11}(s)$ may have more zeros at the distance of the order of $N^{1/3}$ from the origin, but we are not interested in those now.) Let us take

$$\begin{aligned} V_0(s) &= \frac{1}{\sqrt{\det W_0(s)}} W_0(s)\,, \\ W_0(s) &= \begin{pmatrix} d_{12}(s) - b_{21}(s) & b_{11}(s) - d_{11}(s) \\ b_{11}(s) - d_{11}(s) & b_{12}(s) - d_{21}(s) \end{pmatrix}, \end{aligned} \tag{4.60}$$

which is a linear combination of V_1 and V_2 in (4.52). We want $V_0(s)$ to be an analytic, invertible, matrix-valued function. To that end we will slightly correct $W_0(s)$. Consider the matrix elements $w_{ij}(s)$ of $W_0(s)$. From formulae (4.54), (4.55), and (1.66),

$$\begin{aligned} w_{11}(s) &= d_{12}(s) - b_{21}(s) \\ &= 8s^2 + 2v \\ &\quad + N^{-1/3} c_0^{-1} (-1)^{n+1} u(4s^2 + (-1)^n 2w + v) + O(N^{-2/3})\,, \\ w_{12}(s) &= w_{21}(s) = b_{11}(s) - d_{11}(s) \\ &\qquad\qquad = N^{-1/3} c_0^{-1} s(4s^2 + v) + O(N^{-2/3})\,, \\ w_{22}(s) &= b_{12}(s) - d_{21}(s) \\ &= 8s^2 + 2v \\ &\quad + N^{-1/3} c_0^{-1} (-1)^{n+1} u(4s^2 - (-1)^n 2w + v) + O(N^{-2/3})\,. \end{aligned} \tag{4.61}$$

Observe that in the leading order of approximation in $N^{-1/3}$, the zeros of $w_{ij}(s)$ are determined by the same quadratic equation (4.59).

Let us define now

$$\begin{aligned} &W_0(s) = \\ &\begin{pmatrix} d_{12}(s) - b_{21}(s) - N^{-1/3}\alpha_{11} & b_{11}(s) - d_{11}(s) - N^{-2/3}\alpha_{12} s \\ b_{11}(s) - d_{11}(s) - N^{-2/3}\alpha_{21} s & b_{12}(s) - d_{21}(s) - N^{-1/3}\alpha_{22} \end{pmatrix}, \end{aligned} \tag{4.62}$$

where the numbers α_{11}, $\alpha_{12} = \alpha_{21}$, and α_{22} do not depend on s and are chosen in such a way that all the elements in the matrix on the right vanishes at the zeros of $4s^2 + v$. Then the matrix-valued function

$$\widetilde{W}_0(s) = \frac{W_0(s)}{8s^2 + 2v}$$

is analytic in s and for finite s,

(4.63) $$\widetilde{W}_0(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N^{-1/3} \begin{pmatrix} (2c_0)^{-1}(-1)^{n+1}u & (2c_0)^{-1}s \\ (2c_0)^{-1}s & (2c_0)^{-1}(-1)^{n+1}u \end{pmatrix} + O(N^{-2/3})\,.$$

This implies that

$$V_0(s) = \frac{1}{\sqrt{\det W_0(s)}}\, W_0(s) = \frac{1}{\sqrt{\det \widetilde{W}_0(s)}}\, \widetilde{W}_0(s)$$

is analytic in s as well and for finite s,

(4.64) $$V_0(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N^{-1/3} \begin{pmatrix} 0 & (2c_0)^{-1}s \\ (2c_0)^{-1}s & 0 \end{pmatrix} + O(N^{-2/3})\,.$$

If we go back to the global coordinates z and ζ_0, we obtain that

(4.65) $$\begin{aligned} V(z) &= V_0(C^{-1}N^{1/3}z) = \frac{1}{\sqrt{\det W(z)}}\, W(z)\,, \\ W(z) &= C^{-1}N^{-2/3}W_0(C^{-1}N^{1/3}z)\,. \end{aligned}$$

To ensure that $V(z)$ is analytic in the region $(-\Omega_1) \cup \Omega^0 \cup \Omega_1$, we will prove the following lemma.

LEMMA 4.2 *Define*

$$\widetilde{W}(z) = C^{-1}N^{-2/3}\widetilde{W}_0(C^{-1}N^{1/3}z) = \frac{W(z)}{8(C^{-1}N^{1/3}z)^2 + 2v}\,.$$

Then there exists $d_2^0 > 0$ *such that if* $0 < d_2 \le d_2^0$*, then for sufficiently large* N *for all* $z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1$*,* $\det \widetilde{W}(z) \neq 0$*.*

The proof of Lemma 4.2 is given in Appendix E.

Observe that for $|z| \le \varepsilon_0$,

(4.66) $$\begin{aligned} W(z) &= \begin{pmatrix} -\widetilde{a}_{21}(z) + a_{12}^0(z) & \widetilde{a}_{11}(z) - a_{11}^0(z) \\ \widetilde{a}_{11}(z) - a_{11}^0(z) & \widetilde{a}_{12}(z) - a_{21}^0(z) \end{pmatrix} + O(N^{-1})\,, \\ \widetilde{a}_{ij}(z) &= \zeta_0'(z)N^{-2/3}a_{ij}(N^{1/3}\zeta_0(z))\,. \end{aligned}$$

An important feature here is that the correcting terms $N^{-1/3}\alpha_{11}$, $N^{-2/3}\alpha_{12}s$, etc., in (4.62) become $O(N^{-1})$ after multiplication by $N^{-2/3}$ and substitution $s = C^{-1}N^{1/3}z$. From (3.3),

(4.67) $$\begin{aligned} \widetilde{a}_{11}(z) &= -\widetilde{a}_{22}(z) = N^{-1/3}(-1)^n 4u\zeta_0'(z)\zeta_0(z)\,, \\ \widetilde{a}_{12}(z) &= \zeta_0'(z)\big[4\zeta_0^2(z) + N^{-2/3}((-1)^n 2w + v)\big]\,, \\ \widetilde{a}_{21}(z) &= \zeta_0'(z)\big[-4\zeta_0^2(z) + N^{-2/3}((-1)^n 2w - v)\big]\,. \end{aligned}$$

4.4 First-Order Approximation for $\zeta(z)$

We will solve equation (4.9) with an error term of the order of N^{-2}. Using notation (4.11), we write (4.9) as

$$(\zeta')^2 f(\zeta) = \det\left[A_n^0(z) - N^{-1}V'(z)V^{-1}(z)\right]. \tag{4.68}$$

As a suitable approximation to the determinant on the right, we will consider the function

$$\begin{aligned} a(z) \equiv d(z) - N^{-1}\big[&a_{11}^0(z)q_{22}(z) + a_{22}^0(z)q_{11}(z) \\ &- a_{12}^0(z)q_{21}(z) - a_{21}^0(z)q_{12}(z)\big], \end{aligned} \tag{4.69}$$

where $q_{ij}(z)$ are the matrix elements of the matrix

$$Q(z) \equiv V'(z)V^{-1}(z), \tag{4.70}$$

where $V(z)$ is defined in (4.65). We replace (4.68) by the equation

$$\zeta'\sqrt{f(\zeta)} = \sqrt{a(z)}. \tag{4.71}$$

In the local coordinates (s, σ), it is written as

$$\sigma'\sqrt{f_0(\sigma)} = \sqrt{a_0(s)}, \tag{4.72}$$

where

$$f_0(\sigma) \equiv N^{4/3} f(N^{-1/3}\sigma) = \det A(\sigma) = 16\sigma^4 + 8y\sigma^2 + [v^2(y) - 4w^2(y)] \tag{4.73}$$

and

$$a_0(s) = C^2N^{4/3}a(CN^{-1/3}s). \tag{4.74}$$

From (4.69),

$$a_0(s) = d_0(s) - \left[d_{11}(s)q_{22}^0(s) + d_{22}(s)q_{11}^0(s) - d_{12}(s)q_{21}^0(s) - d_{21}(s)q_{12}^0(s)\right], \tag{4.75}$$

where

$$\begin{aligned} d_0(s) &\equiv C^2N^{4/3}d(CN^{-1/3}s) \\ &= 16s^4 + 8c_0^{-1}N^{2/3}\left(\frac{n}{N} - \lambda_c\right)s^2 \\ &\quad + [v^2(y) - 4w^2(y)] - N^{-1/3}c_0^{-1}(-1)^n 2w(y) + O(N^{-2/3}) \end{aligned} \tag{4.76}$$

(use (4.5)), $d_{ij}(s)$ are the matrix elements of the matrix $D_0(s)$ defined in (4.55), and $q_{ij}^0(s)$ are the matrix elements of the matrix

$$\begin{aligned} Q_0(s) \equiv CN^{2/3}Q(CN^{-1/3}s) &= V_0'(s)V^{-1}(s) \\ &= N^{-1/3}\begin{pmatrix} 0 & (2c_0)^{-1} \\ (2c_0)^{-1} & 0 \end{pmatrix} + O(N^{-2/3}) \end{aligned}$$

(use (4.50)). From (4.75) and (4.55),

$$\begin{aligned} a_0(s) &= d_0(s) + N^{-1/3}(2c_0)^{-1}[d_{12}(s) + d_{21}(s)] + O(N^{-2/3}) \\ &= d_0(s) + N^{-1/3}c_0^{-1}(-1)^n 2w(y) + O(N^{-2/3}), \end{aligned}$$

hence by (4.76),

$$a_0(s) = 16s^4 + 8c_0^{-1}N^{2/3}\left(\frac{n}{N} - \lambda_c\right)s^2 + [v^2(y) - 4w^2(y)] + N^{-2/3}r_N(s)\,, \quad r_N(s) = O(1)\,. \tag{4.77}$$

Here the notation $r_N(s) = O(1)$ has the following meaning: For any $R > 0$ there exist $N_0 = N_0(R)$ and $C = C(R)$ such that for all $N \geq N_0$, the function $r_N(s)$ is analytic in the disk $|s| \leq R$ and $|r_N(s)| \leq C(R)$ for all s in the disk.

To get an analytic solution to (4.72) we have to secure the equality of periods of $f_0(s)$ and $a_0(s)$. The function $f_0(s)$ has four zeros $\pm s_j$, $j = 1, 2$ (see Proposition 3.1). The function $a_0(s)$ is an $O(N^{-2/3})$ perturbation of $f_0(s)$, and $a_0(s)$ also has four zeros $\pm t_j$, $j = 1, 2$, such that $|t_j - s_j| \to 0$ as $N \to \infty$.

We have to secure the equality of the periods,

$$\int_0^{s_j} \sqrt{f_0(s)}\, ds = \int_0^{t_j} \sqrt{a_0(s)}\, ds\,, \quad j = 1, 2. \tag{4.78}$$

To that end we need two parameters. As before, we can use the parameter α in (4.26) that slightly changes the relation between n and y. Where can we take the second parameter? The idea is to change $a_0(s)$ by a constant $\beta N^{-2/3}$, putting

$$a_0(s) = 16s^4 + 8(y - \alpha N^{-2/3})s^2 + [v^2(y) - 4w^2(y)] + N^{-2/3}r_N(s) + \beta N^{-2/3}\,. \tag{4.79}$$

Observe that by (4.74),

$$a(z) = C^{-2}N^{-4/3}a_0(C^{-1}N^{1/3}z)\,;$$

therefore, when we go back to the global coordinates, the function $a(z)$ will change by βN^{-2}, which is of the order of the error term. We find the values of the parameters α and β from condition (4.78). Let us analyze this condition more carefully.

Assume that $y \neq y_0$, so that $s_2 \neq 0$. Introduce the auxiliary functions

$$I_j(x_1, x_2, x_3) = \int_0^{t_j} \sqrt{16s^4 + 8ys^2 + [v^2(y) - 4w^2(y)] + x_1 s^2 + x_2 + x_3 r_N(s)}\, ds\,, \tag{4.80}$$

where t_j is the corresponding zero of the function under the radical, $j = 1, 2$. Then (4.78) reads

$$I_j(-8\alpha N^{-2/3}, \beta N^{-2/3}, N^{-2/3}) = I_j(0, 0, 0)\,, \quad j = 1, 2. \tag{4.81}$$

Observe that the Jacobian

$$J \equiv \det \begin{pmatrix} \frac{\partial I_1}{\partial x_1} & \frac{\partial I_1}{\partial x_2} \\ \frac{\partial I_2}{\partial x_1} & \frac{\partial I_2}{\partial x_2} \end{pmatrix} \Bigg|_{x_1 = x_2 = x_3 = 0}$$

is not zero. Otherwise, this would mean that there exists a number c such that the periods of the elliptic integral of the second kind,

$$I(s) = \int_0^s \frac{u^2 + c}{\sqrt{16u^4 + 8yu^2 + [v^2(y) - 4w^2(y)]}}\, du\,,$$

on the genus 1 algebraic curve $w = 16s^4 + 8ys^2 + [v^2(y) - 4w^2(y)]$, are all zero. This in turn would mean that the sum

$$\frac{s}{4} + I(s)$$

makes a meromorphic function on the curve (w, s) having a simple pole at exactly one point, namely, at one of the two points lying over $s = \infty$. This is impossible since none of these points is a Weierstrass point of the elliptic curve (w, s) (see, e.g., [26]). (In fact, using Riemann's bilinear relations for the periods of abelian integrals, one can show that $J = \frac{\pi i}{128}$.) Therefore, the implicit function theorem gives the solvability of equations (4.78) for α and β.

If $y = y_0$, so that $s_2 = 0$ and the curve (w, s) degenerates, the parameter β should be taken equal to $-r_N(0)$ to ensure that $t_2 = 0$ as well. The parameter α is determined from equation (4.78), $j = 1$, which is the only period equation left. Its solvability follows from the obvious inequality (we recall that $y_0 > 0$),

$$\int_0^{s_1} \frac{s^2}{\sqrt{16s^4 + 8y_0 s^2}}\, du \neq 0\,, \quad s_1 = i\sqrt{\frac{y_0}{2}}\,.$$

(One can also argue that since the Jacobian $J = \pi i/128$ and hence does not depends on y, the solvability for $y \neq y_0$ implies as well the solvability for $y = y_0$.)

Let us summarize our calculations. The function $\zeta(z)$ is the unique analytic solution of the initial value problem

$$\begin{aligned}
&\zeta'\sqrt{f(\zeta)} = \sqrt{a(z)}\,, \quad \zeta(0) = 0\,,\\
&f(\zeta) = 16\zeta^4(z) + 8N^{-2/3}y\zeta^2(z) + N^{-4/3}[v^2(y) - 4w^2(y)]\,,\\
&a(z) = d(z) - N^{-1}\big[a_{11}^0(z)q_{22}(z) + a_{22}^0(z)q_{11}(z)\\
&\qquad - a_{12}^0(z)q_{21}(z) - a_{21}^0(z)q_{12}(z)\big] - \beta N^{-2}\,,\\
&y = c_0^{-1}N^{2/3}\left(\frac{n}{N} - \lambda_c\right) + \alpha N^{-2/3}\,,
\end{aligned} \tag{4.82}$$

where $d(z)$ is defined in (4.5), the functions $a_{jk}^0(z)$ are defined in (4.51), the functions $q_{jk}(z)$ are the matrix entries of the matrix $Q(z)$ defined in (4.70), and the constants α and β are uniquely determined by the equation of periods (4.78). The branches of square roots in (4.82) are determined for large N by the condition that they are positive on the interval $[(z_0/4), (3z_0/4)]$. The domain for $\zeta(z)$ is the rectangle $(-\Omega_1) \cup \Omega^0 \cup \Omega_1$.

By exactly the same arguments as in the case of initial problem (4.33) and taking into account that the term in the brackets in (4.82) is of the order of $N^{-2/3}$ (cf. equation (C.10)), we derive from (4.82) the estimate

$$\zeta(z) = \zeta_\infty(z) + N^{-2/3} y \zeta_1(z) + O(N^{-4/3}), \quad z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1, \tag{4.83}$$

that is (cf. (4.48))

$$\zeta(z) = \zeta_0(z) + O(N^{-4/3}), \quad z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1. \tag{4.84}$$

In addition, the following symmetry relations take place:

$$\zeta(-z) = -\zeta(z) \tag{4.85}$$

and

$$V(-z) = \sigma_3 V(z) \sigma_3. \tag{4.86}$$

Equation (4.86) follows from (4.35) and the explicit formulae (4.62) and (4.65) defining $V(z)$. The uniqueness of a solution to initial value problem (4.82) implies equation (4.85).

Remark. Unlike the zeroth-order change-of-variable function $\zeta_0(z)$, the first-order function $\zeta(z)$ cannot be replaced by the first two terms in the right-hand side of (4.83). As we will see in Section 5 (Proposition 5.2 and Theorem 5.4), in order to match the critical point solution and the WKB solution within an $O(N^{-1})$ error, the approximation given by (4.83) is not enough. However, it can be used for a more explicit but less accurate approximation to the solution $\Psi_n(z)$ of Riemann-Hilbert problem (1.35)–(1.38). We discuss this matter in more detail in Appendix F below.

