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Topological Expansion in the Cubic Random Matrix Model

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In this paper, we study the topological expansion in the cubic random matrix model, and we evaluate explicitly the expansion coefficients for genus 0 and 1. For genus 0 our formula coincides with the one in [6]. For higher genus, we obtain the asymptotic behavior of the coefficients in the expansion as the number of vertices of the associated graphs tends to infinity. Our study is based on the Riemann–Hilbert problem, string equations, and the Toda equation.

1 Introduction and Statement of the Main Results

In this paper, we return to the classical work [6], in which, among other things, the authors explicitly calculated the coefficients of the topological expansion in the cubic random matrix model in genus 0. Our main goal will be to rigorously prove the results of Brézin [6] and to obtain an explicit formula for the coefficients of the topological expansion in genus 1. We will also prove some formulae and asymptotic results for the coefficients of the topological expansion in higher genera.

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We consider the random matrix model given by the probability distribution

$$d\mu_N(M) = \frac{1}{\tilde{Z}_N} e^{-N\operatorname{Tr} V(M)}\, dM, \tag{1.1}$$

on the space of $N \times N$ Hermitian matrices M, where

$$V(M) = \frac{M^2}{2} - uM^3, \tag{1.2}$$

and $u > 0$. The model is ill-defined because of the divergence at infinity of the partition function,

$$\tilde{Z}_N(u) = \int e^{-N\operatorname{Tr} V(M)}\, dM. \tag{1.3}$$

The partition function of eigenvalues,

$$Z_N(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \le j < k \le N} (z_j - z_k)^2 \prod_{j=1}^{N} e^{-N(\frac{z_j^2}{2} - uz_j^3)} dz_1 \cdots dz_N \tag{1.4}$$

diverges as well on the real line. To regularize it, we will consider integration on a specially chosen contour Γ in the complex plane:

$$Z_N(u) = \int_{\Gamma} \cdots \int_{\Gamma} \prod_{1 \le j < k \le N} (z_j - z_k)^2 \prod_{j=1}^{N} e^{-N\left(\frac{z_j^2}{2} - uz_j^3\right)} dz_1 \cdots dz_N \tag{1.5}$$

on which the integral converges.

To choose Γ, consider the three sectors on the complex plane,

$$\begin{aligned} S_0 &= \left\{ z \in \mathbb{C} : \frac{5\pi}{6} < \arg z < \frac{7\pi}{6} \right\}, \\ S_1 &= \left\{ z \in \mathbb{C} : \frac{\pi}{6} < \arg z < \frac{\pi}{4} \right\}, \\ S_2 &= \left\{ z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < -\frac{\pi}{6} \right\}, \end{aligned} \tag{1.6}$$

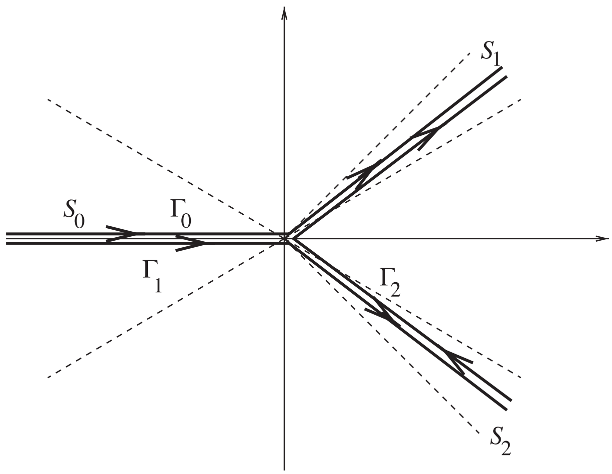

Fig. 1. The sectors S_0, S_1, S_2 and the contours Γ_0, Γ_1, Γ_2.

see Figure 1. Then for any ray

$$R_\theta = \{z \in \mathbb{C} : \arg z = \theta\}, \tag{1.7}$$

lying in the sectors S_0, S_1, and S_2, the integral

$$\int_{R_\theta} z^k \, \mathrm{e}^{-N\left(\frac{z^2}{2} - uz^3\right)} \, \mathrm{d}z \tag{1.8}$$

converges for any $k = 0, 1, \ldots$ and any $u \geq 0$. Therefore, we will use contours consisting of two rays in the sectors S_0, S_1, and S_2. More specifically, we may consider the three contours on the complex plane,

$$\Gamma_0 = R_\pi \cup R_{\pi/5}, \quad \Gamma_1 = R_\pi \cup R_{-\pi/5}, \quad \Gamma_2 = R_{-\pi/5} \cup R_{\pi/5}, \tag{1.9}$$

with orientation from $(-\infty)$ to $(\infty\, \mathrm{e}^{\pi i/5})$ on Γ_0, from $(-\infty)$ to $(\infty\, \mathrm{e}^{-\pi i/5})$ on Γ_1, and from $(\infty\, \mathrm{e}^{-\pi i/5})$ to $(\infty\, \mathrm{e}^{\pi i/5})$ on Γ_2, see Figure 1. However, to get the topological expansion of the free energy, we will use $\Gamma = \Gamma_0$ or $\Gamma = \Gamma_1$, but not $\Gamma = \Gamma_2$.

More generally, following [13], it is convenient to introduce a linear combination of the contours Γ_0 and Γ_1. To that end, let us fix some $\alpha \in \mathbb{C}$ and consider Γ as a linear combination of Γ_0 and Γ_1,

$$\Gamma = \alpha \Gamma_0 + (1 - \alpha)\Gamma_1, \tag{1.10}$$

in the sense that

$$\int_{\Gamma} f(z)\,\mathrm{d}z = \alpha \int_{\Gamma_0} f(z)\,\mathrm{d}z + (1-\alpha)\int_{\Gamma_1} f(z)\,\mathrm{d}z, \tag{1.11}$$

cf. [3]. With this choice of $\Gamma = \Gamma(\alpha)$, the integral $Z_N(u) = Z_N(u;\alpha)$ in (1.5) is convergent for any $u \geq 0$. By the Cauchy theorem, we have some flexibility in the choice of the contours Γ_0, Γ_1 within the sectors S_0, S_1, and S_2.

It is not difficult to see, by differentiation of the integral with respect to u, that the partition function $Z_N(u;\alpha)$ is analytic for $u > 0$, and it is infinitely differentiable for $u \geq 0$. Observe, however, that $Z_N(u;\alpha)$ is not analytic at $u = 0$. Indeed, the analytic continuation of $Z_N(u)$, $u > 0$, to $u = r\,\mathrm{e}^{-3i\theta}$, $r > 0$, can be obtained by the change of variable $z_j = w_j\,\mathrm{e}^{\mathrm{i}\theta}$:

$$\begin{aligned} Z_N(r\,\mathrm{e}^{-3i\theta};\alpha) = \int_{\Gamma} \cdots \int_{\Gamma} \prod_{1\leq j<k\leq N} (w_j\,\mathrm{e}^{\mathrm{i}\theta} - w_k\,\mathrm{e}^{\mathrm{i}\theta})^2 \\ \times \prod_{j=1}^{N} \mathrm{e}^{-N\left(\frac{w_j^2\,\mathrm{e}^{2i\theta}}{2} - r w_j^3\right)}\,\mathrm{d}(w_1\,\mathrm{e}^{\mathrm{i}\theta}) \cdots \mathrm{d}(w_N\,\mathrm{e}^{\mathrm{i}\theta}). \end{aligned} \tag{1.12}$$

By the Cauchy theorem, one can use the same contour $\Gamma = \Gamma(\alpha)$ for all θ. For $\theta = \frac{\pi}{2}$, the exponential term in the latter formula becomes

$$\mathrm{e}^{N\left(\frac{w_j^2}{2} + r w_j^3\right)}$$

and the integral in (1.12) diverges at $r = 0$. Moreover,

$$\lim_{r\to 0} |Z_N(r\,\mathrm{e}^{-3i\theta};\alpha)| = \infty, \qquad \theta = \frac{\pi}{2}, \tag{1.13}$$

hence $Z_N(u;\alpha)$ is not analytic at $u = 0$.