5 Matching CP and WKB Solutions

5.1 WKB Solution

The WKB solution is an approximate solution to (4.1) in Ω^c, and it is defined as (cf. [7])

$$\Psi_{\mathrm{WKB}}(z) = \tilde{C}_0 T(z) e^{-[N \int_\infty^z \mu(u)du\, \sigma_3 + \int_\infty^z \operatorname{diag} T^{-1}(u) T'(u) du + C_1 \sigma_3]} \tag{5.1}$$

where $\tilde{C}_0 \neq 0$ and C_1 are some constants (parameters of the solution),

$$\mu(z) = [-d(z)]^{1/2}, \quad T(z) = \begin{pmatrix} 1 & \frac{a_{12}^0(z)}{\mu - a_{11}^0(z)} \\ -\frac{a_{21}^0(z)}{\mu - a_{11}^0(z)} & 1 \end{pmatrix}, \tag{5.2}$$

and diag A means the diagonal part of the matrix A. Recall that $d(z)$ is a suitable approximation of det $A_n^0(z)$ (see (4.5)), and it is a polynomial of the sixth degree in z. Observe that both the function $\mu_0(z)$ in (4.36) and $\mu(z)$ in (5.2) are defined as $[-d(z)]^{1/2}$. The difference is in their domains: For $\mu_0(z)$ the domain is $((-\Omega_1) \cup \Omega^0 \cup \Omega_1) \setminus \{|z| \le \omega_0\}$, while for $\mu(z)$ it is Ω^c. The function $\mu_0(z)$ is an analytic continuation of $\mu(z)$ along the contour that goes by the positive half-axis from ∞ to z_0+d_1 and then around z_0 from above. For z lying in $[((-\Omega_1)\cup\Omega^0\cup\Omega_1)\setminus\{|z|\le \omega_0\}]\cap\Omega^c$,

$$\mu(z)=\begin{cases}\mu_0(z)\,, & \operatorname{Im} z>0\,,\\ -\mu_0(z)\,, & \operatorname{Im} z<0\,.\end{cases} \tag{5.2$'$}$$

The integral $\int_\infty^z \mu(u)du$ in (5.1) diverges at infinity and its regularization is defined as follows. We obtain from (4.5) that

$$\mu(z)=\mu_3 z^3+\mu_1 z+\mu_{-1}z^{-1}+\widetilde{\mu}(z),\quad \widetilde{\mu}(z)=\sum_{j=0}^{\infty}\mu_{-3-2j}z^{-3-2j}\,,$$
$$\mu_3=\frac{g}{2}\,,\quad \mu_1=\frac{t}{2}\,,\quad \mu_{-1}=-\frac{n}{N}\,,$$

and we define

$$\int_\infty^z \mu(u)du\equiv\frac{\mu_3}{4}z^4+\frac{\mu_1}{2}z^2+\mu_{-1}\ln z+\int_\infty^z\widetilde{\mu}(u)du\,. \tag{5.3}$$

Because of the logarithmic term, this defines $\int_\infty^z \mu(u)du$ as a multivalued function on the complex plane. Observe that the coefficient μ_{-1} at the logarithmic term in (5.3) is $-n/N$; hence the function $e^{-N\int_\infty^z\mu(u)du}$ is one-valued if n is an integer.

The integral $\int_\infty^z \operatorname{diag} T^{-1}(u)T'(u)du$ in (5.1) converges at infinity, because by (4.4) and (4.5), as $z\to\infty$,

$$\frac{a_{12}^0(z)}{\mu(z)-a_{11}^0(z)}=\frac{(R_n^0)^{1/2}}{z}+O(z^{-3})\,,\qquad \frac{a_{21}^0(z)}{\mu(z)-a_{11}^0(z)}=-\frac{(R_n^0)^{1/2}}{z}+O(z^{-3})\,.$$

Formula (5.1) can be brought into the following form:

$$\Psi_{\mathrm{WKB}}(z)=C_0T_0(z)e^{-[N\xi(z)+\tau(z)+C_1]\sigma_3}\,, \tag{5.4}$$

where $C_0=\tilde{C}_0/\sqrt{2}$ and

$$\begin{gathered}\xi(z)=\int_\infty^z\mu(u)du,\quad \tau(z)=\int_\infty^z\frac{a_{12}^0(u)a_{21}^{0\,\prime}(u)-a_{12}^{0\,\prime}(u)a_{21}^0(u)}{4\mu(u)[\mu(u)-a_{11}^0(u)]}\,du\,,\\ T_0(z)=\left(\frac{\mu(z)-a_{11}^0(z)}{\mu(z)}\right)^{1/2}T(z)\,,\quad \det T_0(z)\equiv 2\,.\end{gathered} \tag{5.5}$$

The square root branch in (5.5) is determined by the condition that for large $z>0$,

$$\left(\frac{\mu(z)-a_{11}^0(z)}{\mu(z)}\right)^{1/2}>0\,.$$

The integral for $\tau(z)$ in (5.5) converges at infinity, so no regularization is needed. The reduction of (5.1) to (5.4) is based on the formula

$$\operatorname{diag} T^{-1}(z)T'(z)=\tau'(z)\sigma_3-\frac{1}{2}\left[\ln\left(\frac{\mu(z)-a_{11}^0(z)}{\mu(z)}\right)\right]'I\,, \tag{5.5$'$}$$

which follows from (5.2) by a straightforward calculation (cf. [7, app. A]). The numbers $C_0 \neq 0$ and C_1 in (5.4) are free constants that will be chosen later.

PROPOSITION 5.1 *There exists $N_0 = N_0(d_1, d_2)$ such that for all $N \geq N_0$,* (5.4) *and* (5.5) *define $\Psi_{\mathrm{WKB}}(z)$ as an analytic function in Ω^c. As $z \to \infty$,*

$$\Psi_{\mathrm{WKB}}(z) = \sqrt{2}\, C_0 \left(I + z^{-1} \left(R_n^0\right)^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O(|z|^{-2}) \right) e^{-[N(V(z)/2) - n \ln z + C_1]\sigma_3} . \tag{5.6}$$

In addition, the symmetry relation

$$\Psi_{\mathrm{WKB}}(-z) = (-1)^n \sigma_3 \Psi_{\mathrm{WKB}}(z) \sigma_3 \tag{5.7}$$

holds.

PROOF: From (5.3) and (5.4),

$$\Psi_{\mathrm{WKB}}(z) = C_0 T_0(z) e^{-[N(V(z)/2) - n \ln z + N\tilde{\xi}(z) + \tau(z) + C_1]\sigma_3} , \tag{5.4$'$}$$

where

$$\tilde{\xi}(z) = \int_\infty^z \tilde{\mu}(u) du .$$

Let us prove that for large N,

$$\mu(z) - a_{11}^0(z) \neq 0 , \quad z \in \Omega^c . \tag{5.8}$$

Indeed, by (4.4) and (4.5),

$$\begin{aligned} \mu^2(z) - \left(a_{11}^0(z)\right)^2 &= -d(z) - (a_{11}^0(z))^2 \\ &= \frac{g^2 z^4 (z^2 - z_0^2)}{4} - \frac{g^2 z^6}{4} + O(N^{-1/3}(1 + |z|)^4) \\ &= -\frac{g^2 z^4 z_0^2}{4} + O(N^{-1/3}(1 + |z|)^4) \neq 0 , \quad z \in \Omega^c ; \end{aligned}$$

hence (5.8) holds. In addition, as $z \to \infty$,

$$\frac{\mu(z) - a_{11}^0(z)}{\mu(z)} = 2 + O(|z|^{-2}) . \tag{5.9}$$

Therefore, the function $(\frac{\mu(z) - a_{11}^0(z)}{\mu(z)})^{1/2}$ is analytic in Ω^c. Since $\tilde{\mu}(z)$ is analytic in Ω^c and $\tilde{\mu}(z) = O(z^{-3})$ as $z \to \infty$, it follows that $\tilde{\xi}(z)$ is analytic in Ω^c. Similar arguments prove the analyticity of $\tau(z)$ and hence the analyticity of $\Psi_{\mathrm{WKB}}(z)$. Asymptotics (5.6) follows from (5.4$'$) and (5.9). The equations

$$\mu(-z) = -\mu(z), \quad a_{11}^0(-z) = -a_{11}^0 , \quad a_{12}^0(-z) = a_{12}^0(z) , \quad a_{21}^0(-z) = a_{21}^0(z) ,$$

and

$$\xi(-z) = \pm \frac{n}{N} \pi i + \xi(z) , \quad \tau(-z) = \tau(z) ,$$

imply symmetry (5.7). Proposition 5.1 is proven. $\square$

5.2 Critical Point Solution

We define the critical point solution in the region $\Omega_c \equiv (-\Omega_1) \cup \Omega^0 \cup \Omega_1$ as

$$\Psi_{\mathrm{CP}}(z) = \begin{cases} \tilde{C} V(z) \Phi^u(N^{1/3}\zeta(z))\,, & \operatorname{Im} z \geq 0\,, \\ \tilde{C} V(z) \Phi^d(N^{1/3}\zeta(z))\,, & \operatorname{Im} z \leq 0\,, \end{cases} \tag{5.10}$$

where $\tilde{C}$ is a constant, a parameter of the solution. The functions $\zeta(z)$ and $V(z)$ are defined by (4.68) and (4.51), respectively, and the model solutions $\Phi^{u,d}(z)$ are defined and described in Section 3 (see (3.21)). It is important to notice that equations (4.85), (4.86), and (3.22) yield

$$\Psi_{\mathrm{CP}}(-z) = (-1)^n \sigma_3 \Psi_{\mathrm{CP}}(z) \sigma_3\,, \tag{5.11}$$

i.e., the same symmetry relation as for the function $\Psi_{\mathrm{WKB}}(z)$.

5.3 Matching CP and WKB Solutions

Let $\Gamma_c^{\pm}$ be the horizontal sides of the rectangle Ω_c,

$$\Gamma_c^{\pm} = \{z \in \Omega_c : \operatorname{Im} z = \pm d_2\}\,.$$

Our goal is to show that we can choose constants $C_0 \neq 0$ and C_1 in (5.1) and $\tilde{C} \neq 0$ in (5.10) such that the CP solution $\Psi_{\mathrm{CP}}(z)$ coincides, up to terms of the order of N^{-1}, with the WKB solution $\Psi_{\mathrm{WKB}}(z)$ on $\Gamma_c^{\pm}$. Because of equations (5.7) and (5.11), it is enough to consider Γ_c^{+}. Replacing in (5.10) the model function $\Phi^u(z)$ by its asymptotics (3.23), we obtain that for $z \in \Gamma_c^{+}$,

$$\Psi_{\mathrm{CP}}(z) = \tilde{C} V(z) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} Y_0(z) e^{-i(N(4/3)\zeta^3(z) + N^{1/3} y \zeta(z) - i\gamma)\sigma_3}\,, \tag{5.12}$$

where

$$Y_0(z) = I + \frac{N^{-1/3} m_1}{\zeta(z)} + \frac{N^{-2/3} m_2}{\zeta^2(z)} + O(N^{-1})\,. \tag{5.13}$$

To transform $\Psi_{\mathrm{CP}}(z)$ to the WKB solution, we use the following proposition. Denote, as before, by z_0^N the zero of the potential $U^0(z)$ (see (1.68)) that approaches z_0 as $N \to \infty$.

Proposition 5.2 *For $z \in \Gamma_c^{+}$,*

$$\begin{aligned} i\left(N\left(\frac{4}{3}\right)\zeta^3(z) + N^{1/3} y \zeta(z) + N^{-1/3} \frac{D}{2} \zeta^{-1}(z)\right) = \\ N \int_{\infty}^{z} \mu(u)\,du + \tau(z) + C^0 + \frac{i\pi n}{2} + N^{-2/3} \Delta(z) + O(N^{-1})\,, \end{aligned} \tag{5.14}$$

where $D = D(y)$ is given in (3.6)*,*

$$C^0 = N \int_{z_0^N}^{\infty} \mu^c(u)\,du - \frac{1}{4} \ln R_n^0\,, \tag{5.15}$$

$\mu^c(z) = \sqrt{U^0(z)}$ *(see* (1.68)*), and*

$$\Delta(z) \equiv \frac{c^0}{ia_{11}^0(z) - a_{21}^0(z) - i\mu(z)} - 2i(-1)^n w\zeta'(z)\frac{a_{11}^0(z)}{b(z)}\,,$$
(5.15′)
$$c^0 = \left(\frac{2|t|}{g}\right)^{1/6}(-1)^n w\,, \quad b(z) \equiv -2\mu^2(z) - i\mu(z)\big(a_{12}^0(z) - a_{21}^0(z)\big)\,.$$

In addition,

$$V(z)\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = T_0(z)V_0(z) \tag{5.16}$$

where

$$V_0(z) = I + \frac{N^{-1/3}n_1}{\zeta(z)} + \frac{N^{-2/3}n_2(z)}{\zeta^2(z)} + O(N^{-1})\,,$$
(5.16′)
$$n_1 = (-1)^n\frac{u}{2}\sigma_1, \quad n_2(z) = \frac{u^2}{8}I + (-1)^{n+1}\frac{w}{4}\sigma_2 + \Delta(z)\zeta^2(z)\sigma_3\,.$$

Proof of Proposition 5.2 is given in Appendix C. Applying this proposition to (5.12), we obtain that

$$\Psi_{\mathrm{CP}}(z) = \tilde{C}T_0(z)Y_1(z)e^{-[N\xi(z)+\tau(z)+C^0]\sigma_3}\,, \tag{5.17}$$

where

$$Y_1(z) = V_0(z)Y_0(z)e^{(N^{-1/3}\frac{iD}{2\zeta(z)} - N^{-2/3}\Delta(z) + O(N^{-1}))\sigma_3}\,. \tag{5.18}$$

PROPOSITION 5.3 *$Y_1(z) = I + O(N^{-1})$ for $z \in \Gamma_c^+$.*

PROOF: From (5.18), (5.16′), and (5.13) we obtain that

$$\begin{aligned} Y_1(z) &= \left[I + \frac{N^{-1/3}n_1}{\zeta(z)} + \frac{N^{-2/3}n_2}{\zeta^2(z)}\right]\left[I + \frac{N^{-1/3}m_1}{\zeta(z)} + \frac{N^{-2/3}m_2}{\zeta^2(z)}\right] \\ &\quad \times \left[I + N^{-1/3}\frac{iD}{2\zeta(z)}\sigma_3 - N^{-2/3}\left(\frac{D^2}{8\zeta^2(z)}I + \Delta(z)\sigma_3\right)\right] + O(N^{-1}) \\ &= I + \frac{N^{-1/3}Y_1^{(1)}}{\zeta(z)} + \frac{N^{-2/3}Y_1^{(2)}(z)}{\zeta^2(z)} + O(N^{-1})\,, \end{aligned}$$

where

$$\begin{aligned} Y_1^{(1)} &= n_1 + m_1 + \frac{iD}{2}\sigma_3\,, \\ Y_1^{(2)}(z) &= n_2(z) + m_2(z) + n_1m_1 - \frac{D^2}{8}I - \Delta(z)\zeta^2(z)\sigma_3 + \frac{(n_1+m_1)iD}{2}\sigma_3\,. \end{aligned}$$

By formulae (3.24) and (5.16′),

$$n_1 = \frac{(-1)^n u}{2}\sigma_1\,, \quad m_1 = -\frac{iD}{2}\sigma_3 - \frac{(-1)^n u}{2}\sigma_1\,, \quad D = w^2 - u^4 - yu^2\,,$$

$$n_2(z) = \frac{u^2}{8}I - (-1)^n \frac{w}{4}\sigma_2 + \Delta(z)\zeta^2(z)\sigma_3\,,$$

$$m_2 = \frac{u^2 - D^2}{8}I + (-1)^n \frac{w + uD}{4}\sigma_2\,,$$

$$n_1 m_1 = -\frac{(-1)^n uD}{4}\sigma_2 - \frac{u^2}{4}I\,, \quad \frac{(n_1 + m_1)iD}{2}\sigma_3 = \frac{D^2}{4}I\,.$$

This gives $Y_1^{(1)} = 0$ and $Y_1^{(2)}(z) = 0$. Proposition 5.3 is proven. □

Now we can formulate the main result of this section about the match of $\Psi^u_{\mathrm{CP}}(z)$ with $\Psi_{\mathrm{WKB}}(z)$ on Γ_c^+.