We define the free energy as

$$F_N(u;\alpha) = \frac{1}{N^2} \ln \frac{Z_N(u;\alpha)}{Z_N(0;\alpha)}. \tag{1.14}$$

Observe that $Z_N(0;\alpha)$ is independent of α. The aim of this paper is to investigate the large N asymptotic behavior of the free energy, and, in particular, the structure of the different terms that appear in this asymptotic expansion.

We prove the following results.

Theorem 1.1. Suppose that $\alpha \in \mathbb{C}$ is fixed. Then there exists a critical value $u_c > 0$ such that for any $0 \le u < u_c$, the free energy $F_N(u; \alpha)$ admits an asymptotic expansion in inverse powers of N^2:

$$F_N(u; \alpha) \sim \sum_{g=0}^{\infty} \frac{F^{(2g)}(u)}{N^{2g}}. \tag{1.15}$$

This expansion is uniform in the variable u on any interval $[0, u_c - \varepsilon]$, $\varepsilon > 0$ and can be differentiated on $[0, u_c - \varepsilon]$ with respect to u any number of times, with a uniform estimate of the error term with respect to $u \in [0, u_c - \varepsilon]$. The functions $F^{(2g)}(u)$ do not depend on α and they admit an analytic continuation to the disk $|u| < u_c$ in the complex plane, and if we expand them in powers of u,

$$F^{(2g)}(u) = \sum_{j=1}^{\infty} \frac{f_{2j}^{(2g)} u^{2j}}{(2j)!}, \tag{1.16}$$

then the coefficient $f_{2j}^{(2g)}$ is a positive integer number that counts the number of 3-valent connected graphs with $2j$ vertices on a Riemann surface of genus g. □

Proof. The existence of the $\frac{1}{N^2}$ asymptotic expansion for the free energy is known in the physical literature since the classical work [4] (see also references therein to the earlier works). A rigorous proof of the $\frac{1}{N^2}$ asymptotic expansion for the free energy for a general polynomial $V(M)$ of even degree is given in the paper [15]. Our Theorem 1.1 is an analogous result that follows from that reference, with a small modification due to the fact that the cubic model is defined on the contour Γ in the complex plane and not on the real line.

In the above reference (see Theorem 1.1 therein), the authors prove that the large N asymptotic expansion for the free energy is valid for a small enough perturbation of the Gaussian case. Also, note from the subsequent remark that what is actually required for the asymptotic expansion to hold is that the associated equilibrium measure is supported on a single interval $[a, b]$ on the real axis, together with regularity of the corresponding density function.

In our setting, the deformation from the Gaussian case is controlled by the parameter u, and we show in Section 2 that the conditions on the equilibrium measure stated in [15] are satisfied, provided that $0 \le u < u_c$, for some critical value u_c. Therefore, for small enough values of u, the free energy in the cubic model can be expanded in powers of $1/N^2$ and the coefficients in the expansion have the desired regularity in terms of u. ■

Moreover, a nice feature of the cubic model is that we can give an explicit value of this critical value of the parameter u. Namely, we define

$$u_c = \frac{3^{1/4}}{18}. \tag{1.17}$$

It is worth noticing that the coefficients $F^{(2g)}(u)$ of asymptotic expansion (1.15) in powers of $1/N^2$ do not depend on α for all $0 \le u < u_c$. The dependence on α arises, however, in the double scaling limit as $u \to u_c$. We will return to this question in a subsequent paper. (See also the paper [13], where a similar double scaling limit is studied for the quartic random matrix model.)

In the next theorem, we evaluate explicitly the coefficients $f_{2j}^{(0)}$ in the genus 0 term $F^{(0)}(u)$:

Theorem 1.2. The coefficient $f_{2j}^{(0)}$ can be written as

$$f_{2j}^{(0)} = \frac{72^j \Gamma(\frac{3j}{2})(2j)!}{2\Gamma(j+3)\Gamma(\frac{j}{2}+1)}, \tag{1.18}$$

and it has the following asymptotic behavior as $j \to \infty$:

$$f_{2j}^{(0)} = \frac{K_0 (2j)!}{u_c^{2j} j^{7/2}}(1 + \mathcal{O}(j^{-1})), \quad K_0 = \frac{1}{\sqrt{6\pi}}, \tag{1.19}$$

where u_c is the critical value defined in (1.17). □

From formula (1.18) we obtain the first several terms of the Taylor series of $F^{(0)}(u)$ at $u = 0$:

$$F^{(0)}(u) = 6u^2 + 216u^4 + 13,608u^6 + 1,119,744u^8 + \frac{540,416,448}{5}u^{10} + \mathcal{O}(u^{12}). \tag{1.20}$$

Our formula (1.18) for $f_{2j}^{(0)}$ coincides with the one obtained in the work [6, Section 4] in the planar diagrams approximation.

A more complicated expression, but still explicit, is available when $g=1$:

Theorem 1.3. The coefficient $f_{2j}^{(2)}$ can be written explicitly in terms of a ${}_3F_2$ hypergeometric function as follows:

$$f_{2j}^{(2)} = \frac{5\cdot 72^j \Gamma(\frac{3j}{2})(2j)!}{48(3j+2)\Gamma(j+1)\Gamma(\frac{j}{2}+1)}\, {}_3F_2\left(\begin{matrix} -j+1, 2, 6 \\ 5, -\frac{3j}{2}+1 \end{matrix} ; \frac{3}{2}\right), \tag{1.21}$$

and it has the following asymptotic behavior as $j\to\infty$:

$$f_{2j}^{(2)} = \frac{K_2 (2j)!}{u_c^{2j} j}(1+\mathcal{O}(j^{-1/2})), \quad K_2 = \frac{1}{48}. \tag{1.22}$$

□

We remark that the ${}_3F_2$ function in formula (1.21) can actually be written as a linear combination of two ${}_2F_1$ functions:

$$\begin{aligned} {}_3F_2\left(\begin{matrix} -j+1, 2, 6 \\ 5, -\frac{3j}{2}+1 \end{matrix} ; \frac{3}{2}\right) = \Bigg[{}_2F_1\left(\begin{matrix} -j+1, 2 \\ -\frac{3j}{2}+1 \end{matrix} ; \frac{3}{2}\right) \\ + \frac{6(j-1)}{5(3j-2)}\, {}_2F_1\left(\begin{matrix} -j+2, 3 \\ -\frac{3j}{2}+2 \end{matrix} ; \frac{3}{2}\right)\Bigg]. \end{aligned} \tag{1.23}$$

From formula (1.21), the first several terms of the Taylor series of $F^{(2)}(u)$ at $u=0$ are

$$F^{(2)}(u) = \frac{3}{2}u^2 + 189u^4 + 26,892u^6 + 4,076,568u^8 + \frac{3,213,210,384}{5}u^{10} + \mathcal{O}(u^{12}). \tag{1.24}$$

In the case of genus $g>1$, an explicit formula for the coefficients $f_{2j}^{(2g)}$ becomes complicated, but it is possible to obtain the asymptotic behavior of these coefficients when $j\to\infty$:

Theorem 1.4. For any $g>1$, the coefficient $f_{2j}^{(2g)}$ has the following asymptotic behavior as $j\to\infty$:

$$f_{2j}^{(2g)} = \frac{K_{2g}(2j)!\, j^{\frac{5g-7}{2}}}{u_c^{2j}}(1+\mathcal{O}(j^{-1/2})), \tag{1.25}$$

where

$$K_{2g}=\frac{6\cdot 3^{1/4}C_{2g}}{\Gamma(\frac{5g-1}{2})u_c^g}, \tag{1.26}$$

and C_{2g} satisfies the recurrence relation

$$C_{2g}=\frac{1}{2^{3/2}3^{5/4}}\left(\frac{(5g-6)(5g-4)C_{2g-2}}{48}+54\sum_{\substack{m+m'=g\\ m,m'\le g-1}}C_{2m}C_{2m'}\right) \tag{1.27}$$

for $g\geq 1$, with the initial value

$$C_0=-2^{-1/2}3^{-7/4}. \tag{1.28}$$

□

For $g=2$, we have that

$$F^{(4)}=\frac{8,505u^6}{2}+2,217,618u^8+\frac{3,905,028,468u^{10}}{5}+\mathcal{O}(u^{12}), \tag{1.29}$$

and the constants C_4 and K_4 are

$$C_4=\frac{49\cdot 2^{1/2}3^{3/4}}{17,915,904},\quad K_4=\frac{7}{1440\sqrt{6\pi}}. \tag{1.30}$$

We note that if we write the recursion (1.27) as follows:

$$C_{2g}=\mu(5g-6)(5g-4)C_{2g-2}+\nu\sum_{\substack{m+m'=g\\ m,m'\le g-1}}C_{2m}C_{2m'}, \tag{1.31}$$

and if we construct the following generating function (cf. [2])

$$y(t)=\sum_{g=0}^{\infty}C_{2g}t^{\frac{1-5g}{2}}, \tag{1.32}$$

then $y(t)$ satisfies the Painlevé I differential equation

$$y''(t)=\frac{y^2(t)}{8\mu}-\frac{C_0^2}{8\mu}t. \tag{1.33}$$

We can make the change of variables $t = -c\tau$ and $u = \lambda y$ to bring it to the standard form

$$u''(\tau) = 6u^2(\tau) + \tau, \tag{1.34}$$

see [12]. Explicitly,

$$c = 2^{-3/5}, \quad \lambda = 2^{3/10}3^{5/4}. \tag{1.35}$$

In the sequel, the precise structure of the coefficients $F^{(2g)}(u)$ in the asymptotic expansion of the free energy will be analyzed using the following steps:

(1) First, we apply the Deift–Zhou nonlinear steepest descent method to the associated Riemann–Hilbert problem (RHP), in order to obtain the large N asymptotic expansion of the recurrence coefficients γ_n^2 and β_n of the corresponding orthogonal polynomials $P_n(z)$, when the index n is of the order of N. The coefficients in this asymptotic expansion will be found by using the string equations, which are nonlinear algebraic relations for γ_n^2 and β_n. Simultaneously, this proves the existence of the orthogonal polynomials $P_n(z)$ when the index n is of the order of N and N is large enough.

(2) Then we make a change of variable to reduce the cubic polynomial $V(M) = \frac{M^2}{2} - uM^3$ to $\tilde{V}(M) = tM - \frac{M^3}{3}$, and we derive simple formulae for the partition function and the recurrence coefficients under this change of variable. The change of variable allows us to use the Toda equation that connects the second derivative of the free energy $\tilde{F}_N$ with respect to the parameter t to the recurrence coefficient γ_N^2.

(3) By integrating the large N asymptotic expansion for γ_N^2 term by term, we obtain detailed information about the large N asymptotic expansion of the free energy $F_N(u)$, and, in particular, the form of the different $F^{(2g)}(u)$ terms. Note that in this derivation we only need the recurrence coefficients γ_N^2 and β_N for large values of N, whose existence and asymptotic behavior are proved via the Riemann–Hilbert (RH) analysis before.

We note that an alternative method to derive formulae for $F^{(2g)}$ was proposed in [1], using the so-called loop equations (see [16] for a rigorous derivation of the loop equations). In [1], explicit expressions for $F^{(2)}$ and $F^{(4)}$ are given, in terms of elementary functions that depend on the endpoints of the support of the equilibrium measure associated with the corresponding potential. In [7, 19], a general heuristic formula for computing $F^{(g)}$ is presented for arbitrary potentials and multicut cases.

We would also like to bring attention to the very recent paper [18], which appeared after the present paper had been posted on arXiv, and which cited the present paper. In [18] a different approach, based on the so-called difference string equations, is developed for the calculation of the terms of the topological expansion in the random matrix model with a general potential $V(M)$, including the case when the dominant power in $V(z)$ is odd.

The rest of the paper is organized as follows: in Sections 2 and 3, we analyze the equilibrium measure and the corresponding RHP for orthogonal polynomials with respect to the cubic-type weight. In Section 4, we apply the string equations to the large N expansion of the recurrence coefficients γ_n^2 and β_n. This is used in Section 5 to obtain the large N expansion of the free energy $F_N(u)$, and this leads to the proof of Theorems 1.2 and Theorem 1.3 in Section 6. In Section 7, we prove Theorem 1.4, expanding the recurrence coefficients around the critical point $u=u_c$ and selecting the terms that give the dominant behavior in the free energy. Finally, Section 8 is devoted to the interpretation of the topological expansion in terms of connected graphs embedded in closed Riemann surfaces.

2 Equilibrium Measure and the RH Analysis

2.1 Construction

Let us denote by $\varrho(s)$ the density of the equilibrium measure for this problem, that we assume supported on a certain curve $J\subset\mathbb{C}$ with endpoints $z=a$ and $z=b$. We consider J oriented from a to b, and we take the $+$-side on the left of J and the $-$-side on the right of J, following the standard convention.

The resolvent

$$\omega(z)=\int_J \frac{\varrho(s)\,\mathrm{d}s}{z-s} \tag{2.1}$$

is analytic off J, and it satisfies the Euler–Lagrange equation on J,

$$\omega_+(z)+\omega_-(z)=V'(z), \qquad z\in J, \tag{2.2}$$

and the asymptotics at infinity,

$$\omega(z)=\frac{1}{z}+\mathcal{O}(z^{-2}), \qquad z\to\infty. \tag{2.3}$$

To solve (2.2), we write

$$\omega(z) = \tfrac{1}{2} V'(z) - \tfrac{1}{2}\sqrt{R(z)}\, h(z), \tag{2.4}$$

where $h(z)$ is an analytic function and

$$R(z) = (z-a)(z-b), \tag{2.5}$$

since we are assuming that we are in the one cut case. We take the principal sheet for $\sqrt{R(z)}$, with a cut on J. Due to the Plemelj formula, we have

$$\omega_-(z) - \omega_+(z) = 2\pi\mathrm{i}\,\varrho(z) = \sqrt{R(z)}h(z) \Rightarrow \varrho(z) = \frac{1}{2\pi\mathrm{i}}\sqrt{R(z)}h(z).$$

We easily deduce from (2.4) that $h(z)$ is a polynomial of degree 1. Moreover, if we write

$$a = x - y, \quad b = x + y \tag{2.6}$$

and $h(z) = A(z - z_0)$, then identifying the asymptotics at infinity on both sides of (2.4), we find

$$\begin{aligned} A &= -3u, \\ z_0 &= \frac{1}{3u} - x, \\ 2x^2 + y^2 &= \frac{2x}{3u}, \\ \frac{y^2}{4}(1-6ux) &= 1. \end{aligned} \tag{2.7}$$

Thus,

$$\omega(z) = \frac{z - 3uz^2}{2} - \frac{1}{2}\sqrt{(z-a)(z-b)}\,(1 - 3uz - 3ux) \tag{2.8}$$

and therefore

$$\varrho(z) = \frac{1}{2\pi\mathrm{i}}\sqrt{(z-a)(z-b)}\,(1 - 3uz - 3ux). \tag{2.9}$$

It follows that $\varrho(z)^2$ has a double zero at

$$z_0 = \frac{1}{3u} - x$$

and two simple roots located at $z_{1,2} = a, b$.

Next we will prove the following facts:

(1) For $0 \leq u < u_c$, where u_c is given by (1.17), the equilibrium measure is supported on an interval $J = [a, b]$ of the real axis. The endpoints of this interval are analytic functions of the parameter u.

(2) For $0 \leq u < u_c$, we can extend the interval J to an unbounded contour $\tilde{\Gamma}_0 = \gamma_1 \cup J \cup \gamma_2$ in the upper half-plane,

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z \geq 0\},$$

in such a way that

$$\begin{aligned} \phi_1(z) &> 0, \quad z \in \gamma_1, \\ \phi_2(z) &> 0, \quad z \in \gamma_2, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \phi_1(z) &= \frac{1}{2} \int_a^z \sqrt{R(s)}\, h(s)\, \mathrm{d}s, \\ \phi_2(z) &= \frac{1}{2} \int_b^z \sqrt{R(s)}\, h(s)\, \mathrm{d}s. \end{aligned} \tag{2.11}$$

Additionally, as $|z| \to \infty$ on γ_1 and γ_2, it is possible to estimate the growth of $\Re\phi_1(z)$ and $\Re\phi_2(z)$:

$$\Re\phi_i(z) = -\frac{uz^3}{3} + \mathcal{O}(z^2), \quad i = 1, 2. \tag{2.12}$$

This last result is relevant in the steepest descent method applied to the RHP, in order to show that the jump matrices outside of J tend to the identity when $N \to \infty$.