THEOREM 5.4 *If we take*

$$C_0 = \tilde{C}\,, \quad C_1 = N\int_{z_0^N}^{\infty} \sqrt{U^0(u)}\,du - \frac{1}{4}\ln R_n^0\,, \tag{5.19}$$

where the potential function $U^0(z)$ *is defined in* (1.68)*, then*

$$\Psi_{\mathrm{CP}}(z) = (I + O(N^{-1}))\Psi_{\mathrm{WKB}}(z) \tag{5.20}$$

uniformly with respect to $z \in \Gamma_c^+$.

PROOF: From (5.17) and Proposition 5.3,

$$\Psi_{\mathrm{CP}}(z) = (I + O(N^{-1}))\tilde{C}T_0(z)Y_1(z)e^{-[N\xi(z)+\tau(z)+C^0]\sigma_3}\,. \tag{5.21}$$

Therefore, if we take the constants C_0 and C_1 in formula (5.4) as in (5.19), then equation (5.20) follows. Theorem 5.4 is proven. □

Remark. Since the parameters d_1 and d_2 of the rectangle Ω can vary, estimate (5.20) is valid uniformly in an ε-neighborhood of Γ_c^+.

6 Matching TP and WKB Solutions

6.1 Turning Point Solution

Let $\Omega_0 = \Omega_1 \cup \Omega^1$ (see Figure 1.2), and let $\Omega_0^{u,d}$ be the upper and lower halves of Ω_0,

$$\Omega_0^{u,d} = \{z \in \Omega_0 \pm \operatorname{Im} z \geq 0\}\,.$$

The turning point solution $\Psi_{\mathrm{TP}}(z)$ is defined in Ω_0 as

$$\Psi_{\mathrm{TP}}(z) = \begin{cases} \tilde{C}_1 W(z) Y_u(w(z))\,, & z \in \Omega_0^u\,, \\ \tilde{C}_1 W(z) Y_d(w(z))\,, & z \in \Omega_0^d\,. \end{cases} \tag{6.1}$$

The constant $\tilde{C}_1 \neq 0$ is a parameter of the solution. The function of change of variable $w(z)$ and the gauge matrix $W(z)$ have been defined by formulae (1.78) and (1.79). The matrix-valued functions $Y_{u,d}(z)$ are model solutions defined in terms of the Airy function

$$Y_{u,d}(z) = \begin{pmatrix} N^{1/6} & 0 \\ 0 & N^{-1/6} \end{pmatrix} \begin{pmatrix} y_0(N^{2/3}z) & y_{1,2}(N^{2/3}z) \\ y_0'(N^{2/3}z) & y_{1,2}'(N^{2/3}z) \end{pmatrix}, \tag{6.2}$$

where

$$\begin{gathered} y_0(z) = \mathrm{Ai}(z)\,, \quad y_1(z) = e^{-\pi i/6}\,\mathrm{Ai}(e^{-2\pi i/3}z)\,, \\ y_2(z) = e^{\pi i/6}\,\mathrm{Ai}(e^{2\pi i/3}z)\,. \end{gathered} \tag{6.3}$$

Remember that $\mathrm{Ai}(z)$ is a solution to the Airy equation $y'' = zy$, which has the following asymptotics as $z \to \infty$:

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}z^{1/4}} \exp\left(-\frac{2z^{3/2}}{3} + O(|z|^{-3/2})\right), \quad -\pi+\varepsilon \le \arg z \le \pi - \varepsilon\,. \tag{6.4}$$

The functions $y_j(z)$ satisfy the relation

$$y_1(z) - y_2(z) = -iy_0(z)\,. \tag{6.5}$$

Let us explain (6.1). The turning point solution can be obtained by solving the Schrödinger equation (see (1.50))

$$-\eta'' + N^2 U^0 \eta = 0 \tag{6.6}$$

near the turning point. Namely, we are looking for solutions in the form

$$\eta(z) = \frac{\tilde{C}_1 N^{1/6}}{\sqrt{w'(z)}} y_j(N^{2/3}w(z))\,, \quad j = 0, 1, 2, \tag{6.7}$$

(cf. [5]). Equation (6.6) then reduces to the following equation on $w(z)$:

$$(w'(z))^2 w(z) = U^0(z) + \frac{1}{2N^2}\{w, z\} \tag{6.8}$$

where $\{w, z\}$ is the Schwarzian

$$\{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2. \tag{6.9}$$

Dropping the Schwarzian term, we get

$$(w'(z))^2 w(z) = U^0(z)\,, \tag{6.10}$$

or taking the square root,

$$\left(\frac{2}{3} w^{3/2}(z)\right)' = \sqrt{U^0(z)}\,. \tag{6.11}$$

To secure the analyticity of $w(z)$ at $z = z_0^N$, we take a solution as

$$w(z) = \left(\frac{3}{2}\int_{z_0^N}^{z} \sqrt{U^0(u)}\,du\right)^{2/3}. \tag{6.12}$$

The function $U^0(z)$ is positive for $z > z_0^N$, and we take the positive branch for $\sqrt{U^0(u)}$ for $u > z_0^N$ and also the positive branch for the power 2/3. Formula (6.12) defines $w(z)$ as an analytic function in Ω_0 for large N. Observe that because the dropped Schwarzian term in (6.8) is of the order of N^{-2}, (6.7) and (6.12) solve (6.6) with an error $O(N^{-2})$.

To obtain the gauge matrix $W(z)$, recall that equation (6.6) was derived from the system

$$\vec{\Psi}' = NA_n^0\vec{\Psi}\,, \quad \vec{\Psi} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

by solving ψ_2 in terms of ψ_1,

$$\psi_2 = \frac{1}{a_{12}^0}\left(N^{-1}\psi_1' - a_{11}\psi_1\right),$$

and by substitution $\psi_1 = \sqrt{a_{12}^0}\,\eta$ (cf. (1.49)). For ψ_1 and ψ_2, we obtain from (6.7) the following formulae:

$$\begin{aligned}\psi_1(z) &= \tilde{C}_1 N^{1/6}\left(\frac{a_{12}^0(z)}{w'(z)}\right)^{1/2} y_j(N^{2/3}w(z))\,, \quad j = 0, 1, 2,\\ \psi_2(z) &= \tilde{C}_1\left(\frac{a_{12}^0(z)}{w'(z)}\right)^{1/2}\Bigg[N^{-1/6}\frac{w'(z)}{a_{12}^0(z)}y_j'(N^{2/3}w(z))\\ &\qquad - N^{1/6}\frac{a_{11}^0(z)}{a_{12}^0(z)}y_j(N^{2/3}w(z))\Bigg]\end{aligned} \tag{6.13}$$

(we omit the term of the order of $N^{-5/6}$ in ψ_2), or in the vector form,

$$\vec{\Psi}(z) = \tilde{C}_1 W(z)\begin{pmatrix}N^{1/6} & 0\\ 0 & N^{-1/6}\end{pmatrix}\begin{pmatrix}y_j(N^{2/3}w(z))\\ y_j'(N^{2/3}w(z))\end{pmatrix}, \tag{6.14}$$

where

$$W(z) = \left(\frac{a_{12}^0(z)}{w'(z)}\right)^{1/2}\begin{pmatrix}1 & 0\\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{w'(z)}{a_{12}^0(z)}\end{pmatrix}. \tag{6.15}$$

This gives the gauge matrix $W(z)$.

It is important to notice that the turning point solution $\Psi_{\mathrm{TP}}(z)$ has the “right” jump on the real axis:

$$\Psi_{\mathrm{TP}}^+(z) = \Psi_{\mathrm{TP}}^-(z)\begin{pmatrix}1 & -i\\ 0 & 1\end{pmatrix}, \quad z \in \Omega_0 \cap \{\operatorname{Im} z = 0\}\,, \tag{6.16}$$

which follows from equation (6.5). We want to check that if we choose appropriately the constants C_0 and C_1 in (5.1), then the turning point solution $\Psi_{\mathrm{TP}}(z)$ matches the WKB solution $\Psi_{\mathrm{WKB}}(z)$ on the boundary of Ω_0 excluding a neighborhood of the interval $[0, z_0]$.

Let Γ_0^r be the union of the three sides of the square Ω_0 excluding the one crossing $[0, z_0]$,

$$\Gamma_0^r = \{z \in \Omega_0 : |\mathrm{Im}\, z_0| = d_2 \text{ or } \mathrm{Re}\, z = z_0 + d_1\}\,. \tag{6.17}$$

LEMMA 6.1 *If we take the WKB solution with the constants*

$$C_0 = \frac{\tilde{C}_1}{2\pi^{1/2}}\,, \quad C_1 = N\int_{z_0^N}^{\infty} \sqrt{U^0(u)}\,du - \frac{1}{4}\ln R_n^0\,, \tag{6.18}$$

then

$$\Psi_{\mathrm{TP}}(z) = (I + O(N^{-1}))\Psi_{\mathrm{WKB}}(z)\,, \quad z \in \Gamma_0^r\,, \tag{6.19}$$

uniformly with respect to $z \in \Gamma_0^r$.

A proof of Lemma 6.1 is given in Appendix B. It will be based on an alternative form of the WKB solution in Ω^c.

6.2 Alternative Form of the WKB Solution

The standard WKB form for a solution of the Schrödinger equation (6.6) is

$$\eta(z) = \frac{C_0}{(\kappa'(z))^{1/2}} e^{\pm N\kappa(z)}\,. \tag{6.20}$$

Equation (6.6) then reduces to the following equation on $\kappa(z)$:

$$(\kappa'(z))^2 = U^0(z) + \frac{1}{2N^2}\{\kappa, z\}\,, \tag{6.21}$$

where $\{\kappa, z\}$ is the Schwarzian derivative. Dropping the term with the Schwarzian derivative, we obtain

$$\kappa'(z) = (U^0(z))^{1/2}\,, \tag{6.22}$$

which gives

$$\eta(z) = \frac{C_0}{(U^0(z))^{1/4}} e^{\pm(N\int_{z_0^N}^{z}(U^0(u))^{1/2}du + \hat{C}_1)} \tag{6.23}$$

and the WKB solution in the form

$$\hat{\Psi}_{\mathrm{WKB}}(z) = C_0 T^c(z) E(N\xi^c(z))\,, \tag{6.24}$$

where the gauge matrix $T^c(z)$ is defined in (1.70),

$$\xi^c(z) = \int_{z_0^N}^{z} \mu^c(u)du + \hat{C}_1\,, \quad \mu^c(z) = (U^0(z))^{1/2}\,, \tag{6.25}$$

and the model solution $E(z)$ is

$$E(z) = \begin{pmatrix} e^{-z} & e^{z} \\ -e^{-z} & e^{z} \end{pmatrix}. \tag{6.26}$$

Lemma 6.1 is an obvious corollary of the following result, which will be proven in Appendix B.

LEMMA 6.2 *If we take C_0 and C_1 as in Lemma 6.1 and $\hat{C}_1 = 0$, then for $z \in \Omega^c$,*

$$\Psi_{\mathrm{WKB}}(z) = (1 + O(N^{-1}|z|^{-1}))\hat{\Psi}_{\mathrm{WKB}}(z)\,, \quad z \in \Omega^c\,, \tag{6.27}$$

and for $z \in \Gamma_0^r$,

$$\Psi_{\mathrm{TP}}(z) = (I + O(N^{-1}))\hat{\Psi}_{\mathrm{WKB}}(z)\,, \quad z \in \Gamma_0^r\,. \tag{6.28}$$

It is worth noticing that the derivation of (6.28) is based on WKB-type asymptotics for the Airy function. In vector form, the latter is formulated as follows: Let

$$\vec{\mathrm{Ai}}(z) = \begin{pmatrix} \mathrm{Ai}(z) \\ \mathrm{Ai}'(z) \end{pmatrix}, \quad \vec{E}(z) = \begin{pmatrix} e^{-z} \\ -e^{-z} \end{pmatrix}, \quad \vec{C}(z) = \begin{pmatrix} \cos z \\ -\sin z \end{pmatrix}. \tag{6.29}$$

PROPOSITION 6.3 *For any $\varepsilon > 0$, as $z \to \infty$,*

$$\vec{\mathrm{Ai}}(z) = (1 + O(|z|^{-3/2}))\frac{1}{2\sqrt{\pi}} B(z)\vec{E}\left(\frac{2z^{3/2}}{3}\right), \qquad -\pi + \varepsilon \le \arg z \le \pi - \varepsilon\,, \tag{6.30}$$

where

$$B(z) = \begin{pmatrix} z^{-1/4} & 0 \\ 0 & z^{1/4} \end{pmatrix}, \tag{6.31}$$

and

$$\vec{\mathrm{Ai}}(-z) = (1 + O(|z|^{-3/2}))\frac{1}{\sqrt{\pi}} B(z)\vec{C}\left(\frac{2z^{3/2}}{3} - \frac{\pi}{4}\right), \qquad -\frac{2\pi}{3} + \varepsilon \le \arg z \le \frac{2\pi}{3} - \varepsilon\,. \tag{6.32}$$

The notation $O(|z|^{-3/2})$ in (6.31) and (6.32) means a 2×2 matrix-valued function $r(z)$ such that $|r(z)| \le C_0|z|^{-3/2}$ for $|z| \ge C_1$, where $C_0, C_1 > 0$ are some constants. The branches for fractional powers are fixed by the condition that they are positive on the positive half-axis. We will not prove Proposition 6.3 because it is just a reformulation of well-known asymptotics for the Airy function.

7 Matching CP and TP Solutions

In this section we will define a WKB solution $\Psi_{\mathrm{WKB}}(z)$ in the region Ω_1, and we will show that it matches both $\Psi_{\mathrm{CP}}(z)$ and $\Psi_{\mathrm{TP}}(z)$. We begin by introducing an auxiliary CP solution. Let Ω_c^r be the part of $\Omega_c = \Omega^0 \cup \Omega_1$ between the diagonals of the first and fourth quadrants,

$$\Omega_c^r = \{z \in \Omega_c : |\mathrm{Im}\, z| \le \mathrm{Re}\, z\}\,.$$

We define the auxiliary CP solution in Ω_c^r as

$$\Psi_{\mathrm{CP}}^a(z) = \tilde{C}V(z)\big(\vec{\Phi}_2(N^{1/3}\zeta(z)), \vec{\Phi}_1(N^{1/3}\zeta(z))\big)\,, \tag{7.1}$$

where $\vec{\Phi}_1(z)$ and $\vec{\Phi}_2(z)$ are the Painlevé II ψ functions defined in Proposition 3.2. Using (5.10) and (3.21) we can relate $\Psi^a_{\mathrm{CP}}(z)$ to $\Psi_{\mathrm{CP}}(z)$ as

$$\Psi_{\mathrm{CP}}(z) = \Psi^a_{\mathrm{CP}}(z) S_{1,2}\,, \quad \pm \operatorname{Im} z \geq 0\,, \tag{7.2}$$

where

$$S_1 = \begin{pmatrix} -i & 0 \\ i & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -i & 1 \\ i & 0 \end{pmatrix}. \tag{7.3}$$

Observe that $\Psi^a_{\mathrm{CP}}(z)$ is defined in terms of $\vec{\Phi}_1(z)$ and $\vec{\Phi}_2(z)$ for which we know from (3.18) their asymptotics in the sector

$$-\frac{\pi}{3} + \varepsilon \leq \arg z \leq \frac{\pi}{3} - \varepsilon\,.$$

This allows us to find asymptotics of $\Psi^a_{\mathrm{CP}}(z)$ in Ω_1, assuming that d_2 is small enough. Namely, $\zeta'(0) \geq \varepsilon > 0$, and if d_2 is small enough, then $z \in \Omega_1$ implies that

$$-\frac{\pi}{3} + \varepsilon \leq \arg \zeta(z) \leq \frac{\pi}{3} - \varepsilon\,.$$

Using the asymptotics of $\Phi_{1,2}(z)$, we obtain the following result:

PROPOSITION 7.1 *If we take C_0 and C_1 as in* (5.19)*, then, uniformly with respect to $z \in \Omega_1$,*

$$\Psi^a_{\mathrm{CP}}(z) = (I + O(N^{-1}))\Psi^0_{\mathrm{WKB}}(z) S_0\,, \quad S_0 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.4}$$

where the function $\Psi^0_{\mathrm{WKB}}(z)$ is an analytic continuation of the function $\Psi_{\mathrm{WKB}}(z)$ in (5.4) *from the upper half-plane to Ω_1.*

Proof of Proposition 7.1 is like the proof of Theorem 5.4, and we omit it (take into account that formula (5.14) is extended to Ω_1; see Appendix A). Formula (7.4) is easy to check without any calculations if $\operatorname{Im} z \geq \varepsilon > 0$, $z \in \Omega_1$. In this case $\vec{\Phi}_1(N^{1/3}\zeta(z))$ is exponentially small, while $\vec{\Phi}_2(N^{1/3}\zeta(z))$ is exponentially big (cf. Proposition 3.2); hence in (7.1) the function

$$\vec{\Phi}_2(N^{1/3}\zeta(z)) = \vec{\Phi}_1(N^{1/3}\zeta(z)) + i\vec{\Phi}(N^{1/3}\zeta(z))$$

can be replaced by $i\vec{\Phi}(N^{1/3}\zeta(z))$ with an exponentially small error. This gives $\Psi_{\mathrm{CP}}(z)S_0$, which can be further replaced by $\Psi_{\mathrm{WKB}}(z)S_0$, with an $O(N^{-1})$ error, due to Lemma 6.1. Thus, we obtain (7.4).