2.2 Support of the equilibrium measure

Combining the last two equations in (2.7), we obtain the following cubic equation for the variable x as a function of the parameter u:

$$18u^2x^3 - 9ux^2 + x - 6u = 0. \tag{2.13}$$

If $u > 0$ is small, then this equation has three solutions,

$$x_1 = 6u + \mathcal{O}(u^2), \quad x_2 = \frac{1}{6u} + \mathcal{O}(1), \quad x_3 = \frac{1}{3u} + \mathcal{O}(1). \tag{2.14}$$

In what follows, we will be interested in the first root, $x = x_1$, which is the solution that remains bounded when u is small. More terms in the expansion of x are

$$x = x_1(u) = 6u + 324u^3 + 31,104u^5 + \mathcal{O}(u^7). \tag{2.15}$$

From the last equation in (2.7), we find

$$y = 2 + 36u^2 + 2,916u^4 + \mathcal{O}(u^6). \tag{2.16}$$

The support of the equilibrium measure is the interval $[a, b] = [x - y, x + y]$. As $u \to 0$, it converges to $[-2, 2]$, which is the support of the equilibrium measure for equilibrium measure for the gaussian unitary ensemble.

The series that we obtain in (2.15) can be proved to be convergent for small values of u:

Proposition 2.1. The series (2.15) is convergent for $|u| < u_c$, where u_c is given by (1.17). □

Proof. Since by (2.14) the root $x = x_1(u)$ is isolated for small u, $x_1(u)$ is analytic at $u = 0$. The discriminant of cubic equation (2.13) is $\Delta = 9u^2(1 - 34,992u^4)$, and it vanishes at the critical value $u = u_c$. Hence it does not vanish for $0 < |u| < u_c$, and the root $x_1(u)$ remains isolated. This proves that $x_1(u)$ is analytic in the disk $|u| < u_c$, and hence the Taylor series for $x_1(u)$ is convergent in this disk. ■

It follows from the last equation of (2.7) that a similar result is valid for the variable $y = y(u) > 0$. Namely, $y(u)$ is obviously analytic at $u = 0$. Suppose that $1 - 6ux_1(u) = 0$, then from Equation (2.13) we obtain that $u = 0$, which contradicts $1 - 6ux_1(u) = 0$. Therefore,

$$1 - 6ux_1(u) \neq 0, \quad |u| < u_c, \tag{2.17}$$

hence $y(u)$ is analytic in the disk $|u| < u_c$. This shows that for real $0 \le u < u_c$, both $x = x_1(u)$ and $y(u) > 0$ are real, so the support of the equilibrium measure is the interval

$[a, b]$ on the real axis. Moreover, the endpoints $z = a$ and b are analytic functions of the parameter u.

In order to prove that we are in the one-cut regular case, we also need to show that the double root z_0 lies outside $[a, b]$:

Proposition 2.2. For $0 \le u < u_c$, we have that $z_0 > b$, so that the double root is outside the interval $[a, b]$. □

Proof. By the second equation in (2.7),

$$z_0 = \frac{1}{3u} - x \to +\infty,$$

as $u \to 0^+$, so $z_0 > b$ for small $u > 0$. Suppose that, for some $0 < u < u_c$, we have $z_0 = b$. Then

$$\frac{1}{3u} - x = x + y,$$

hence

$$\left(\frac{1}{3u} - 2x\right)^2 = y^2 = \frac{4}{1 - 6ux}$$

and

$$(1 - 6ux)^3 = 36u^2.$$

Denote $v = ux$. Then the latter equation and (2.13) give two cubic equations on v:

$$(1 - 6v)^3 - 36u^2 = 0, \quad 18v^3 - 9v^2 + v - 6u^2 = 0.$$

By applying the Euclidean algorithm to these two cubic equations, we obtain that $u^2(1 - 34,992u^4) = 0$, which is not true for $0 < u < u_c$. Therefore, $z_0 > b$ for $0 < u < u_c$. ■

When $u = u_c$, the support of the equilibrium measure will be the interval

$$[a, b] = [3^{3/4} - 3^{5/4}, 3^{3/4} + 3^{1/4}], \tag{2.18}$$

and the double root of the function $\varrho(z)^2$ is located at

$$z_c = 3^{3/4} + 3^{1/4}, \tag{2.19}$$

which coincides with the right endpoint of the support of the equilibrium measure.

We prove now that there exists an extension of the interval J to a curve Γ in $\mathbb{C}$ with right properties for the RH analysis.

Proposition 2.3. There exist two curves γ_1 and γ_2 in $\mathbb{C}$ such that conditions (2.10) hold true. □

Proof. If $b<z<z_0$, then $(z-a)(z-b)>0$ and $z-z_0<0$ again, so $\Re\phi_2(z)>0$ and we can take the segment (b,z_0) as part of γ_2. However, we cannot take the whole interval (b,∞) as γ_2, since $\Re\phi_2(z)<0$ if $z>z_0$. Nevertheless, if we define the quadratic differential $-Q(z)\,\mathrm{d}z^2$, where

$$Q(z)=\tfrac{1}{4}R(z)h^2(z), \tag{2.20}$$

then it follows from the general theory (see [24, Section 8.2] or [26, Theorem 7.1]) that, since $z=z_0$ is a double zero of $-Q(z)\,\mathrm{d}z^2$, then there are exactly four trajectories that emanate from $z=z_0$, along which $\Im Q(z)=0$. Locally, these trajectories leave the point $z=z_0$ with angles $k\pi/2$, $0\le k\le 3$. We choose the trajectory that leaves $z=z_0$ with angle $\pi/2$ into the upper half plane. This trajectory must go to ∞, since $\phi_2(z_0)>0$ and $\phi_2(z)$ is increasing on γ_2, so it cannot go back to the real axis. Now,

$$\begin{aligned}0<\phi_2(z)-\phi_2(z_0)&=\frac{1}{2}\int_{z_0}^{z}\sqrt{R(s)}h(s)\,\mathrm{d}s\\&=\int_{z_0}^{z}\left(\frac{V'(s)}{2}-\omega(s)\right)\mathrm{d}s\\&=\frac{V(z)-V(z_0)}{2}+\mathcal{O}(\ln|z|)=-\frac{uz^3}{3}+\mathcal{O}(z^2),\end{aligned} \tag{2.21}$$

when $|z|\to\infty$. Since $\phi_2(z)$ is real for $z\in\gamma_2$, we have $\Re\phi_2(z)=\phi_2(z)$, and

$$\Re\phi_2(z)=-\frac{uz^3}{3}+\mathcal{O}(z^2)>0. \tag{2.22}$$

This also shows that γ_2 goes to infinity with angle $\pi/3$.

We can take $\gamma_1=(-\infty,a)$, since for $z<a$ we have $(z-a)(z-b)>0$ and $z-z_0<0$, so

$$\frac{1}{2}\sqrt{R(z)}h(z)=\frac{3u}{2}\sqrt{(z-a)(z-b)}(z-z_0)<0, \tag{2.23}$$

and hence $\Re\,\phi_1(z)>0$ for $z<a$, after integrating. Note that on $(-\infty,a)$ we need to take the square root with a negative sign. On $(-\infty,a)$, with a similar computation as before,

we have

$$\Re\phi_1(z) = \phi_1(z) = -\frac{uz^3}{3} + \mathcal{O}(z^2), \tag{2.24}$$

and the result follows. ■

2.3 The RH analysis

We will consider monic orthogonal polynomials (OPs) $P_n(z) = z^n + \cdots$ with respect to the weight function

$$w(z) = \mathrm{e}^{-NV(z)}, \tag{2.25}$$

so that

$$\int_\Gamma P_n(z) z^k w(z)\,\mathrm{d}z = 0, \quad k = 0, 1, \ldots, n-1. \tag{2.26}$$

The existence of the OPs is not obvious, since the contour Γ is complex, but the RH analysis provides us, among other properties, with the existence and uniqueness of $P_n(z)$ for large enough values of n. The RH analysis also gives us various properties of the OPs. In particular, we will see that the OPs satisfy the three-term recurrence relation,

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n^2 P_{n-1}(z), \tag{2.27}$$

and that the recurrence coefficients β_n and γ_n^2 satisfy string and deformation equations.