We will call the function on the right in (7.4) the auxiliary WKB solution

$$\Psi^a_{\mathrm{WKB}}(z) = \Psi^0_{\mathrm{WKB}}(z) S_0\,, \tag{7.5}$$

so that

$$\Psi^a_{\mathrm{CP}}(z) = (I + O(N^{-1}))\Psi^a_{\mathrm{WKB}}(z)\,, \quad z \in \Omega_1\,.$$

With the help of $\Psi^a_{\mathrm{WKB}}(z)$, we introduce the WKB solution in Ω_1 by pattern (7.2),

$$\Psi_{\mathrm{WKB}}(z) = \Psi^a_{\mathrm{WKB}}(z) S_{1,2}\,, \quad \pm \operatorname{Im} z \geq 0\,. \tag{7.6}$$

It shares the following nice properties:

PROPOSITION 7.2 *Uniformly with respect to* $z \in \Omega_1$,

$$\Psi_{\mathrm{CP}}(z) = (1 + O(N^{-1}))\Psi_{\mathrm{WKB}}(z)\,. \tag{7.7}$$

On the real axis,

$$\Psi^{+}_{\mathrm{WKB}}(z) = \Psi^{-}_{\mathrm{WKB}}(z)\begin{pmatrix}1 & -i\\ 0 & 1\end{pmatrix}, \quad d_1 \le z \le z_0 - d_1\,. \tag{7.8}$$

Finally, on the horizontal sides of Ω_1,

$$\Psi^{+}_{\mathrm{WKB}}(z) = (1 + O(N^{-1}))\Psi^{-}_{\mathrm{WKB}}(z)\,, \quad z \in \Gamma_1^{+} \cup \Gamma_1^{-}\,, \tag{7.9}$$

where

$$\Gamma_1^{\pm} = \{z \in \Omega_1 : \operatorname{Im} z = \pm d_2\}\,.$$

PROOF: Equation (7.7) follows from Proposition 7.1, (7.8) from (7.6), and (7.9) from (7.7) and Theorem 5.4. □

We introduce next the auxiliary TP solution, similar to the auxiliary CP one (cf. (7.1)),

$$\Psi^{a}_{\mathrm{TP}}(z) = \tilde{C}_1 W(z) Y_a(w(z))\,, \quad z \in \Omega_0^{l}\,,$$

where

$$Y_a(z) = \begin{pmatrix}N^{1/6} & 0\\ 0 & N^{-1/6}\end{pmatrix}\begin{pmatrix}y_2(N^{2/3}z) & y_1(N^{2/3}z)\\ y_2'(N^{2/3}z) & y_1'(N^{2/3}z)\end{pmatrix}. \tag{7.10}$$

The function $\Psi^{a}_{\mathrm{TP}}(z)$ is defined in the domain

$$\Omega_0^{l} = \{z = \Omega_0 : |\operatorname{Im} z| \le z_0 - \operatorname{Re} z\}$$

between two 45° lines through z_0 to the left. Using (6.1), (6.2), and (6.5), we can relate $\Psi^{a}_{\mathrm{TP}}(z)$ to $\Psi_{\mathrm{TP}}(z)$ as

$$\Psi_{\mathrm{TP}}(z) = \Psi^{a}_{\mathrm{TP}}(z) S_{1,2}\,, \quad \pm \operatorname{Im} z \ge 0\,. \tag{7.11}$$

By (6.3) and (6.4), we know the asymptotics of $y_{1,2}(z)$ in the sector $|\arg z - \pi| \le 2\pi/3 - \varepsilon$, and from these asymptotics we obtain, as in Lemma 6.1, that $\Psi^{a}_{\mathrm{TP}}(z)$ matches $\Psi^{a}_{\mathrm{WKB}}(z)$ in Ω_1,

$$\Psi^{a}_{\mathrm{TP}}(z) = (I + O(N^{-1}))\Psi^{a}_{\mathrm{WKB}}(z)\,, \quad z \in \Omega_1\,.$$

This, together with (7.11), proves the following addition to Proposition 7.2:

PROPOSITION 7.3 (Addition to Proposition 7.2) *Uniformly with respect to* $z \in \Omega_1$,

$$\Psi_{\mathrm{TP}}(z) = (1 + O(N^{-1}))\Psi_{\mathrm{WKB}}(z)\,. \tag{7.12}$$

8 Proof of the Main Theorem

Define the matrix-valued function $\Psi_n^0(z)$ on the complex plane by the formulae

$$\Psi_n^0(z) = \begin{cases} \Psi_{\mathrm{WKB}}(z)\,, & z \in \Omega^c \cup \Omega_1\,, \\ (-1)^n \sigma_3 \Psi_{\mathrm{WKB}}(-z)\sigma_3\,, & z \in (-\Omega_1)\,, \\ \Psi_{\mathrm{TP}}(z)\,, & z \in \Omega^1\,, \\ (-1)^n \sigma_3 \Psi_{\mathrm{TP}}(-z)\sigma_3\,, & z \in (-\Omega^1)\,, \\ \Psi_{\mathrm{CP}}(z)\,, & z \in \Omega^0\,. \end{cases} \tag{8.1}$$

Let

$$X_n(z) = \Psi_n(z)[\Psi_n^0(z)]^{-1}\,. \tag{8.2}$$

From (1.37) and (5.6) we obtain that as $z \to \infty$, $X_n(z)$ admits the asymptotic expansion

$$X_n(z) \sim \sum_{j=0}^{\infty} \frac{\Theta_j}{z^j} \tag{8.3}$$

with

$$\begin{aligned} \Theta_0 &= \frac{1}{\sqrt{2}} C_0^{-1} \Gamma_0 e^{(C_1-\lambda_n)\sigma_3}\,, \\ \Theta_1 &= \frac{1}{\sqrt{2}} C_0^{-1} \big[\Gamma_1 e^{(C_1-\lambda_n)\sigma_3} - \Gamma_0 e^{(C_1-\lambda_n)\sigma_3} (R_n^0)^{1/2} \sigma_1\big]\,. \end{aligned} \tag{8.4}$$

In particular,

$$\lim_{z\to\infty} X_n(z) = \Theta_0\,. \tag{8.5}$$

From (8.1) we obtain that $\Psi_n^0(z)$ satisfies the equation

$$\Psi_n^0(-z) = (-1)^n \sigma_3 \Psi_n^0(z)\sigma_3\,, \tag{8.6}$$

the same as $\Psi_n(z)$ (cf. (1.42)); hence

$$X_n(-z) = \sigma_3 X_n(z) \sigma_3\,. \tag{8.7}$$

The function $X_n(z)$ has multiplicative jumps on a number of contours because of the piecewise definition (8.1) and because of the jump of $\Psi_n(z)$ on the real axis. Let us discuss this situation more carefully. Let γ_0 be the union of boundaries of the regions Ω, Ω_1, and $-\Omega_1$,

$$\gamma_0 = \partial\Omega \cup \partial\Omega_1 \cup (-\partial\Omega_1)\,.$$

Then by Proposition 7.2,

$$[X_{n-}(z)]^{-1} X_{n+}(z) = \big[\Psi_{n-}^0(z)\big]^{-1} \Psi_{n+}^0(z) = I + O(N^{-1})\,, \quad z \in \gamma_0\,.$$

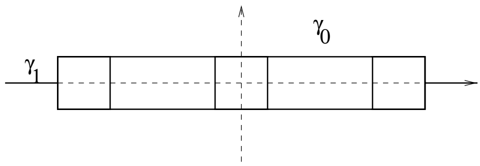

FIGURE 8.1. The contour $\gamma = \gamma_0 \cup \gamma_1$.

The function $X_n(z)$ has no jump on the segment $[-z_0 - d_1, z_0 + d_1]$ because here the jumps of $\Psi_n^0(z)$ and $\Psi_n(z)$ cancel out each other. Consider also the set

$$\gamma_1 = \{z \le -z_0 - d_1\} \cup \{z \ge z_0 + d_1\},$$

and let $\gamma = \gamma_0 \cup \gamma_1$ (see Figure 8.1). On γ_1,

$$[X_{n-}(z)]^{-1} X_{n+}(z) = [\Psi_n^0(z)]^{-1} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \Psi_n^0(z).$$

Observe that if $z \ge z_0 + d_1$, then

$$\Psi_n^0(z) = \Psi_{\mathrm{WKB}}(z) = \hat{T}(z) e^{-N\xi^c(z)\sigma_3},$$

where the matrix $\hat{T}(z)$ is uniformly bounded; hence

$$[X_{n-}(z)]^{-1} X_{n+}(z) = [\hat{T}(z)]^{-1} \begin{pmatrix} 1 & -ie^{-2N\xi^c(z)} \\ 0 & 1 \end{pmatrix} \hat{T}(z) = I + O(e^{-aN|z|})$$

with some $a > 0$. Thus, we obtain the following proposition:

PROPOSITION 8.1 *The function $X_n(z)$ is an analytic, matrix-valued function on the complex plane with multiplicative jumps on the contours γ_0 and γ_1 such that*

$$[X_{n-}(z)]^{-1} X_{n+}(z) = \begin{cases} I + O(N^{-1}), & z \in \gamma_0, \\ I + O(e^{-aN|z|}), & z \in \gamma_1. \end{cases} \tag{8.8}$$

At infinity $X_n(z)$ admits asymptotic expansion (8.3).

Proposition 8.1 implies that the function $X_n(z)$ solves the Riemann-Hilbert problem on the contour $\gamma = \gamma_0 \cup \gamma_1$,

$$X_n(\infty) \equiv \lim_{z\to\infty} X_n(z) = \Theta_0, \tag{8.9}$$

$$X_{n+}(z) = X_{n-}(z) G(z), \quad z \in \gamma, \tag{8.10}$$

with the jump matrix given by the equations

$$\begin{aligned} G(z) &= \left[\Psi_{n-}^0(z)\right]^{-1} \Psi_{n+}^0(z), & z \in \gamma_0, \\ G(z) &= \left[\Psi_n^0(z)\right]^{-1} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \Psi_n^0(z), & z \in \gamma_1, \end{aligned} \tag{8.11}$$

and satisfying the estimates

$$\|I - G(z)\|_{L_2(\gamma)\cap L_\infty(\gamma)} = O(N^{-1}). \tag{8.12}$$

The Riemann-Hilbert problem shares a remarkable property of well-posedness (see, e.g., [3, 14, 51, 66]). Namely, estimate (8.12) implies the following estimate of $X_n(z)$ on the full complex plane (see [7, app. D]):

PROPOSITION 8.2 *For all $z \in \mathbb{C}$,*

$$X_n(z) = \Theta_0\left(I + O\left(\frac{1}{N(1+|z|)}\right)\right). \tag{8.13}$$

Comparing (8.13) with (8.3), we obtain that

$$\Theta_0^{-1}\Theta_1 = O(N^{-1}),$$

that is, according to (8.4),

$$e^{-(C_1-\lambda_n)\sigma_3}\Gamma_0^{-1}\Gamma_1 e^{(C_1-\lambda_n)\sigma_3} - (R_n^0)^{1/2}\sigma_1 = O(N^{-1}). \tag{8.14}$$

This is a 2×2 matrix equation. If we write it for the matrix elements, we obtain from (1.38) that two equations are trivial, $0 = 0$, and the other two are

$$\begin{aligned} e^{-2C_1+2\lambda_n} &= \left(R_n^0\right)^{1/2} + O(N^{-1}), \\ R_n e^{2C_1-2\lambda_n} &= \left(R_n^0\right)^{1/2} + O(N^{-1}). \end{aligned} \tag{8.15}$$

Observe that by (4.2),

$$R_n^0 = -\frac{t}{2g} + O(N^{-1/3}),$$

so that $R_n^0 > 0$ is uniformly bounded and separated from 0 as $N \to \infty$; hence from the first equation in (8.15) we obtain that

$$e^{\lambda_n - C_1} = (R_n^0)^{1/4} + O(N^{-1}). \tag{8.16}$$

Substituting this into the second equation in (8.15) gives

$$R_n = R_n^0 + O(N^{-1}). \tag{8.17}$$

From (8.16) we also obtain that

$$e^{\lambda_n} = e^{C_1}(R_n^0)^{1/4}(1 + O(N^{-1})).$$

Since $h_n = e^{2\lambda_n}$ (see Proposition 1.1), we obtain that

$$h_n = e^{2C_1}(R_n^0)^{1/2}(1 + O(N^{-1})). \tag{8.18}$$

The constant C_1 is given in (5.19). Substituting its value into (8.18), we obtain that

$$h_n = e^{2N\int_{z_0^N}^{\infty} \mu^c(u)du}(1 + O(N^{-1})). \tag{8.19}$$

This proves the part of Theorem 1.2 concerning the asymptotics of R_n and h_n.

Let us take

$$C_0 = \frac{1}{2^{1/2}(R_n^0)^{1/4}}. \tag{8.20}$$

Then from (8.16) and (8.4) we obtain that

$$\Theta_0 = I + O(N^{-1});$$

hence

$$\Psi_n(z) = \left(I + O\left(\frac{1}{N(1+|z|)}\right)\right)\Psi_n^0(z)\,. \tag{8.21}$$

Due to (8.1) and (1.96), this proves asymptotic relations (1.64), (1.76), (1.80), and (1.89). Theorem 1.2 is proven.

9 Universality

9.1 Critical Point

The double scaling limit at the critical point $z = 0$ is determined by the kernel

$$Q_c(u, v) = \lim_{N\to\infty} \frac{1}{cN^{1/3}} Q_N\left(\frac{u}{cN^{1/3}}, \frac{v}{cN^{1/3}}\right), \tag{9.1}$$

where

$$Q_N(z, w) = R_N^{1/2} \frac{\psi_N(z)\psi_{N-1}(w) - \psi_{N-1}(z)\psi_N(w)}{z - w}\,,$$

and $c > 0$ is a normalizing constant. To evaluate (9.1), we apply CP solution (1.88), which gives that modulo $O(N^{-1/3})$ terms,

$$\begin{aligned}
\psi_N(z) &= \frac{1}{\pi^{1/2}}\,(R_N^0)^{1/4}\left[V_{11}(z)\Phi^1(N^{1/3}\zeta(z)) + V_{12}(z)\Phi^2(N^{1/3}\zeta(z))\right],\\
\psi_{N-1}(z) &= \frac{1}{\pi^{1/2}}\,(R_N^0)^{-1/4}\left[V_{21}(z)\Phi^1(N^{1/3}\zeta(z)) + V_{22}(z)\Phi^2(N^{1/3}\zeta(z))\right].
\end{aligned} \tag{9.2}$$

To evaluate (9.1) we can replace $V_{ij}(z)$ by $V_{ij} \equiv V_{ij}(0)$ and $\zeta(z)$ by $\zeta'(0)z$. We take

$$c = \zeta'(0)\,.$$

Then, modulo $O(N^{-1/3})$ terms,

$$\begin{aligned}
&\frac{1}{cN^{1/3}} Q_N\left(\frac{u}{cN^{1/3}}, \frac{v}{cN^{1/3}}\right)\\
&\quad = C(u-v)^{-1}\Big[\big(V_{11}\Phi^1(u) + V_{12}\Phi^2(u)\big)\big(V_{21}\Phi^1(v) + V_{22}\Phi^2(v)\big)\\
&\qquad\qquad - \big(V_{11}\Phi^1(v) + V_{12}\Phi^2(v)\big)\big(V_{21}\Phi^1(u) + V_{22}\Phi^2(u)\big)\Big]\\
&\quad = C'\frac{\Phi^1(u)\Phi^2(v) - \Phi^1(v)\Phi^2(u)}{u - v}\,,
\end{aligned} \tag{9.3}$$

where $C = 1/\pi$ and

$$C' = C(V_{11}V_{22} - V_{12}V_{22}) = C\det V = C\,.$$

Observe that as they are defined in Section 3, the functions $\Phi^j(z)$ depend on n mod 4. Nevertheless, the combination

$$\Phi^1(u)\Phi^2(v) - \Phi^1(v)\Phi^2(u) \tag{9.4}$$

does not depend on n. Indeed, by formula (A.5),

$$\vec{\Phi}(z) = U_0 \vec{B}_0(z)\,, \quad U_0 = \begin{pmatrix} i^n & (-i)^n \\ i^{n-1} & (-i)^{n-1} \end{pmatrix},$$

where $\vec{B}_0(z) = \left(\begin{smallmatrix} B_0^1(z) \\ B_0^2(z) \end{smallmatrix}\right)$ does not depend on n. A direct computation gives that

$$\Phi^1(u)\Phi^2(v) - \Phi^1(v)\Phi^2(u) = (\det U_0)\big[B_0^1(u)B_0^2(v) - B_0^1(v)B_0^2(u)\big]\,.$$

Since $\det U_0 = 2i$, this proves that combination (9.4) does not depend on n. Thus,

$$Q_c(u, v) = \frac{\Phi^1(u)\Phi^2(v) - \Phi^1(v)\Phi^2(u)}{\pi(u - v)}\,. \tag{9.5}$$

9.2 Edge of the Spectrum

The double scaling limit at the edge is determined by the kernel

$$Q_e(u, v) = \lim_{N\to\infty} \frac{1}{cN^{2/3}} Q_N\left(z_0 + \frac{u}{cN^{2/3}}, z_0 + \frac{v}{cN^{2/3}}\right), \tag{9.6}$$

where $Q_n(z, w)$ is the same as in (9.2) and $c > 0$ is a normalizing constant. To evaluate (9.6) we use the TP solution

$$\begin{aligned} &\psi_n(z) = \\ &\quad \frac{(R_n^0)^{1/4}}{2\pi^{1/2}}\big[W_{11}(z)N^{1/6}\,\mathrm{Ai}(N^{2/3}\zeta(z)) + W_{12}(z)N^{-1/6}\,\mathrm{Ai}'(N^{1/3}\zeta(z))\big]\,, \\ &\psi_{n-1}(z) = \\ &\quad \frac{(R_n^0)^{1/4}}{2\pi^{1/2}}\big[W_{21}(z)N^{1/6}\,\mathrm{Ai}(N^{1/3}\zeta(z)) + W_{22}(z)N^{-1/6}\,\mathrm{Ai}'(N^{1/3}\zeta(z))\big]\,. \end{aligned} \tag{9.7}$$

If we take $c = \zeta'(z_0)$ and repeat the above calculation at the critical point, we obtain that

$$Q_e(u, v) = \frac{\mathrm{Ai}(u)\,\mathrm{Ai}'(v) - \mathrm{Ai}(v)\,\mathrm{Ai}'(u)}{u - v}\,, \tag{9.8}$$

the Airy kernel.