It is known from the work of Fokas et al. [20], that the family of orthogonal polynomials can be characterized as the solution of the following 2×2 RHP:

Find a 2×2 matrix-valued $Y_n : \mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1) \to \mathbb{C}^{2\times 2}$ such that

(1) $Y_n(z)$ is analytic for $z \in \mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1)$ and the limits

$$\lim_{s \to z \pm 0} Y_n(s) = Y_{n\pm}(z)$$

exist as s approaches z from the $\pm$-side of $\Gamma_0 \cup \Gamma_1$. As usual, we assume that the $+$-side is on the left of an oriented contour.

(2) For $z \in \Gamma_0 \cup \Gamma_1$,

$$Y_{n+}(z) = Y_{n-}(z) \begin{pmatrix} 1 & \alpha(z)\,\mathrm{e}^{-NV(z)} \\ 0 & 1 \end{pmatrix}, \tag{2.28}$$

where

$$\alpha(z) = \alpha\chi_{\Gamma_0}(z) + (1-\alpha)\chi_{\Gamma_1}(z), \tag{2.29}$$

and $\chi_A(z)$ denotes the characteristic function of the set A.

(3) As $z \to \infty$,

$$Y_n(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{2.30}$$

We will call n the *degree* of the RHP. This RHP has a unique solution if and only if the monic polynomial $P_n(z)$, orthogonal with respect to the weight function $w(z)e^{-NV(z)}$, uniquely exists (see the next section). If additionally $P_{n-1}(z)$ uniquely exists, then the solution of the RHP is given by

$$Y_n(z) = \begin{pmatrix} P_n(z) & (\mathcal{C}P_n w)(z) \\ -\dfrac{2\pi i}{h_{n-1}} P_{n-1}(z) & -\dfrac{2\pi i}{h_{n-1}} (\mathcal{C}P_{n-1} w)(z) \end{pmatrix}, \tag{2.31}$$

where

$$(\mathcal{C}f)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s-z}\,ds \tag{2.32}$$

is the Cauchy transform on Γ, and the coefficient h_{n-1} is defined as

$$h_{n-1} = \int_\Gamma P_{n-1}^2(s) w(s)\,ds. \tag{2.33}$$

In the asymptotic analysis of the RHP, it is convenient to deform the RHP analytically to the contour $\tilde{\Gamma} = \alpha\tilde{\Gamma}_0 + (1-\alpha)\tilde{\Gamma}_1$, where $\tilde{\Gamma}_0$ is shown in Figure 2 and $\tilde{\Gamma}_1$ is the complex conjugation of $\tilde{\Gamma}_0$.

The nonlinear steepest descent method, due to Deift and Zhou (see [11] and also [10]), consists of a series of explicit and invertible transformations of this original RHP, with the objective of arriving at a transformed problem for a 2×2 matrix $R_n(z)$, where the matrix jumps tend to the identity uniformly in $\mathbb{C}$ as $n \to \infty$. These steps are briefly as follows:

(1) $Y_n \mapsto T_n$, normalization at ∞, using the equilibrium measure on J.

(2) $T_n \mapsto S_n$, opening of lenses around J to convert the highly oscillatory jump on J into a combination of exponentially close to identity and constant matrix jumps.

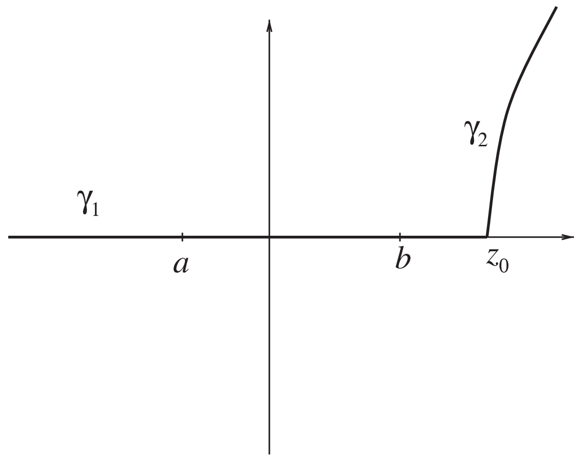

Fig. 2. The contour $\tilde{\Gamma}_0 = \gamma_1 \cup [a, b] \cup \gamma_2$.

(3) $S_n \mapsto R_n$, construction of global and local parametrices, which are (resp.) an approximation near the interval J (but away from the endpoints of J), and an approximation close to the endpoints of J. In this case, the local approximation near the endpoints $z = a$ and $z = b$ are given in terms of Airy functions.

(4) The 2×2 matrix-valued function $R_n(z)$ satisfies the RHP on some contours on the complex plane, with jump matrices approaching the unit matrix as $n \to \infty$. At this point, we use conditions (2.10) on the contour $\tilde{\Gamma}_0$, which are established in Proposition 2.3. A solution of the RHP for R_n is then obtained by a series of the perturbation theory.

We refer the reader to the works of [9, 10, 15] for the details. In this case, the RH analysis gives the following result:

Theorem 2.4. For any $\delta > 0$, there exist $N_0(\delta) > 0$ and $\varepsilon(\delta) > 0$ such that for any $u \in [0, u_c - \delta]$ and for any $N \geq N_0(\delta)$, a solution $Y_n(z)$ to the RHP (2.28)–(2.30) exists for n in the interval

$$I_N = \left\{ n \colon 1 - \varepsilon(\delta) \leq \frac{n}{N} < 1 + \varepsilon(\delta) \right\}. \tag{2.34}$$

□

One of the key results also given by the RH analysis is that the recurrence coefficients γ_n^2 and β_n admit an asymptotic expansion in inverse powers of N for large N. We will return to these expansions later, in Section 4, but first we need to discuss various properties of the orthogonal polynomials, which follow from the RHP.

3 Orthogonal Polynomials $P_n(z)$

The orthogonality conditions (2.26) are equivalent to a linear system of equations for the coefficients of the orthogonal polynomial

$$P_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0. \tag{3.1}$$

Namely, if

$$c_j = \int_\Gamma z^j w(z)\, \mathrm{d}z, \tag{3.2}$$

then

$$\sum_{j=0}^{n-1} c_{j+k} a_j = -c_{n+k}, \quad k = 0, 1, \ldots, n-1. \tag{3.3}$$

Therefore, the orthogonal polynomial $P_n(z)$ exists and it is unique if and only if linear system (3.3) is nondegenerate, so that

$$D_{n-1} = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} \neq 0. \tag{3.4}$$

In this case,

$$P_n(z) = \frac{D_n(z)}{D_{n-1}}, \tag{3.5}$$

where

$$D_n(z) = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & c_{2n-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{vmatrix}, \tag{3.6}$$

see [27]. In addition, we have the following proposition.

Proposition 3.1. If there exists a unique orthogonal polynomial $P_n(z)$, then

$$D_n = h_n D_{n-1}, \tag{3.7}$$

where

$$h_n = \int_\Gamma P_n(z)^2 w(z)\, dz. \tag{3.8}$$

□

For a proof of this and subsequent propositions of this section see Appendix 1 at the end of the paper. Another useful formula for D_{n-1} is

$$D_{n-1} = \frac{1}{n!}\int_\Gamma \cdots \int_\Gamma \Delta(z)^2 \prod_{j=1}^{n} e^{-NV(z_j)}\, dz_1 \cdots dz_n, \tag{3.9}$$

where

$$\Delta(z) = \prod_{1\le j<k\le n} (z_j - z_k)^2. \tag{3.10}$$

see [27].

Now we will relate the existence of a solution to RHP (2.28)–(2.30) to the existence and the uniqueness of orthogonal polynomials.