9.3 Bulk of the Spectrum

The double scaling limit in the bulk is determined by the kernel

$$Q(u, v) = \lim_{N\to\infty} \frac{1}{cN} Q_N\left(z + \frac{u}{cN}, z + \frac{v}{cN}\right), \tag{9.9}$$

where $c = p(z)$, the value of the density function at z. To evaluate (9.9), we use the WKB solution, and in the same way as above we obtain that

$$Q_b(u, v) = \frac{\sin \pi(u - v)}{\pi(u - v)}\,,$$

the sine kernel.

Appendix A: Proof of Proposition 3.2

Let $\widetilde{\Psi}_j(z),\ j = 1, \dots, 6$, be the Stokes solutions to equation (3.11). We recall that the solutions $\widetilde{\Psi}_j(z)$ are uniquely determined by the asymptotic condition

$$\lim_{|z|\to\infty} \widetilde{\Psi}_j(z) e^{i((4/3)z^3+yz)\sigma_3} = I \quad \text{if } \left|\arg z - \frac{(j-1)\pi}{3}\right| \le \frac{\pi}{3} - \varepsilon\,. \tag{A.1}$$

and that they are related as follows:

$$\begin{aligned} &\widetilde{\Psi}_1(z) = \widetilde{\Psi}_6(z)\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \quad \widetilde{\Psi}_2(z) = \widetilde{\Psi}_1(z)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad \widetilde{\Psi}_3(z) = \widetilde{\Psi}_2(z)\,,\\ &\widetilde{\Psi}_4(z) = \widetilde{\Psi}_3(z)\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}, \quad \widetilde{\Psi}_5(z) = \widetilde{\Psi}_4(z)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \widetilde{\Psi}_6(z) = \widetilde{\Psi}_5(z)\end{aligned} \tag{A.2}$$

(see (3.13) and (3.14)).

Let us rewrite these equations in terms of vector solutions. Let

$$\widetilde{\Psi}_2(z) = (\vec{B}_0(z), \vec{B}_1(z))\,.$$

Then by (A.2),

$$\begin{aligned} &\widetilde{\Psi}_1(z) = \widetilde{\Psi}_4(z) = (\vec{B}_2(z), \vec{B}_1(z))\,, \quad \vec{B}_2(z) = \vec{B}_0(z) - \vec{B}_1(z)\,,\\ &\widetilde{\Psi}_2(z) = \widetilde{\Psi}_3(z) = (\vec{B}_0(z), \vec{B}_1(z))\,,\\ &\widetilde{\Psi}_5(z) = \widetilde{\Psi}_6(z) = (\vec{B}_2(z), \vec{B}_0(z))\,, \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} &\lim_{|z|\to\infty} \vec{B}_0(z) e^{i((4/3)z^3+yz)} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{if } \varepsilon < \arg z < \pi - \varepsilon\,,\\ &\lim_{|z|\to\infty} \vec{B}_0(z) e^{-i((4/3)z^3+yz)} = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{if } \pi + \varepsilon < \arg z < 2\pi - \varepsilon\,,\\ &\lim_{|z|\to\infty} \vec{B}_1(z) e^{-i((4/3)z^3+yz)} = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{if } -\frac{\pi}{3} + \varepsilon < \arg z < \frac{4\pi}{3} - \varepsilon\,,\\ &\lim_{|z|\to\infty} \vec{B}_2(z) e^{i((4/3)z^3+yz)} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{if } \frac{2\pi}{3} + \varepsilon < \arg z < \frac{7\pi}{3} - \varepsilon\,. \end{aligned} \tag{A.4}$$

Define

$$\begin{aligned} \vec{\Phi}(z) &= i^n U \vec{B}_0(z) = \begin{pmatrix} i^n & (-i)^n\\ i^{n-1} & (-i)^{n-1}\end{pmatrix} \vec{B}_0(z)\,,\\ \vec{\Phi}_1(z) &= i^{n-1} U \vec{B}_1(z) = \begin{pmatrix} i^{n-1} & (-i)^{n+1}\\ i^{n-2} & (-i)^{n}\end{pmatrix} \vec{B}_1(z)\,,\\ \vec{\Phi}_2(z) &= i^{n+1} U \vec{B}_2(z) = \begin{pmatrix} i^{n+1} & (-i)^{n-1}\\ i^{n} & (-i)^{n-2}\end{pmatrix} \vec{B}_2(z)\,. \end{aligned} \tag{A.5}$$

Then

$$\vec{\Phi}_j'(z) = A(z)\vec{\Phi}_j(z) \tag{A.6}$$

and

$$
\text{(A.7)}\quad
\begin{aligned}
&\lim_{|z|\to\infty} \vec{\Phi}(z) e^{i((4/3)z^3+yz)} = \begin{pmatrix} i^n \\ i^{n-1} \end{pmatrix} && \text{if } \varepsilon < \arg z < \pi - \varepsilon\,, \\
&\lim_{|z|\to\infty} \vec{\Phi}(z) e^{-i((4/3)z^3+yz)} = \begin{pmatrix} (-i)^n \\ (-i)^{n-1} \end{pmatrix} && \text{if } \pi + \varepsilon < \arg z < 2\pi - \varepsilon\,, \\
&\lim_{|z|\to\infty} \vec{\Phi}_1(z) e^{-i((4/3)z^3+yz)} = \begin{pmatrix} (-i)^{n+1} \\ (-i)^n \end{pmatrix} && \text{if } -\frac{\pi}{3} + \varepsilon < \arg z < \frac{4\pi}{3} - \varepsilon\,, \\
&\lim_{|z|\to\infty} \vec{\Phi}_2(z) e^{i((4/3)z^3+yz)} = \begin{pmatrix} i^{n+1} \\ i^n \end{pmatrix} && \text{if } \frac{2\pi}{3} + \varepsilon < \arg z < \frac{7\pi}{3} - \varepsilon\,.
\end{aligned}
$$

Observe that the function $\overline{\vec{\Phi}(\overline{z})}$ satisfies equation (A.6), because the matrix $A(z)$ is real and, in addition, $\overline{\vec{\Phi}(\overline{z})}$ has the same asymptotics (A.7) as $\vec{\Phi}(z)$. Hence (3.15) holds. Similarly, $\overline{\vec{\Phi}_1(\overline{z})}$ satisfies equation (A.6), and it has the same asymptotics as $\vec{\Phi}_2(z)$. Hence (iii) holds.

The function $V\vec{\Phi}(-z)$, where

$$
V = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^{n-1} \end{pmatrix},
$$

satisfies (A.6), and it has the same asymptotics as $\vec{\Phi}(z)$. Hence (3.16) holds. A similar argument proves (3.19). Finally, (3.20) follows from (A.3). Proposition 3.2 is proven.

Appendix B: Proof of Lemma 4.1

The proof is a direct computation:

$$
V_1 B = \begin{pmatrix} d_{12} & 0 \\ b_{11} - d_{11} & b_{12} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix} = \begin{pmatrix} b_{11}d_{12} & b_{12}d_{12} \\ b_{11}^2 - b_{11}d_{11} + b_{12}b_{21} & -b_{12}d_{11} \end{pmatrix}
$$

and

$$
DV_1 = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & -d_{11} \end{pmatrix} \begin{pmatrix} d_{12} & 0 \\ b_{11} - d_{11} & b_{12} \end{pmatrix} = \begin{pmatrix} b_{11}d_{12} & b_{12}d_{12} \\ d_{11}^2 - b_{11}d_{11} + d_{12}d_{21} & -b_{12}d_{11} \end{pmatrix}.
$$

Since

$$
b_{11}^2 + b_{12}b_{21} = -\det B = -\det D = d_{11}^2 + d_{12}d_{21}\,,
$$

we obtain that $V_1 B = DV_1$, which was stated. Similarly, $V_2 B = DV_2$. Lemma 4.1 is proven.

Appendix C: Proof of Proposition 5.2

Let $z \in \Gamma_c^+$. By (4.37) and (5.2′),

$$\mu(z) = 4i\zeta_0'(z)\zeta_0^2(z) + N^{-2/3} iy\zeta_0'(z) + O(N^{-4/3})\,; \tag{C.1}$$

hence from (4.66) and (4.67), we obtain the following asymptotics of the matrix $W(z)$ as $N \to \infty$:

$$\begin{aligned} W(z) = &\begin{pmatrix} a_{12}^0(z) - i\mu(z) & -a_{11}^0(z) \\ -a_{11}^0(z) & -a_{21}^0(z) - i\mu(z) \end{pmatrix} \\ &+ N^{-1/3}(-1)^n 4u\zeta_0'(z)\zeta_0(z)\sigma_1 \\ &+ N^{-2/3}\big[-(-1)^n 2w\zeta_0'(z)\sigma_3 + (v-y)\zeta_0'(z) I\big] + O(N^{-1})\,. \end{aligned} \tag{C.1′}$$

Substituting this formula into (4.65), we obtain the equation

$$\begin{aligned} &V(z)\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \\ &\frac{1}{\sqrt{b(z)}}\begin{pmatrix} a_{12}^0(z) - i\mu(z) + ia_{11}^0(z) & -ia_{12}^0(z) - \mu(z) - a_{11}^0(z) \\ -a_{11}^0(z) + ia_{21}^0(z) - \mu(z) & ia_{11}^0(z) - a_{21}^0(z) - i\mu(z) \end{pmatrix} U_0(z)\,, \end{aligned} \tag{C.2}$$

where

$$b(z) = -2\mu^2(z) - i\mu(z)\big[a_{12}^0(z) - a_{21}^0(z)\big]\,, \tag{C.2′}$$

and the matrix-valued function $U_0(z)$ admits the asymptotic representation

$$\begin{aligned} U_0(z) = I &+ N^{-1/3}(-1)^n \frac{u}{2\zeta_0(z)}\sigma_1 \\ &+ N^{-2/3}\left[\frac{u^2}{8\zeta_0^2(z)} I + (-1)^{n+1}\frac{w}{4\zeta_0^2(z)}\sigma_2 - 2i(-1)^n \frac{a_{11}^0(z) w\zeta_0'(z)}{b(z)}\sigma_3\right] \\ &+ O(N^{-1})\,. \end{aligned} \tag{C.3}$$

When deriving (C.3), we use (C.1), the formula

$$b(z) = -2i\mu(z)\big[a_{12}^0(z) - i\mu(z)\big] + O(N^{-2/3})\,,$$

which follows from (C.2′), and the formula

$$\frac{a_{12}^0(z) + a_{21}^0(z)}{2} = N^{-2/3} c^0 + O(N^{-1})\,, \quad c^0 = (2g|t|)^{1/6}(-1)^n w\,, \tag{C.4}$$

which follows from (4.4). Taking into account that, by (4.5′),

$$\mu^2(z) = \big(a_{11}^0\big)^2(z) + a_{12}^0(z)a_{21}^0(z) + O(N^{-4/3})\,,$$

simple algebra shows that

$$\begin{aligned} &\frac{1}{\sqrt{b(z)}}\begin{pmatrix} a_{12}^0(z) - i\mu(z) + ia_{11}^0(z) & -ia_{12}^0(z) - \mu(z) - a_{11}^0(z) \\ -a_{11}^0(z) + ia_{21}^0(z) - \mu(z) & ia_{11}^0(z) - a_{21}^0(z) - i\mu(z) \end{pmatrix} = \\ &\qquad T_0(z)\delta(z)^{\sigma_3} + O(N^{-4/3})\,, \end{aligned} \tag{C.4′}$$

where

$$\delta(z) = \left(\frac{a_{12}^0(z) - i\mu(z) + ia_{11}^0(z)}{ia_{11}^0(z) - a_{21}^0(z) - i\mu(z)}\right)^{1/2} \equiv \left(1 + \frac{a_{12}^0(z) + a_{21}^0(z)}{ia_{11}^0(z) - a_{21}^0(z) - i\mu(z)}\right)^{1/2},$$

and $T_0(z)$ is the WKB gauge matrix (see (5.5)). From (C.4),

$$\delta(z) = 1 + N^{-2/3}\frac{c^0}{ia_{11}^0(z) - a_{21}^0(z) - i\mu(z)} + O(N^{-1}).$$

Using this equation, equations (C.3) and (C.4′), and also equation (4.84), we can rewrite (C.2) as

$$V(z)\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = T_0(z)V_0(z), \tag{C.5}$$

where

$$\begin{aligned} V_0(z) &= I + N^{-1/3}(-1)^n\frac{u}{2\zeta(z)}\sigma_1 \\ &\quad + N^{-2/3}\left[\frac{u^2}{8\zeta^2(z)}I + (-1)^{n+1}\frac{w}{4\zeta^2(z)}\sigma_2 + \Delta(z)\sigma_3\right] + O(N^{-1}), \end{aligned} \tag{C.6}$$

and

$$\Delta(z) = \frac{c^0}{ia_{11}^0(z) - a_{21}^0(z) - i\mu(z)} - 2i(-1)^n w\zeta_0'(z)\frac{a_{11}^0(z)}{b(z)}. \tag{C.6′}$$

This proves equation (5.16).

Let us prove (5.14). By (4.68) and (4.69),

$$a(z) = \det\left[A_n^0(z) - N^{-1}V'(z)V^{-1}(z)\right] + O(N^{-2}).$$

Using (C.5) and the diagonalizing property of the matrix $T_0(z)$,

$$T_0^{-1}A_n^0T_0 = -\mu\sigma_3,$$

we obtain that

$$\begin{aligned} &\det\left[A_n^0 - N^{-1}V'V^{-1}\right] \\ &= \det\left[A_n^0 - N^{-1}T_0'T_0^{-1} - N^{-1}T_0V_0'V_0^{-1}T_0^{-1}\right] \\ &= \det\left[T_0^{-1}A_n^0T_0 - N^{-1}T_0^{-1}T_0' - N^{-1}V_0'V_0^{-1}\right] \\ &= \det\left[-\mu\sigma_3 - N^{-1}T_0^{-1}T_0' - N^{-1}V_0'V_0^{-1}\right] \\ &= -\mu^2\det\left[I + \frac{1}{\mu}N^{-1}\sigma_3T_0^{-1}T_0' + \frac{1}{\mu}N^{-1}\sigma_3V_0'V_0^{-1}\right] \\ &= -\mu^2 - \mu N^{-1}\operatorname{tr}\left[\sigma_3T_0^{-1}T_0'\right] - \mu N^{-1}\operatorname{tr}\left[\sigma_3V_0'V_0^{-1}\right] + O(N^{-2}). \end{aligned} \tag{C.7}$$

From (C.6) we conclude at once that

$$\operatorname{tr}\left[\sigma_3 V_0' V_0^{-1}\right] = 2\Delta'(z) N^{-2/3} + O(N^{-1})\,. \tag{C.8}$$

At the same time, equation (5.5′) and the identity (following from the equations $\det T_0 \equiv 2$ and $(T_0)_{11} = (T_0)_{22}$)

$$\operatorname{diag} T_0^{-1} T_0' = \frac{1}{2}\big(\operatorname{tr}[\sigma_3 T_0^{-1} T_0']\big)\sigma_3$$

imply

$$\operatorname{tr}\left[\sigma_3 T_0^{-1} T_0'\right] = 2\tau'(z)\,. \tag{C.9}$$

From (C.7)–(C.9), and taking into account our convention about the branches of the square roots in (4.82), we derive the asymptotic formula

$$\begin{aligned}\sqrt{a(z)} &= \left[\det(A_n^0(z) - N^{-1}V'(z)V^{-1}(z)) + O(N^{-2})\right]^{1/2} \\ &= -i\mu(z) - iN^{-1}\tau'(z) - iN^{-5/3}\Delta'(z) + O(N^{-2})\,,\end{aligned} \tag{C.10}$$

as $N \to \infty$, $z \in \Gamma_c^+$.

From (4.11) it follows that when $z \in \Gamma_c^+$,

$$\begin{aligned}\sqrt{f(\zeta)} &= \left\{16\zeta^4 + 8N^{-2/3}y\zeta^2 + N^{-4/3}[v^2(y) - 4w^2(y)]\right\}^{1/2} \\ &= 4\zeta^2\left\{1 + N^{-2/3}\frac{y}{4\zeta^2} + N^{-4/3}\frac{v^2(y) - 4w^2(y) - y^2}{32\zeta^4} + O(N^{-2})\right\} \\ &= 4\zeta^2 + N^{-2/3}y - N^{-4/3}\frac{D}{2\zeta^2} + O(N^{-2})\,,\end{aligned} \tag{C.11}$$

where (cf. (3.6))

$$D \equiv D(y) = \frac{y^2 - v^2(y) + 4w^2(y)}{4}\,.$$

Replacing both sides of (4.82) by their asymptotics (C.10) and (C.11) and integrating the resulting equation, we arrive at the formula

$$\begin{aligned}i\left(\frac{4}{3}\zeta^3 + N^{-2/3}y\zeta + N^{-4/3}\frac{D}{2}\zeta^{-1}\right) &= \int_\infty^z \mu(u)du + N^{-1}\tau(z) \\ &\quad + N^{-5/3}\Delta(z) + c + O(N^{-2})\,.\end{aligned} \tag{C.12}$$

Let us find the constant c. We will use the fact that the function $\zeta(z)$ is odd in z. Consider the contour γ that goes along the imaginary axis from $i\infty$ to id_2, then around zero on the right to $-id_2$, and then to $-i\infty$. Denote by γ_1 the part of γ from $i\infty$ to id_2, by γ_2 the part of γ from id_2 to $-id_2$, and by γ_3, from $-id_2$ to $-\infty$ (see Figure C.1).

Let $\mu_0(z) = \sqrt{-d(z)}$ with two cuts $(-\infty, -z^N)$ and (z^N, ∞). We will consider $\mu_0(z)$ outside of a small disk $D_{\omega_0} = \{|z| \le \omega_0\}$ around the origin. For $z \in \Omega^c$,

$$\mu_0(z) = \begin{cases}\mu(z)\,, & \operatorname{Im} z > 0\,, \\ -\mu(z)\,, & \operatorname{Im} z < 0\,.\end{cases} \tag{C.13}$$

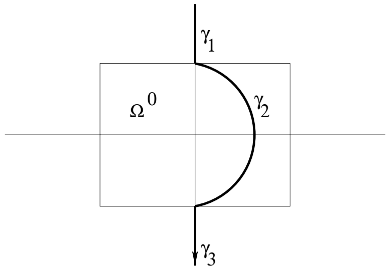

FIGURE C.1. The contour γ.

In contrast to $\mu(z)$, which is an odd function, the function $\mu_0(z)$ is even. The function $\tau(z)$ is defined in (5.5). Let $\tau_0(z)$ be the function on $\mathbb{C} \setminus D_{\omega_0}$ defined by the same formula (5.5) with two cuts $(-\infty, -z^N)$ and (z^N, ∞). Replacing in all formulae (C.1)–(C.11) the functions $\mu(z)$ and $\tau(z)$ by the functions $\mu_0(z)$ and $\tau_0(z)$, respectively, we can extend the validity of these formulae to the points z of the annulus $\Omega_c \setminus D_{\omega_0}$. Hence equation (C.12) can be extended to the equation

$$\text{(C.12}'\text{)}\quad i\left(\frac{4}{3}\zeta^3 + N^{-2/3}y\zeta + N^{-4/3}\frac{D}{2}\zeta^{-1}\right) = \int_\infty^z \mu_0(u)du + N^{-1}\tau_0(z) + N^{-5/3}\Delta_0(z) + c + O(N^{-2}), \quad z \in \Omega_c \setminus D_{\omega_0},$$

where

$$\Delta_0(z) \equiv \Delta(z)\big|_{\mu(z)\to\mu_0(z)},$$

and it is an analytic continuation of $\Delta(z)$ to $\mathbb{C} \setminus D_{\omega_0}$ with two cuts $(-\infty, -z^N)$ and (z^N, ∞).

We will split the function $\tau_0(z)$ as

$$\text{(C.14)}\qquad \tau_0(z) = \tilde{\tau}_0(z) + \hat{\tau}_0(z) + c_0 + O(N^{-1}),$$

where

$$\text{(C.15)}\qquad \begin{aligned} &\tilde{\tau}_0(z) = \frac{1}{2}\ln\left[\frac{\mu_0(z) - a_{11}^0(z)}{a_{12}^0(z)}\right], \qquad \hat{\tau}_0(z) = \frac{1}{2}\int_\infty^z \frac{U^1(u)}{\mu_0(u)}\,du, \\ &c_0 = \frac{1}{4}\ln R_n^0, \qquad U^1(z) = a_{11}^{0\,\prime}(z) - a_{11}^0(z)\frac{a_{12}^{0\,\prime}(z)}{a_{12}^0(z)} \end{aligned}$$

(we will justify (C.14) at the end of this section). The branch of the logarithm in $\tilde{\tau}_0(z)$ is determined by the condition that for sufficiently large N,

$$0 < \operatorname{Im}\tilde{\tau}_0(z) < \frac{\pi}{2}, \qquad z \in \mathbb{C} \setminus [D_{\omega_0} \cup (-\infty, -z^N] \cup [z^N, \infty)].$$

Note that

$$\frac{\mu_0(z) - a_{11}^0(z)}{a_{12}^0(z)} = \phi(z) + O(N^{-1/3}) \quad \text{where} \quad \phi(z) = \frac{z + \sqrt{z^2 - z_0^2}}{2[R_n^0]^{1/2}}$$

and $\phi : \mathbb{C} \setminus [(-\infty, -z_0] \cup [z_0, \infty)] \mapsto \{\operatorname{Re} \phi > 0\}$.

Using (C.14), we rewrite (C.12′) as

$$\text{(C.16)} \quad i\left(\frac{4}{3}\zeta^3 + N^{-2/3} y\zeta + N^{-4/3}\frac{D}{2}\zeta^{-1}\right) =$$
$$\int_\infty^z \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right) du + N^{-1}\tilde{\tau}_0(z) + N^{-5/3}\Delta_0(z) + c_1 + O(N^{-2}),$$
$$c_1 = c + N^{-1}c_0 .$$

where $z \in \Omega_c \setminus D_{\omega_0}$. The integral $\int_\infty^z \mu_0(u)du$ diverges at infinity, so a regularization is needed. Let us describe the regularization we use.

Let γ_0 be any contour that goes from ∞ to z in the region

$$S_0 \equiv \mathbb{C} \setminus (D_{\omega_0} \cup (-\infty, -z^N) \cup (z^N, \infty)),$$

and which starts at ∞ in the upper half-plane. For our purposes it will be useful to introduce a slightly more general regularization than (5.3), which in fact will be equivalent to (5.3). Take any point a on γ_0 and define the regularized integral as

$$\text{(C.17)} \quad \int\limits_{\gamma_0} \mu_0(u)du \equiv \frac{\mu_3}{4}a^4 + \frac{\mu_1}{2}a^2 + \mu_{-1}\ln a + \int_\infty^a \widetilde{\mu}_0(u)du + \int_a^z \mu_0(u)du ,$$

where

$$\widetilde{\mu}_0(z) = \mu_0(z) - \mu_3 z^3 - \mu_1 z - \mu_{-1} z^{-1} ,$$
$$\mu_3 = \frac{g}{2}, \quad \mu_1 = \frac{t}{2}, \quad \mu_{-1} = -\frac{n}{N} .$$

The branch of $\ln a$ is taken as follows. By our assumption γ_0 starts at ∞ in the upper half-plane. If the whole piece of γ_0 from ∞ to a lies in the upper half-plane, then take $\ln a$ on the main branch of the logarithm, with a cut at $(-\infty, 0)$ and $\ln 1 = 0$. For other a's, extend $\ln a$ continuously along γ_0. Definition (C.17) yields the following important properties:

(1) The right-hand side of (C.17) does not depend on a.
(2) The contour of integration γ_0 can be deformed by the Cauchy theorem.
(3) The integral $\int_\infty^z \mu_0(u)du$ does not depend on the contour of integration in S_0.

Observe that property (3) is not automatic because S_0 is not simply connected. It follows obviously from the fact that $\mu_0(z)$ is an even function.

The function $\frac{U^1(z)}{2\mu_0(z)}$ has the following asymptotics as $z \to \infty$, $\operatorname{Im} z > 0$:

$$\frac{U^1(z)}{2\mu_0(z)} = -\frac{1}{2}z^{-1} + r(z)\,, \quad r(z) = O(z^{-3})\,, \tag{C.18}$$

(see (1.71)), and we define the regularized integral of this function as

$$\int_\infty^z \frac{U^1(u)}{2\mu_0(u)}\,du \equiv -\frac{\ln a}{2} + \int_\infty^a r(u)du + \int_a^z \frac{U^1(u)}{2\mu_0(u)}\,du\,. \tag{C.19}$$

It also shares properties (1), (2), and (3).

Observe that the left-hand side of (C.16) is an odd function in the annulus $\Omega^0 \setminus D_{\omega_0}$. Hence the right-hand side is odd as well. Applying this to $z = id_2$, we obtain the equation

$$\begin{aligned}
&\int\limits_{\gamma_1} \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right)du + N^{-1}\tilde{\tau}_0(id_2) + N^{-5/3}\Delta_0(id_2) + c_1 \\
&\quad = -\Bigg[\int\limits_{\gamma_1\cup\gamma_2} \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right)du \\
&\qquad + N^{-1}\tilde{\tau}_0(-id_2) + N^{-5/3}\Delta_0(-id_2) + c_1\Bigg] + O(N^{-2})\,.
\end{aligned} \tag{C.20}$$

The functions $\mu_0(z)$ and $U^1(z)$ are even; hence

$$\begin{aligned}
&\int\limits_{\gamma_1} \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right)du = \\
&\qquad \int\limits_{\gamma_3} \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right)du - \frac{\pi i n}{N} - \frac{\pi i}{2N}\,,
\end{aligned} \tag{C.21}$$

with the term $-(\pi i n)/N - (\pi i)/(2N)$ coming from regularization. Indeed, due to (C.13), as $z \to \infty$, $\operatorname{Im} z < 0$,

$$\mu_0(z) = -\mu_3 z^3 - \mu_1 z - \mu_{-1}z^{-1} + \widetilde{\mu}_0(z)\,, \quad \widetilde{\mu}_0(z) = O(z^{-3})\,, \tag{C.22}$$

and we define the regularization of $\int_{\gamma_3} \mu_0(u)du$ as

$$\int\limits_{\gamma_3} \mu_0(u)du \equiv \frac{\mu_3}{4}a^4 + \frac{\mu_1}{2}a^2 + \mu_{-1}\ln a + \int_a^\infty \widetilde{\mu}_0(u)du + \int_z^a \mu_0(u)du\,, \tag{C.23}$$

where $z = -id_2$ and a is any point on γ_3. The integral $\int_a^\infty$ is taken over the part of γ_3 from a to ∞ (in the lower half-plane). Similarly, as $z \to \infty$, $\operatorname{Im} z < 0$,

$$\frac{U^1(z)}{2\mu_0(z)} = \frac{1}{2}z^{-1} + r(z)\,, \quad r(z) = O(z^{-3})\,, \tag{C.24}$$

and we define

$$\int_{\gamma_3} \frac{U^1(u)}{2\mu_0(u)}\, du \equiv -\frac{\ln a}{2} + \int_a^\infty r(u)du + \int_z^a \frac{U^1(u)}{2\mu_0(u)}\, du\,. \tag{C.25}$$

We take the main branch of logarithm for $\ln a$ in (C.23) and (C.25). The remainder functions $\widetilde{\mu}_0(z)$ and $r(z)$ are even. Since

$$\ln(id_2) = \ln(-id_2) + \pi i \tag{C.26}$$

and $\mu_{-1} = -n/N$,

$$\begin{aligned}
&\int_{\gamma_1} \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right) du \\
&\quad= \frac{\mu_3}{4}(id_2)^4 + \frac{\mu_1}{2}(id_2)^2 + \left(\mu_{-1} - \frac{1}{2N}\right)\ln(id_2) \\
&\qquad + \int_\infty^{id_2} \left[\widetilde{\mu}_0(u) + N^{-1}r(u)\right] du \\
&\quad= \frac{\mu_3}{4}(-id_2)^4 + \frac{\mu_1}{2}(-id_2)^2 - \left(\frac{n}{N} + \frac{1}{2N}\right)[\ln(-id_2) + \pi i] \\
&\qquad + \int_{-id_2}^\infty \left[\widetilde{\mu}_0(u) + N^{-1}r(u)\right] du \\
&\quad= \int_{\gamma_3} \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right) du - \frac{\pi i n}{N} - \frac{\pi i}{2N}\,;
\end{aligned}$$

hence (C.21) is proven.

We obtain from (C.20) and (C.21) that

$$\begin{aligned}
c_1 = &-\frac{1}{2}\int_\gamma \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right) du + \frac{\pi i n}{2N} + \frac{\pi i}{4N} \\
&- N^{-1}\frac{\tilde{\tau}_0(id_2) + \tilde{\tau}_0(-id_2)}{2} \\
&- N^{-5/3}\frac{\Delta_0(id_2) + \Delta_0(-id_2)}{2} + O(N^{-2})\,.
\end{aligned} \tag{C.27}$$

For $u \in \gamma$,

$$\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)} = [-d(u) + N^{-1}U^1(u)]^{1/2} + O(N^{-2}u^{-2})\,; \tag{C.28}$$

hence

$$\int_\gamma \left(\mu_0(u) + N^{-1}\frac{U^1(u)}{2\mu_0(u)}\right)du = \int_\gamma [-d(u) + N^{-1}U^1(u)]^{1/2}du + O(N^{-2})\,. \tag{C.29}$$

Deforming the contour of integration, we obtain that

$$\begin{aligned}\int_\gamma [-d(u) + N^{-1}U^1(u)]^{1/2}\,du &= -2\int_{z_0^N}^{\infty} [-d(u) + N^{-1}U^1(u)]^{1/2}\,du \\ &= -2\int_{z_0^N}^{\infty} \mu^c(u)du\,.\end{aligned} \tag{C.30}$$

Thus, (C.27) reduces to

$$c_1 = \int_{z_0^N}^{\infty} \mu^c(u)du + \frac{\pi i n}{2N} + \frac{\pi i}{4N} + N^{-1}a + N^{-5/3}b + O(N^{-2})\,, \tag{C.31}$$

where

$$a = -\frac{\tilde\tau_0(id_2) + \tilde\tau_0(-id_2)}{2}\,, \quad b = -\frac{\Delta_0(id_2) + \Delta_0(-id_2)}{2}\,. \tag{C.32}$$

Let us evaluate a and b. From (C.15),

$$\begin{aligned}\tilde\tau_0(-z) &= \frac{1}{2}\ln\frac{\mu_0(-z) - a_{11}^0(-z)}{a_{12}^0(-z)} = \frac{1}{2}\ln\frac{\mu_0(z) + a_{11}^0(z)}{a_{12}^0(z)} \\ &= \frac{1}{2}\ln\frac{a_{21}^0(z)}{\mu_0(z) - a_{11}^0(z)} = -\tilde\tau_0(z) + \frac{1}{2}\ln\frac{a_{21}^0(z)}{a_{12}^0(z)}\,.\end{aligned} \tag{C.33}$$