Proposition 3.2. Suppose that RHP (2.28)–(2.30) has a solution $Y_n(z)$. Then there exists a unique orthogonal polynomial $P_n(z)$. □

The following proposition proves the existence of the three-term recurrence relation:

Proposition 3.3. Suppose that RHP (2.28)–(2.30) has solutions $Y_{n-1}(z)$, $Y_n(z)$, and $Y_{n+1}(z)$ with degrees $n-1$, n, and $n+1$, respectively. Then the orthogonal polynomials $P_{n-1}(z)$, $P_n(z)$, and $P_{n+1}(z)$, which uniquely exist by Proposition 3.2, satisfy the three-term recurrence relation,

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n^2 P_{n-1}(z). \tag{3.11}$$

□

Observe that under the conditions of Proposition 3.3, $D_n \neq 0$, $D_{n-1} \neq 0$, and $D_{n-2} \neq 0$ and hence, by (3.7),

$$h_n = \frac{D_n}{D_{n-1}} \neq 0, \quad h_{n-1} = \frac{D_{n-1}}{D_{n-2}} \neq 0. \tag{3.12}$$

From (3.11) we obtain, by multiplying by $P_{n-1}(z)$ and integrating with respect to $w(z)\,dz$, that

$$\gamma_n^2 = \frac{h_n}{h_{n-1}} \neq 0. \tag{3.13}$$

Note that the proof of Propositions 3.1–3.3 does not use the specific form (2.25) of the weight $w(z)$, and therefore they hold in a much more general situation. In the two subsequent propositions, we assume that the weight $w(z)$ has the form (2.25).

Proposition 3.4. Suppose that RHP (2.28)–(2.30) has solutions $Y_j(z)$ with degrees $j = n, n \pm 1, n \pm 2$. Then the string equations hold:

$$\begin{aligned} 3u(\gamma_{n+1}^2 + \beta_n^2 + \gamma_n^2) &= \beta_n, \\ \gamma_n^2(1 - 3u(\beta_n + \beta_{n-1})) &= \frac{n}{N}. \end{aligned} \tag{3.14}$$

□

By Theorem 2.4, for any $\delta > 0$, Propositions 3.1–3.4 are applied for $u \in [0, u_c - \delta]$, provided $\{n, n \pm 1, n \pm 2\} \subset I_N = I_N(\delta)$. Observe that $I_N(\delta)$ does not depend on u.

Proposition 3.5. For any $\delta > 0$, if $\{n, n \pm 1, n \pm 2\} \subset I_N(\delta)$, then the coefficients β_n and γ_n^2 are C^∞-functions of u for $u \in [0, u_c - \delta]$ and they are analytic functions of u for $u \in (0, u_c - \delta]$. □

Remark. In what follows, we use the recurrence coefficient γ_n^2, which is uniquely defined, and we will not use γ_n, which is defined up to a sign. If needed, the sign of γ_n can be uniquely defined by the analytic continuation of γ_n in u starting from the value $\gamma_n(0) = \sqrt{n/N}$. Note that, in general, the recurrence coefficients β_n and γ_n^2 are complex-valued. □

4 String Equations and the Large N Expansion of the Recurrence Coefficients

4.1 The large N expansion of the recurrence coefficients

From the RH analysis, we know that γ_n^2 and β_n can be expanded in inverse powers of N, as functions of the parameter $s = n/N$. Moreover, in the expansion of γ_n^2 the odd coefficients vanish, and we have an expansion in inverse powers of N^{-2}. Similarly, for β_n we also have an expansion in inverse powers of N^{-2}, if we consider β_n as a function of the shifted parameter $s = \frac{n}{N} + \frac{1}{2N}$. Namely, we have the following result:

Theorem 4.1 (See [5]). For any $\delta > 0$, if $n \subset I_N(\delta/2)$, then the coefficients β_n and γ_n^2 can be expanded in uniform asymptotic series,

$$\begin{aligned} \gamma_n^2 &\sim \sum_{k=0}^{\infty} \frac{1}{N^{2k}} g_{2k}\left(\frac{n}{N}, u\right), \\ \beta_n &\sim \sum_{k=0}^{\infty} \frac{1}{N^{2k}} b_{2k}\left(\frac{n}{N} + \frac{1}{2N}, u\right), \end{aligned} \tag{4.1}$$

where the functions $g_{2k}(s,u)$, $b_{2k}(s,u)$, $k = 0, 1, \ldots$, do not depend on n and N, and they are C^∞-smooth in s and u on the set $\{0 \le u \le u_c - \delta, |s-1| \le \varepsilon(\delta)/2\}$. □

The proof of this theorem in [5] is based on the string equations and is applied to the current case without any changes. To find the functions $g_{2k}(s,u)$ and $b_{2k}(s,u)$ iteratively, we substitute expansions (4.1) into string equations (3.14) and equate coefficients that multiply equal powers of N^{-2}. Let us consider the first equation in (3.14). If we take $s = \frac{n}{N} + \frac{1}{2N}$, then from the first equation in (4.1) we have the asymptotic expansions

$$\begin{aligned} \gamma_n^2 &\sim \sum_{k=0}^{\infty} \frac{1}{N^{2k}} g_{2k}\left(s - \frac{1}{2N}, u\right), \\ \gamma_{n+1}^2 &\sim \sum_{k=0}^{\infty} \frac{1}{N^{2k}} g_{2k}\left(s + \frac{1}{2N}, u\right), \quad s = \frac{n}{N} + \frac{1}{2N}. \end{aligned} \tag{4.2}$$

We substitute these expansions into the first equation in (3.14) and expand functions the g_{2k} into Taylor series at s:

$$3u\left[2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{g_{2k}^{(2j)}(s,u)}{(2j)!2^{2j}N^{2k+2j}} + \left(\sum_{k=0}^{\infty} \frac{b_{2k}(s,u)}{N^{2k}}\right)^2\right] = \sum_{k=0}^{\infty} \frac{b_{2k}(s,u)}{N^{2k}}, \tag{4.3}$$

where

$$g_{2k}^{(2j)}(s,u) = \frac{\partial^{2j} g_{2k}(s,u)}{\partial s^{2j}}. \tag{4.4}$$

By equating the coefficients at the same powers of N^{-2} on both sides, we obtain a series of equations for the functions g_{2k}, b_{2k}.

Similarly, if we take $s = \frac{n}{N}$, then from the second equation in (4.1) we obtain that

$$\left(\sum_{k=0}^{\infty} \frac{g_{2k}(s,u)}{N^{2k}}\right)\left[1 - 6u\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{b_{2k}^{(2j)}(s,u)}{(2j)!2^{2j}N^{2k+2j}}\right)\right] = s. \tag{4.5}$$

Let us analyze the equations obtained when we equate the coefficients at the same powers of N^{-2} on both sides of Equations (4.3) and (4.5).

4.2 Zeroth order terms in the recurrence coefficients

At the zeroth order in N^{-2} we obtain the system of equations,

$$\begin{aligned} &3u[2g_0(s,u) + b_0(s,u)^2] = b_0(s,u), \\ &g_0(s,u)[1 - 6ub_0(s,u)] = s. \end{aligned} \tag{4.6}$$

These equations can be solved to yield

$$\begin{aligned} &72u^2 g_0(s,u)^3 - g_0(s,u)^2 + s^2 = 0, \\ &b_0(s,u) = \frac{g_0(s,u) - s}{6ug_0(s,u)} \end{aligned} \tag{4.7}$$

or alternatively,

$$\begin{aligned} &18b_0(s,u)^3u^2 - 9b_0(s,u)^2u + b_0(s,u) - 6su = 0, \\ &g_0(s,u) = \frac{s}{1 - 6ub_0(s,u)}. \end{aligned} \tag{4.8}$$

Consider s in a small neighborhood of the point $s = 1$ on the complex plane, say, $|s-1| \le c \ll 1$. Then the cubic equation (4.7) admits three different solutions, which behave as follows when $u \to 0$:

$$g_0^{(1)}(s,u) = s + \mathcal{O}(u^2),$$
$$g_0^{(2)}(s,u) = -s + \mathcal{O}(u^2), \tag{4.9}$$
$$g_0^{(3)}(s,u) = \frac{1}{72u^2} + \mathcal{O}(1).$$

The solution we are interested in is $g_0(s,u) = g_0^{(1)}(s,u)$. Since $g_0^{(1)}(s,u)$ is an isolated solution of the cubic equation, it is analytic at $u=0$. In fact, it is analytic in the disk $|u| < u_c s^{-1/2}$ on the complex plane. In the following proposition, we explicitly evaluate the Taylor series of $g_0(s,u) = g_0^{(1)}(s,u)$ at $u=0$.