From (4.4) and (C.4),

$$\frac{a_{21}^0(z)}{a_{12}^0(z)} = -1 + \frac{a_{21}^0(z) + a_{12}^0(z)}{a_{12}^0(z)} = -1 + N^{-2/3}\frac{2c^0}{a_{12}^0(z)} + O(N^{-1})\,; \tag{C.34}$$

hence

$$\ln\frac{a_{21}^0(z)}{a_{12}^0(z)} = \pi i - N^{-2/3}\frac{2c^0}{a_{12}^0(z)} + O(N^{-1})\,. \tag{C.35}$$

Thus, from (C.33) (and the inequality $0 < \operatorname{Im}\tilde\tau_0(z) < \frac{\pi}{2}$),

$$\tilde\tau_0(z) + \tilde\tau_0(-z) = \frac{\pi i}{2} - N^{-2/3}\frac{c^0}{a_{12}^0(z)} + O(N^{-1}) \tag{C.36}$$

and

$$N^{-1}a = -\frac{\pi i}{4N} + N^{-5/3}\frac{c^0}{2a_{12}^0(z)} + O(N^{-2})\,, \quad z = id_2\,. \tag{C.37}$$

Let us evaluate $\Delta_0(z)+\Delta_0(-z)$. From (C.6′),

$$\Delta_0(z)=\tilde{\Delta}_0(z)+\hat{\Delta}_0(z)\,, \tag{C.38}$$

where

$$\begin{aligned}\tilde{\Delta}_0(z)&=\frac{c^0}{ia^0_{11}(z)-a^0_{21}(z)-i\mu(z)}\,,\\ \hat{\Delta}_0(z)&=-2i(-1)^n w\zeta_0'(z)\frac{a^0_{11}(z)}{b(z)}\,.\end{aligned} \tag{C.39}$$

The function $\hat{\Delta}_0(z)$ is odd; hence $\hat{\Delta}_0(z)+\hat{\Delta}_0(-z)=0$. For $\tilde{\Delta}_0(z)$ we have that modulo $O(N^{-2/3})$ terms,

$$\begin{aligned}\tilde{\Delta}_0(z)+\tilde{\Delta}_0(-z)&=c^0\left(\frac{1}{ia^0_{11}-a^0_{21}-i\mu_0}+\frac{1}{-ia^0_{11}-a^0_{21}-i\mu_0}\right)\\ &=-2c^0\left(\frac{i\mu_0+a^0_{21}}{-\mu_0^2+2i\mu_0a^0_{21}+(a^0_{21})^2+(a^0_{11})^2}\right)\\ &=-2c^0\left(\frac{i\mu_0+a^0_{21}}{-a^0_{12}a^0_{21}+2i\mu_0a^0_{21}+(a^0_{21})^2}\right)\\ &=-2c^0\left(\frac{i\mu_0+a^0_{21}}{2a^0_{21}(i\mu_0+a^0_{21})}\right)=-\frac{c^0}{a^0_{21}(z)}=\frac{c^0}{a^0_{12}(z)}\,.\end{aligned} \tag{C.40}$$

Thus,

$$N^{-5/3}b=-N^{-5/3}\frac{c^0}{2a^0_{12}(z)}+O(N^{-2})\,,\quad z=id_2\,. \tag{C.41}$$

Combining (C.37) and (C.41), we obtain that

$$N^{-1}a+N^{-5/3}b=-\frac{\pi i}{4N}+O(N^{-2})\,; \tag{C.42}$$

hence by (C.31),

$$c_1=\int_{z_0^N}^{\infty}\mu^c(u)du+\frac{\pi i n}{2N}+O(N^{-2})\,,$$

so that by (C.15),

$$c=c_1-N^{-1}c_0=\int_{z_0^N}^{\infty}\mu^c(u)du-\frac{1}{4N}\ln R_n^0+\frac{\pi i n}{2N}+O(N^{-2})\,. \tag{C.43}$$

From (C.12) and (C.43), we obtain (5.14). Proposition 5.2 is proven. It remains to prove (C.14).

PROOF OF (C.14): To simplify notation we will drop "0" from super- and subscripts. In the proof we will neglect $O(N^{-1})$ terms. We will prove first that

$$\tau'=\tilde{\tau}'+\hat{\tau}'\,, \tag{C.44}$$

and then we will check (C.14) at infinity. From (C.15),

$$2(\tilde{\tau}' + \hat{\tau}') = \frac{\mu' - a_{11}'}{\mu - a_{11}} - \frac{a_{12}'}{a_{12}} + \frac{a_{11}' a_{12} - a_{11} a_{12}'}{\mu a_{12}} = \frac{a_{12}(\mu\mu' - a_{11}a_{11}') + a_{12}'(a_{11}^2 - \mu^2)}{a_{12}\mu(\mu - a_{11})}.$$

Since

$$\mu^2 = a_{11}^2 + a_{12}a_{21}, \quad 2\mu\mu' = 2a_{11}a_{11}' + a_{12}'a_{21} + a_{12}a_{21}',$$

we obtain that

$$4(\tilde{\tau}' + \hat{\tau}') = \frac{a_{12}(a_{12}'a_{21} + a_{12}a_{21}') - 2a_{12}'a_{12}a_{21}}{a_{12}\mu(\mu - a_{11})} = \frac{a_{12}a_{21}' - a_{12}'a_{21}}{\mu(\mu - a_{11})},$$

which coincides with $4\tau'$; see (5.5). This proves (C.44).

As $z \to \infty$, $\operatorname{Im} z > 0$,

$$\tau(z) = O(z^{-2}), \quad \tilde{\tau}(z) = \frac{1}{2}\ln z - \frac{1}{4}\ln R_n^0 + O(z^{-2}), \quad \hat{\tau}'(z) = -\frac{1}{2}z^{-1} + \hat{r}(z), \quad \hat{r}(z) = O(z^{-3});$$

hence

$$\hat{\tau}(z) = -\frac{1}{2}\ln z + \int_{\infty}^{z} \hat{r}(u)du.$$

This implies that

$$\lim_{z\to\infty} \tau(z) = \lim_{z\to\infty}\left[\tilde{\tau}(z) + \hat{\tau}(z) + \frac{1}{4}\ln R_n^0\right] = 0, \quad \operatorname{Im} z > 0.$$

Equation (C.14) is proven. □

Appendix D: Proof of Lemma 6.2

PROOF OF (6.27): Let us rewrite (6.24) as

$$\hat{\Psi}_{\mathrm{WKB}}(z) = C_0 \widetilde{T}^c(z) e^{-N\xi^c(z)\sigma_3}, \tag{D.1}$$

where

$$\widetilde{T}^c(z) \equiv T^c(z)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \left(\frac{a_{12}^0(z)}{\mu^c(z)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{\mu^c(z)}{a_{12}^0(z)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \left(\frac{a_{12}^0(z)}{\mu^c(z)}\right)^{1/2} \begin{pmatrix} 1 & 1 \\ -\frac{\mu^c(z)+a_{11}^0(z)}{a_{12}^0(z)} & \frac{\mu^c(z)-a_{11}^0(z)}{a_{12}^0(z)} \end{pmatrix}, \tag{D.2}$$

and

$$\xi^c(z) = \int_{z_0^N}^{z} \mu^c(u)du, \quad \mu^c(z) = (U^0(z))^{1/2}. \tag{D.3}$$

From (1.68),

$$\text{(D.4)}\qquad \begin{aligned} \mu^c(z) &= (U^0(z))^{1/2} = (-d(z) + N^{-1}U^1(z))^{1/2} \\ &= \mu(z) + N^{-1}\frac{1}{2}\,\frac{U^1(z)}{\mu(z)} + O(N^{-2}|z|^{-2})\,, \\ U^1(z) &= (a_{11}^0)'(z) - a_{11}^0(z)\frac{(a_{12}^0)'(z)}{a_{12}^0(z)}\,; \end{aligned}$$

therefore we can replace $\mu^c(z)$ for $\mu(z)$ in formula (D.2) for $\widetilde{T}^c(z)$, with an error term of the order of $N^{-1}|z|^{-1}$, which we will omit. This gives

$$\text{(D.5)}\qquad \begin{aligned} \widetilde{T}^c(z) &= \left(\frac{a_{12}^0(z)}{\mu(z)}\right)^{1/2}\begin{pmatrix} 1 & 1 \\ -\frac{\mu(z)+a_{11}^0(z)}{a_{12}^0(z)} & \frac{\mu(z)-a_{11}^0(z)}{a_{12}^0(z)} \end{pmatrix} \\ &= \left(\frac{\mu(z)-a_{11}^0(z)}{\mu(z)}\right)^{1/2}\begin{pmatrix} 1 & \frac{a_{12}^0(z)}{\mu(z)-a_{11}^0(z)} \\ -\frac{a_{21}^0(z)}{\mu(z)-a_{11}^0(z)} & 1 \end{pmatrix} e^{-\tilde{\tau}(z)\sigma_3} \\ &= T_0(z)e^{-\tilde{\tau}(z)\sigma_3}\,, \end{aligned}$$

where

$$\text{(D.6)}\qquad \tilde{\tau}(z) = \frac{1}{2}\ln\left[\frac{\mu(z)-a_{11}^0(z)}{a_{12}^0(z)}\right],$$

hence by (D.1),

$$\hat{\Psi}_{\mathrm{WKB}}(z) = C_0T_0(z)e^{-[N\xi^c(z)+\tilde{\tau}(z)]\sigma_3}\,.$$

From (D.3),

$$\text{(D.7)}\qquad \xi^c(z) = \int_\infty^z \mu^c(u)du + C^1\,,$$

where

$$\text{(D.8)}\qquad C^1 = \int_{z_0^N}^\infty \mu^c(u)du\,.$$

From (D.4),

$$\text{(D.9)}\qquad \begin{aligned} \int_\infty^z \mu^c(u)du &= \int_\infty^z \mu(u)du + N^{-1}\hat{\tau}(z) + O(N^{-2}|z|^{-1}) \\ &= \xi(z) + N^{-1}\hat{\tau}(z) + O(N^{-2}|z|^{-1})\,, \end{aligned}$$

where

$$\text{(D.10)}\qquad \hat{\tau}(z) = \frac{1}{2}\int_\infty^z \frac{U^1(u)}{\mu(u)}\,du\,.$$

Thus, modulo terms of the order of $N^{-1}|z|^{-1}$,

$$\text{(D.11)}\qquad \hat{\Psi}_{\mathrm{WKB}}(z) = C_0T_0(z)e^{-[N\xi(z)+\tilde{\tau}(z)+\hat{\tau}(z)+C^1]\sigma_3}\,.$$

From (5.5), (D.6), and (D.10) we obtain that

$$\tilde{\tau}'(z) + \hat{\tau}'(z) = \tau'(z) \tag{D.12}$$

(cf. the proof of Proposition 5.2). In addition, as $z \to \infty$,

$$\tilde{\tau}(z) = \frac{1}{2}\ln z - \frac{1}{4}\ln R_n^0 + O(z^{-2}), \quad \hat{\tau}'(z) = -\frac{1}{2}z^{-1}, \quad \tau'(z) = O(z^{-3}).$$

Hence, integrating (D.12), we obtain that

$$\tilde{\tau}(z) + \hat{\tau}(z) = \tau(z) - \frac{1}{4}\ln R_n^0, \tag{D.13}$$

where integral (D.10) is regularized as

$$\hat{\tau}(z) = -\frac{1}{2}\ln z + \frac{1}{2}\int_\infty^z \left(\frac{U^1(u)}{\mu(u)} + \frac{1}{u}\right)du.$$

Thus, we obtain that, modulo terms of the order of $N^{-1}|z|^{-1}$,

$$\hat{\Psi}_{\mathrm{WKB}}(z) = C_0 T_0(z) e^{-[N\xi(z)+\tau(z)+C_1]\sigma_3} = \Psi_{\mathrm{WKB}}(z), \quad C_1 = C^1 - \frac{1}{4}\ln R_n^0,$$

which was stated. □

PROOF OF (6.28): Let $z \in \Gamma_0^r$. For the sake of definiteness, assume that $\operatorname{Im} z \geq 0$, so that

$$\Psi_{\mathrm{TP}}(z) = \tilde{C}_1 W(z) Y_u(w(z)),$$

where

$$Y_u(z) = \begin{pmatrix} N^{1/6} & 0 \\ 0 & N^{-1/6} \end{pmatrix} \begin{pmatrix} y_0(N^{2/3}z) & y_1(N^{2/3}z) \\ y_0'(N^{2/3}z) & y_1'(N^{2/3}z) \end{pmatrix}. \tag{D.14}$$

From Proposition 6.3,

$$\begin{aligned} &\begin{pmatrix} y_0(N^{2/3}z) \\ y_0'(N^{2/3}z) \end{pmatrix} \\ &= \vec{\mathrm{Ai}}(N^{2/3}z) \\ &= (1 + O(N^{-1}))\frac{1}{2\sqrt{\pi}} \begin{pmatrix} N^{-1/6}z^{-1/4} & 0 \\ 0 & N^{1/6}z^{1/4} \end{pmatrix} \begin{pmatrix} e^{-2Nz^{3/2}/3} \\ -e^{-2Nz^{3/2}/3} \end{pmatrix}, \\ &\quad -\pi + \varepsilon \leq \arg z \leq \pi - \varepsilon \end{aligned} \tag{D.15}$$

and

$$
\begin{aligned}
&\begin{pmatrix} y_1(N^{2/3}z) \\ y_1'(N^{2/3}z) \end{pmatrix} \\
&\quad = \begin{pmatrix} e^{-\pi i/6} & 0 \\ 0 & e^{-5\pi i/6} \end{pmatrix} \vec{\mathrm{A}}\mathrm{i}(N^{2/3}e^{-2\pi i/3}z) \\
&\quad = (1 + O(N^{-1}))\frac{1}{2\sqrt{\pi}} \begin{pmatrix} N^{-1/6}z^{-1/4} & 0 \\ 0 & -N^{1/6}z^{1/4} \end{pmatrix} \begin{pmatrix} e^{2Nz^{3/2}/3} \\ -e^{2Nz^{3/2}/3} \end{pmatrix}, \\
&\qquad -\pi + \varepsilon \le \arg z - \frac{2\pi}{3} \le \pi - \varepsilon\,.
\end{aligned} \tag{D.16}
$$

Thus,

$$
\begin{aligned}
Y_u(z) = (1 + O(N^{-1}))\frac{1}{2\sqrt{\pi}} \begin{pmatrix} z^{-1/4} & 0 \\ 0 & z^{1/4} \end{pmatrix} \begin{pmatrix} e^{-2Nz^{3/2}/3} & e^{2Nz^{3/2}/3} \\ -e^{-2Nz^{3/2}/3} & e^{2Nz^{3/2}/3} \end{pmatrix}, \\
-\frac{\pi}{3} + \varepsilon \le \arg z \le \pi - \varepsilon\,.
\end{aligned} \tag{D.17}
$$

From (6.12),

$$
\frac{2w(z)^{3/2}}{3} = \int_{z_0^N}^{z} \mu^c(u)du = \xi^c(z)\,, \tag{D.18}
$$

and from (6.10),

$$
w(z) = \frac{\mu^c(z)^2}{w'(z)^2}\,; \tag{D.19}
$$

hence

$$
\begin{aligned}
&\Psi_{\mathrm{TP}}(z) = \\
&\tilde{C}_1 W(z)(1 + O(N^{-1}))\frac{1}{2\sqrt{\pi}} \begin{pmatrix} \frac{w'(z)^{1/2}}{\mu^c(z)^{1/2}} & 0 \\ 0 & \frac{\mu^c(z)^{1/2}}{w'(z)^{1/2}} \end{pmatrix} \begin{pmatrix} e^{-N\xi^c(z)} & e^{N\xi^c(z)} \\ -e^{-N\xi^c(z)} & e^{N\xi^c(z)} \end{pmatrix}, \\
&\qquad -\frac{\pi}{3} + \varepsilon \le \arg w(z) \le \pi - \varepsilon\,.
\end{aligned} \tag{D.20}
$$

Observe that if $z \in \Gamma_r^0$ and $\operatorname{Im} z \ge 0$, then

$$
-\frac{\pi}{3} + \varepsilon \le \arg w(z) \le \pi - \varepsilon\,;
$$

hence (D.20) holds. From (1.79),

$$
\begin{aligned}
W(z)&\begin{pmatrix} \frac{w'(z)^{1/2}}{\mu^c(z)^{1/2}} & 0 \\ 0 & \frac{\mu^c(z)^{1/2}}{w'(z)^{1/2}} \end{pmatrix} \\
&= \left(\frac{a_{12}^0(z)}{w'(z)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{w'(z)}{a_{12}^0(z)} \end{pmatrix} \begin{pmatrix} \frac{w'(z)^{1/2}}{\mu^c(z)^{1/2}} & 0 \\ 0 & \frac{\mu^c(z)^{1/2}}{w'(z)^{1/2}} \end{pmatrix} \\
&= \left(\frac{a_{12}^0(z)}{\mu^c(z)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0(z)}{a_{12}^0(z)} & \frac{\mu^c(z)}{a_{12}^0(z)} \end{pmatrix} = T^c(z)\,;
\end{aligned}
\tag{D.21}
$$

hence from (D.20),

$$
\begin{aligned}
\Psi_{\mathrm{TP}}(z) &= (1+O(N^{-1}))\frac{\tilde{C}_1}{2\sqrt{\pi}} T^c(z) E(N\xi^c(z)) \\
&= (1+O(N^{-1}))\hat{\Psi}_{\mathrm{WKB}}(z)\,,
\end{aligned}
\tag{D.22}
$$

if $C_0 = \tilde{C}_1/(2\sqrt{\pi})$. Lemma 6.2 is proven. □

Appendix E: Proof of Lemma 4.2

PROOF: Let $R > 0$ be a large number, independent of N, which will be chosen later. Consider two cases: (1) $|z| \le RN^{-1/3}$ and (2) $|z| > RN^{-1/3}$. In case (1) $\det \widetilde{W}(z) \ne 0$ by (4.63).

Consider case (2). First we notice that by (4.48),

$$
4\zeta_0'(z)\zeta_0^2(z) = \frac{gz^2}{2}\sqrt{z_0^2 - z^2} + O(N^{-2/3})\,, \quad z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1\,.
\tag{E.1}
$$

Therefore by (4.67),

$$
\begin{aligned}
\widetilde{a}_{11}(z) &= -\widetilde{a}_{22}(z) = O(N^{-1/3}z)\,, \\
\widetilde{a}_{12}(z) &= \frac{gz^2}{2}\sqrt{z_0^2 - z^2} + O(N^{-2/3})\,, \\
\widetilde{a}_{21}(z) &= -\frac{gz^2}{2}\sqrt{z_0^2 - z^2} + O(N^{-2/3})\,.
\end{aligned}
\tag{E.2}
$$

By (4.4),

$$
\begin{aligned}
a_{11}^0(z) &= -a_{22}^0(z) = -\frac{gz^3}{2} + O(N^{-1/3}z)\,, \\
a_{12}^0(z) &= (R_n^0)^{1/2} g z^2 + O(N^{-2/3})\,, \\
a_{21}^0(z) &= -(R_n^0)^{1/2} g z^2 + O(N^{-2/3})\,.
\end{aligned}
\tag{E.3}
$$

By (4.66) this gives the matrix elements $w_{ij}(z)$ of $W(z)$ as

$$
\begin{aligned}
w_{11}(z) &= \frac{gz^2}{2}\sqrt{z_0^2 - z^2} + (R_n^0)^{1/2} g z^2 + O(N^{-2/3})\,, \\
w_{12}(z) &= w_{21}(z) = \frac{gz^3}{2} + O(N^{-1/3} z)\,, \\
w_{22}(z) &= \frac{gz^2}{2}\sqrt{z_0^2 - z^2} + (R_n^0)^{1/2} g z^2 + O(N^{-2/3})\,,
\end{aligned}
\tag{E.4}
$$

and these estimates are valid for all $z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1$. Observe that by (1.45) and (1.53),

$$
R_n^0 = \frac{|t|}{2g} + O(N^{-1/3}) = \frac{z_0^2}{4} + O(N^{-1/3})\,.
$$

Also,

$$
N^{-1/3} = O(R^{-1}) \quad \text{for all } |z| > RN^{-1/3}\,;
$$

hence

$$
W(z) = \frac{gz^2}{2}\left[\begin{pmatrix} \sqrt{z_0^2 - z^2} + z_0 & z \\ z & \sqrt{z_0^2 - z^2} + z_0 \end{pmatrix} + O(R^{-1})\right]
\tag{E.5}
$$

for all $z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1$ and $|z| > RN^{-1/3}$. Since

$$
\det \begin{pmatrix} \sqrt{z_0^2 - z^2} + z_0 & z \\ z & \sqrt{z_0^2 - z^2} + z_0 \end{pmatrix} =
2\sqrt{z_0^2 - z^2}\left(\sqrt{z_0^2 - z^2} + z_0\right) \neq 0
\tag{E.6}
$$

for $z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1$, we obtain that $\det W(z) \neq 0$ for sufficiently large R if $z \in (-\Omega_1) \cup \Omega^0 \cup \Omega_1$ and $|z| > RN^{-1/3}$. Lemma 4.2 is proven. □

Appendix F: Alternative Forms for $\Psi_n^0(z)$

Recall that each of the functions $\zeta(z)$ and $\zeta_0(z)$ defines an analytic change of variable in the rectangle $\Omega_c \equiv (-\Omega_1) \cup \Omega^0 \cup \Omega_1$, and according to (4.84) we have that

$$
\zeta(z) = \zeta_0(z) + O(N^{-4/3})\,, \quad z \in \Omega_c\,.
\tag{F.1}
$$

Let us use the function $\zeta_0(z)$ instead of the function $\zeta(z)$ in the right-hand side of (5.10). This will lead us to the matrix function

$$
\Psi_{\mathrm{CP}}^0(z) = \begin{cases} \tilde{C} V(z) \Phi^u(N^{1/3}\zeta_0(z))\,, & \operatorname{Im} z \geq 0\,, \\ \tilde{C} V(z) \Phi^d(N^{1/3}\zeta_0(z))\,, & \operatorname{Im} z \leq 0\,. \end{cases}
\tag{F.2}
$$

It is clear that the functions $\Psi_{\mathrm{CP}}^0(z)$ and $\Psi_{\mathrm{CP}}(z)$ have exactly the same Stokes matrices. Taking also into account that, by (F.1),

$$e^{-i(\frac{4}{3}N\zeta^3(z)+N^{1/3}y\zeta(z))\sigma_3}e^{i(\frac{4}{3}N\zeta_0^3(z)+N^{1/3}y\zeta_0(z))\sigma_3} = I + O(N^{-1/3}),$$

we arrive at the asymptotic formula

$$\Psi_{\mathrm{CP}}(z)\big[\Psi_{\mathrm{CP}}^0(z)\big]^{-1} = I + O(N^{-1/3}), \quad z \in \Omega^0. \tag{F.3}$$

The last equation together with (8.21) yields the following representation for the solution $\Psi_n(z)$ in the domain Ω^0:

$$\Psi_n(z) = (I + O(N^{-1/3}))\Psi_{\mathrm{CP}}^0(z), \quad z \in \Omega^0. \tag{F.4}$$

Of course, equation (F.4) has a bigger error term than the similar equation

$$\Psi_n(z) = (I + O(N^{-1}))\Psi_{\mathrm{CP}}(z), \quad z \in \Omega^0, \tag{F.5}$$

which is given by Theorem 1.2. Sometimes, however, using (F.4) offers a certain advantage. Indeed, the function $\zeta_0(z)$, unlike the function $\zeta(z)$, is given by an elementary explicit formula (4.49). It is also worth noting that estimate (F.4) can be improved as follows:

Consider the matrix ratio

$$R(z) \equiv \Psi_{\mathrm{CP}}(z)[\Psi_{\mathrm{CP}}^0(z)]^{-1}.$$

The function $R(z)$ does not have a jump on the interval $[-d_1, d_1]$, and hence it is analytic in Ω^0. Therefore,

$$R(z) = \int_{\partial\Omega^0} \frac{\Psi_{\mathrm{CP}}(u)[\Psi_{\mathrm{CP}}^0(u)]^{-1}}{u-z}\frac{du}{2\pi i}, \quad z \in \Omega^0.$$

At the same time, on the boundary of the domain Ω^0, the function $\Psi_{\mathrm{CP}}(z)$ can be replaced by $\Psi_{\mathrm{WKB}}(z)$ with an error of the order of N^{-1},

$$\Psi_{\mathrm{CP}}(z) = (I + O(N^{-1}))\Psi_{\mathrm{WKB}}(z), \quad z \in \partial\Omega^0, \tag{F.6}$$

and therefore

$$R(z) = \int_{\partial\Omega^0} \frac{(I + O(N^{-1}))\Psi_{\mathrm{WKB}}(u)[\Psi_{\mathrm{CP}}^0(u)]^{-1}}{u-z}\frac{du}{2\pi i}. \tag{F.7}$$

In addition, it follows from (F.6) and (F.3) that

$$\Psi_{\mathrm{WKB}}(u)\big[\Psi_{\mathrm{CP}}^0(u)\big]^{-1} = I + O(N^{-1/3}), \quad z \in \partial\Omega^0, \tag{F.8}$$

and hence equation (F.7) can be rewritten as

$$R(z) = \int_{\partial\Omega^0} \frac{\Psi_{\mathrm{WKB}}(u)[\Psi_{\mathrm{CP}}^0(u)]^{-1}}{u-z}\frac{du}{2\pi i} + O\left(\frac{1}{N\,\mathrm{dist}(z;\partial\Omega^0)}\right). \tag{F.9}$$

The last equation, together with the standard arguments based on the flexibility in the choice of the basic geometric parameters d_1 and d_2, yields the following improvement of estimate (F.4):

$$\Psi_n(z) = (I + O(N^{-1}))\Psi^1_{\mathrm{CP}}(z)\,, \quad z \in \Omega^0\,, \tag{F.10}$$

where

$$\Psi^1_{\mathrm{CP}}(z) = \left\{ \int_{\partial\Omega^0} \frac{\Psi_{\mathrm{WKB}}(u)[\Psi^0_{\mathrm{CP}}(u)]^{-1}}{u - z} \frac{du}{2\pi i} \right\} \Psi^0_{\mathrm{CP}}(z)\,. \tag{F.11}$$

The only nonexplicit element—apart from the Painlevé functions, of course—that is left in formulae (F.10) and (F.11) is parameter α in definition (4.82) of the variable y. To determine it, one needs the equation of periods (4.78). However, the terms of order $N^{-1/3}$ can be extracted from (F.10) and (F.11) in the very explicit form. Indeed, by straightforward but slightly involved calculations, one can see that

$$\begin{aligned} \Psi_{\mathrm{WKB}}(z)[\Psi^0_{\mathrm{CP}}(z)]^{-1} &= \\ &I - N^{-1/3}\frac{ir_1(z)}{\sqrt{z_0^2 - z^2}}\big(z\sigma_3 - 2i(R^0_n)^{1/2}\sigma_2\big) + O(N^{-2/3})\,, \quad z \in \partial\Omega^0\,, \end{aligned} \tag{F.12}$$

where

$$r_1(z) = \frac{iD}{2\zeta_\infty(z)} + \frac{1}{g}\int_z^\infty \left(c_3 - \frac{y^2 c_0^2}{u^2 - z_0^2}\right)\frac{du}{u^2\sqrt{u^2 - z_0^2}} + iy^2\frac{\zeta_1(z)D_1(z)}{\zeta_\infty(z)}\,.$$

By a direct calculation (this time much less involved), we check that $r_1(z)$ has no pole at $z = 0$; i.e., it is an analytic function in Ω^0. Therefore, equation (F.11) yields the estimate

$$\Psi^1_{\mathrm{CP}}(z) = \left(I - N^{-1/3}\frac{ir_1(z)}{\sqrt{z_0^2 - z^2}}\big(z\sigma_3 - 2i(R^0_n)^{1/2}\sigma_2\big) + O(N^{-2/3})\right)\Psi^0_{\mathrm{CP}}(z)\,. \tag{F.13}$$

(F.13) can be further improved. However, when we go beyond the order $N^{-1/3}$, we have to take into account the parameter α, and hence representation (F.10) will not give us any advantage over our basic formula (8.21).

In principle, it is possible to avoid the use of the period equation (4.78). To this end, let us define the approximate solution $\Psi^0_n(z)$ of the Riemann-Hilbert problem by relation (8.1), where $\Psi_{\mathrm{CP}}(z)$ is replaced by $\Psi^0_{\mathrm{CP}}(z)$ and α in the definition of y is set to zero. Then for the ratio $X_n(z) \equiv \Psi_n(z)[\Psi^0_n(z)]^{-1}$, we will arrive at the Riemann-Hilbert problem whose jump matrix $G(z) \equiv [X_{n-}(z)]^{-1}X_{n+}(z)$ is given explicitly in terms of $\Psi^0_n(z)$ and satisfies the uniform estimates (cf. (8.11) and (8.12))

$$\|I - G(z)\|_{L_2(\gamma)\cap L_\infty(\gamma)} = O(N^{-1/3})\,.$$

This would mean replacing the final estimate (8.21) by the estimate

$$\Psi_n(z) = \left(I + O\left(\frac{1}{N^{1/3}(1+|z|)}\right)\right)\Psi_n^0(z)\,. \tag{F.14}$$

Theoretically, this estimate can be improved by iterating the corresponding singular integral equation (cf. [21] and [16]). In particular, the first iteration leads to the function $\Psi_{\mathrm{CP}}^1(z)$ satisfying (F.13). The second iteration, however, looks extremely cumbersome. In fact, our basic construction involving the nontrivial change-of-variable function $\zeta(z)$ can be thought of as a way to bring the second iteration of the singular integral operator with the weight $I - G(z)$ to a compact form.

Let us finally discuss the dependence of $\zeta_0(z)$ on the parameter y, $\zeta_0(z) \equiv \zeta_0(z; y)$. To that end, put

$$\begin{gathered} M_1(z, \zeta_0(z; y), y) \equiv \Psi_{\mathrm{CP}}^0(z, \zeta_0(z; y), y)e^{(N\frac{4i}{3}\zeta_0^3(z;y)+N^{1/3}iy\zeta_0(z;y)-i\gamma)\sigma_3}\,, \\ z \in \{\epsilon \le \arg z \le \pi - \epsilon\} \cup \{-\pi + \epsilon \le \arg z \le -\epsilon\}\,, \end{gathered} \tag{F.15}$$

and

$$\begin{gathered} M_2(z, \zeta_0(z; y), y) \equiv \Psi_{\mathrm{CP}}^0(z, \zeta_0(z; y), y)S_{1,2}^{-1}e^{(N\frac{4i}{3}\zeta_0^3(z;y)+N^{1/3}iy\zeta_0(z;y)-i\gamma)\sigma_3}\,, \\ z \in \left\{-\frac{\pi}{3} + \epsilon \le \arg z \le \frac{\pi}{3} - \epsilon, \pm \operatorname{Im} z \ge 0\right\} \end{gathered} \tag{F.16}$$

(the matrices $S_{1,2}^{-1}$ are the Stokes matrices from (7.3)). It is not difficult to see that

$$M_j(z, \zeta_0(z; y), y) = (I + O(N^{-1/3}))M_j(z, \zeta_0(z; 0), y)\,. \tag{F.17}$$

This implies that we can redefine $\Psi_{\mathrm{CP}}^0(z)$ as

$$\begin{gathered} \Psi_{\mathrm{CP}}^0(z) = M_1(z, \zeta_0(z; 0), y)e^{-(N\frac{4i}{3}\zeta_0^3(z;y)+N^{1/3}iy\zeta_0(z;y)-i\gamma)\sigma_3}\,, \\ z \in \{\epsilon \le \arg z \le \pi - \epsilon\} \cup \{-\pi + \epsilon \le \arg z \le -\epsilon\}\,, \end{gathered} \tag{F.18}$$

and

$$\begin{gathered} \Psi_{\mathrm{CP}}^0(z) = M_2(z, \zeta_0(z; 0), y)e^{-(N\frac{4i}{3}\zeta_0^3(z;y)+N^{1/3}iy\zeta_0(z;y)-i\gamma)\sigma_3}S_{1,2}\,, \\ z \in \left\{-\frac{\pi}{3} + \epsilon \le \arg z \le \frac{\pi}{3} - \epsilon, \pm \operatorname{Im} z \ge 0\right\}, \end{gathered} \tag{F.19}$$

and still have (F.4). Observe that by (4.49) and (4.44),

$$\zeta_0(z; 0) = \left[\frac{3}{4}\int_0^z \frac{gu^2}{2}\sqrt{z_0^2 - u^2}\,du\right]^{1/3}. \tag{F.20}$$

Formulae (F.18)–(F.20) are of the type that would appear if for the analysis of the Riemann-Hilbert problem (1.35)–(1.38) we used the nonlinear steepest descent approach similar to the method used in [2] for the orthogonal polynomials on the circle. The advantage of such a scheme would be the appearance of the Painlevé Riemann-Hilbert problem in a completely deductive way, i.e., without using any prior information about the structure of the asymptotics of the coefficients R_n^0. The price, however, would be a much subtler structure of the relevant contour deformation and a much more complicated analysis of the error terms. In our approach, we take full advantage of the prior heuristic study of the Freud equation (1.24), which leads us to the choice of the approximate solution, whose justification needs only a very "light" Riemann-Hilbert analysis. Moreover, we obtain the error term up to the order $O(N^{-1})$. To get this accuracy in the framework of the [2] scheme, avoiding cumbersome multiple integrals for the correcting terms would in fact require, a posteriori, a WKB-type analysis of the associated Lax pair. In our method, we use the WKB formulae from the very beginning.

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