Proposition 4.2. The Taylor series

$$g_0(s,u) = \sum_{j=0}^{\infty} a_{2j}(s) u^{2j} \tag{4.10}$$

converges in the disk $|u| < u_c s^{-1/2}$ and its coefficients are

$$a_{2j}(s) = \frac{72^j s^{j+1} \Gamma(\frac{3j+1}{2})}{2\Gamma(j+1)\Gamma(\frac{j+3}{2})}, \qquad j \geq 0. \tag{4.11}$$

□

Proof. We make the change of variables,

$$w = su^2, \quad v = u^2 g_0(s,u). \tag{4.12}$$

Then $v = v(w)$ satisfies the cubic equation

$$72v^3 - v^2 + w^2 = 0, \tag{4.13}$$

and we are looking for the root $v = v(w)$ such that $v(0) = 0$ and $v(w) > 0$ for small $w > 0$. Observe that $v(w)$ admits an expansion in powers of w,

$$v(w) = \sum_{j=1}^{\infty} c_j w^j. \tag{4.14}$$

Using Cauchy integral formula, if γ denotes a smooth closed contour around $w=0$, we have

$$c_j = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{v(w)}{w^{j+1}}\, \mathrm{d}w. \tag{4.15}$$

Since $v(0)=0$, we make a change of variables

$$c_j = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{v}{w(v)^{j+1}} \frac{\mathrm{d}w}{\mathrm{d}v}\, \mathrm{d}v. \tag{4.16}$$

We have

$$w(v) = \frac{v}{s}\sqrt{1-72v}, \qquad \frac{\mathrm{d}w}{\mathrm{d}v} = \frac{1-108v}{\sqrt{1-72v}}, \tag{4.17}$$

so

$$c_j = \frac{1}{2\pi \mathrm{i}} \oint_\gamma v^{-j}(1-108v)(1-v)^{-\frac{j}{2}-1}\, \mathrm{d}v. \tag{4.18}$$

We expand the binomial series:

$$\begin{aligned}(1-108v)(1-v)^{-\frac{j}{2}-1} &= (1-108v)\sum_{k=0}^{\infty} \binom{-\frac{j}{2}-1}{k} (-1)^k (72v)^k \\ &= (1-108v)\sum_{k=0}^{\infty} \binom{k+\frac{j}{2}}{k} (72v)^k,\end{aligned} \tag{4.19}$$

and we pick up the residues, that correspond to $k=j-1$ in the first term and $k=j-2$ in the second one. Then

$$c_j = 72^{j-1}\left[\binom{\frac{3j}{2}-1}{j-1} - \frac{3}{2}\binom{\frac{3j}{2}-2}{j-2}\right] = \frac{\Gamma(\frac{3j}{2}-1)72^{j-1}}{2\Gamma(j)\Gamma(\frac{j}{2}+1)}. \tag{4.20}$$

Now,

$$\begin{aligned}g_0(s,u) &= \frac{s}{w}v(w) = \frac{s}{w}\sum_{j=1}^{\infty} c_j w^j = s\sum_{j=0}^{\infty} c_{j+1} w^j \\ &= \sum_{j=0}^{\infty} \frac{72^j s^{j+1}\Gamma(\frac{3j+1}{2})u^{2j}}{2\Gamma(j+1)\Gamma(\frac{j+3}{2})},\end{aligned} \tag{4.21}$$

which gives the expression for the coefficients $a_{2j}(s)$. ■

It is easy to generate several coefficients using the general formula (4.11):

$$g_0(s,u) = s + 36s^2u^2 + 3,240s^3u^4 + 373,248s^4u^6 + 48,498,912s^5u^8 + \mathcal{O}(u^{10}). \tag{4.22}$$

As a consequence of Proposition 4.2, we have the following.

Corollary 4.3. The coefficient $b_0(s,u)$ is an analytic function of u for $|u| < u_c s^{-1/2}$. □

Proof. From the second equation in (4.7) and formula (4.22), it is clear that $b_0(s,u)$ is analytic at $u=0$. In addition, $g_0(s,u) \neq 0$ for $|u| < u_c s^{-1/2}$, because if $g_0(s,u)=0$, then, by the first equation in (4.7), we would have $s=0$, but we assume that $|s-1| \leq c \ll 1$. Therefore, the analyticity of $g_0(s,u)$ implies the analyticity of $b_0(s,u)$ in the disk $|u| < u_c s^{-1/2}$. ■

To obtain the asymptotics of γ_N^2 and β_N as $N \to \infty$, we set $n=N$ in (4.1), so that

$$\gamma_N^2 \sim g_0(1,u) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} g_{2k}(1,u), \tag{4.23}$$

$$\beta_N \sim b_0\left(1+\frac{1}{2N}, u\right) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} b_{2k}\left(1+\frac{1}{2N}, u\right). \tag{4.24}$$

In the sequel, we denote

$$g_{2k}(u) = g_{2k}(1,u), \quad b_{2k}(u) = b_{2k}(1,u)$$

for brevity. When $s=1$, the cubic equation (4.7) for $g_0(u)$ becomes

$$72u^2 g_0(u)^3 - g_0(u)^2 + 1 = 0, \tag{4.25}$$

and the solution admits the power-series expansion

$$g_0(u) = \sum_{j=0}^{\infty} \frac{72^j \Gamma(\frac{3j+1}{2})\, u^{2j}}{2\Gamma(j+1)\Gamma(\frac{j+3}{2})}, \tag{4.26}$$

valid when $|u| < u_c$. Formula (4.22) reduces for $s = 1$ to

$$g_0(u) = 1 + 36u^2 + 3,240u^4 + 373,248u^6 + 48,498,912u^8 + \mathcal{O}(u^{10}). \tag{4.27}$$

4.3 **Analysis of higher-order terms**

An important consequence of string equations (4.3) and (4.5) is that higher order terms $g_{2k} = g_{2k}(s, u)$ and $b_{2k} = b_{2k}(s, u)$, $k \geq 1$, can be expressed in terms of $g_0(s, u)$, $b_0(s, u)$, and their derivatives with respect to s. More precisely:

(1) We can solve for g_{2k} and b_{2k} in terms of the previous coefficients through a *linear* system of two equations.

(2) The analysis of the determinant of this linear system shows that higher-order terms have the same singularity $u = \pm u_c$ as the initial terms g_0 and b_0, and that no other singularities appear when increasing k. In this respect, this property is analogous to the one presented in [14, 15, 17] for general even potentials $V(M)$.

(3) Although an explicit formula for g_{2k} and b_{2k} is rather complex when $k \geq 2$, in order to determine the large j asymptotic behavior of the coefficients $f_{2j}^{(2g)}$ in the expansion of the free energy, it is enough to evaluate the leading order term at the singular points $u = \pm u_c$, and this can be extracted from the string equations.

For convenience, we make the change of variable

$$z = \frac{\xi}{u}, \tag{4.28}$$

so that the potential $V(z)$ becomes

$$\hat{V}(\xi) = \frac{1}{u^2}\left(\frac{\xi^2}{2} - \xi^3\right). \tag{4.29}$$

With this change of variable, we introduce a new family of orthogonal polynomials

$$\hat{P}_n(\xi) = u^n P_n\left(\frac{\xi}{u}\right), \tag{4.30}$$

and the corresponding recurrence coefficients are

$$\hat{\beta}_n = u\beta_n, \quad \hat{\gamma}_n^2 = u^2\gamma_n^2. \tag{4.31}$$

Now the string equations (3.14) read

$$\begin{aligned} 3(\hat{\gamma}_{n+1}^2 + \hat{\beta}_n^2 + \hat{\gamma}_n^2) &= \hat{\beta}_n, \\ \hat{\gamma}_n^2(1 - 3(\hat{\beta}_n + \hat{\beta}_{n-1})) &= \frac{nu^2}{N}. \end{aligned} \tag{4.32}$$

We will use the same scaled parameter as before,

$$w = su^2. \tag{4.33}$$

From (4.1) and (4.32), we have an asymptotic expansion for these new coefficients in powers of $\frac{u^2}{N}$:

$$\begin{aligned} \hat{\gamma}_n^2 &\sim \sum_{k=0}^{\infty} \frac{u^{4k}}{N^{2k}} \hat{g}_{2k}\left(\frac{nu^2}{N}\right), \\ \hat{\beta}_n &\sim \sum_{k=0}^{\infty} \frac{u^{4k}}{N^{2k}} \hat{b}_{2k}\left(\frac{nu^2}{N} + \frac{u^2}{2N}\right), \end{aligned} \tag{4.34}$$

where

$$\hat{g}_{2k}(w) = u^{-4k+2} g_{2k}(s, u), \quad \hat{b}_{2k}(w) = u^{-4k+1} b_{2k}(s, u). \tag{4.35}$$

For $s = 1$ this reduces to

$$\hat{g}_{2k}(u^2) = u^{-4k+2} g_{2k}(u), \quad \hat{b}_{2k}(u^2) = u^{-4k+1} b_{2k}(u), \tag{4.36}$$

and, in particular, when $k = 0$, we have

$$\hat{g}_0(w) = u^2 g_0(u), \quad \hat{b}_0(w) = u b_0(u). \tag{4.37}$$

It follows from (4.7) that

$$\hat{b}_0(w) = \frac{\hat{g}_0(w) - w}{6\hat{g}_0(w)} \tag{4.38}$$

and also that $\hat{g}_0(w)$ satisfies the following cubic equation:

$$72\hat{g}_0^3(w) - \hat{g}_0^2(w) + w^2 = 0, \tag{4.39}$$

which is identical to (4.13). The critical value now becomes

$$w_c = u_c{}^2 = \frac{\sqrt{3}}{324}. \tag{4.40}$$

By (4.14) and (4.20), the function $\hat{g}_0(w)$ is analytic at $w = 0$, and it has the following Taylor expansion:

$$\hat{g}_0(w) = \sum_{j=1}^{\infty} \frac{\Gamma(\frac{3j}{2} - 1)72^{j-1}w^j}{2\Gamma(j)\Gamma(\frac{j}{2} + 1)}. \tag{4.41}$$

From Equations (4.3) and (4.5), we obtain the string equations for $\hat{g}_{2k}(w)$ and $\hat{b}_{2k}(w)$:

$$6 \sum_{m+j=k} \frac{\hat{g}_{2m}^{(2j)}(w)}{(2j)!2^{2j}} + 3 \sum_{m+m'=k} \hat{b}_{2m}(w)\hat{b}_{2m'}(w) = \hat{b}_{2k}(w) \tag{4.42}$$

and

$$\hat{g}_{2k}(w) - 6 \sum_{m+m'+j=k} \frac{\hat{g}_{2m}(w)\hat{b}_{2m'}^{(2j)}(w)}{(2j)!2^{2j}} = 0, \quad k \geq 1. \tag{4.43}$$

The advantage of Equations (4.38)–(4.43) is that they do not contain the parameter u anymore: it is hidden in the argument $w = su^2$.

By solving these equations for $\hat{g}_{2k}(w)$ and $\hat{b}_{2k}(w)$, we obtain the following linear system of equations for $k \geq 1$:

$$6\hat{g}_{2k}(w) + (6\hat{b}_0(w) - 1)\hat{b}_{2k}(w) = -6 \sum_{\substack{m+j=k \\ m \leq k-1}} \frac{\hat{g}_{2m}^{(2j)}(w)}{(2j)!2^{2j}} - 3 \sum_{\substack{m+m'=k \\ m,m' \leq k-1}} \hat{b}_{2m}(w)\hat{b}_{2m'}(w) \tag{4.44}$$

and

$$(1 - 6\hat{b}_0(w))\hat{g}_{2k}(w) - 6\hat{g}_0(w)\hat{b}_{2k}(w) = 6 \sum_{\substack{m+m'+j=k \\ m,m' \leq k-1}} \frac{\hat{g}_{2m}(w)\hat{b}_{2m'}^{(2j)}(w)}{(2j)!2^{2j}}. \tag{4.45}$$

The determinant of this system is

$$D(w) = -36\hat{g}_0(w) + (1 - 6\hat{b}_0(w))^2. \tag{4.46}$$

By using Equations (4.38) and (4.39), it can be simplified to

$$D(w) = 1 - 108\hat{g}_0(w). \tag{4.47}$$

This leads to the following result:

Proposition 4.4. For $k \geq 1$, the functions $\hat{g}_{2k}(w)$ and $\hat{b}_{2k}(w)$ are analytic functions of w in the disk $|w| < w_c$ in the complex plane. □

Proof. The proof of this proposition is by induction in k, if we show first that the determinant $D(w)$ does not vanish for $|w| < w_c$.

Suppose $D(w) = 0$ for some w such that $|w| < w_c$. Then by (4.47), we have

$$\hat{g}_0(w) = \frac{1}{108}. \tag{4.48}$$

Substituting this into the cubic equation (4.39) we obtain that

$$w^2 = \frac{1}{3 \cdot 108^2} = {w_c}^2, \tag{4.49}$$

hence $D(w) \neq 0$ inside the disk $|w| < w_c$.

The proposition now follows, since it is clear that the only singularities of the higher-order terms for $|w| < w_c$ are those inherited from $\hat{g}_0(w)$ and $\hat{b}_0(w)$, but $\hat{g}_0(w)$ and $\hat{b}_0(w)$ are analytic in the disk $|w| < w_c$. ■

As a consequence of Proposition 4.4, we have the following result:

Corollary 4.5. The functions $g_{2k}(s, u)$ and $b_{2k}(s, u)$ are analytic in the disk $|u| < u_c\, s^{-1/2}$ in the complex plane. Moreover, $g_{2k}(s, u)$ is even in u and $b_{2k}(s, u)$ is odd in u. □

Proof. Analyticity follows from Proposition 4.4 and the change of variable $w = su^2$, and the parity, from rewriting Equation (4.35):

$$g_{2k}(s, u) = u^{4k-2}\hat{g}_{2k}(w), \quad b_{2k}(s, u) = u^{4k-1}\hat{b}_{2k}(w). \tag{4.50}$$

■

5 Large N Expansion of the Free Energy F_N

5.1 Another change of variable and analysis of the auxiliary potential

To evaluate the large N expansion of the free energy F_N, we will use the Toda equation, which relates F_N to the recurrence coefficient γ_N^2. To this end, we will consider a different form of the partition function. Under the change of variable

$$z = (3u)^{-1/3}\zeta + \frac{1}{6u}, \tag{5.1}$$

where we assume that $u > 0$ and $(3u)^{-1/3} > 0$, we have, by straightforward algebra, that

$$\frac{z^2}{2} - uz^3 - \frac{1}{108u^2} = -\frac{\zeta^3}{3} + t\zeta, \tag{5.2}$$

where

$$t = \frac{1}{4(3u)^{4/3}}. \tag{5.3}$$

Let us denote

$$\tilde{Z}_N = \int_{\tilde{\Gamma}} \cdots \int_{\tilde{\Gamma}} \prod_{1\le j<k\le N} (\zeta_j - \zeta_k)^2 \prod_{j=1}^{N} \mathrm{e}^{-N\tilde{V}(\zeta_j)} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_N, \tag{5.4}$$

where

$$\tilde{V}(\zeta) = -\frac{\zeta^3}{3} + t\zeta, \tag{5.5}$$

and $\tilde{\Gamma}$ is the image of Γ under the change of variable (5.1). We observe that this cubic model was used in [8] with $t = 0$, in the context of complex Gaussian quadrature of integrals with high order stationary points. The corresponding free energy is

$$\tilde{F}_N = \frac{1}{N^2} \ln \tilde{Z}_N. \tag{5.6}$$

Formula (5.2) implies that

$$\tilde{F}_N = \frac{1}{108u^2} + \frac{\ln(3u)}{3} + F_N = \frac{2\,t^{3/2}}{3} - \frac{\ln(4t)}{4} + F_N. \tag{5.7}$$

By Theorem 1.1, for any $\varepsilon > 0$, $F_N(u)$ admits a uniform asymptotic expansion for $0 \le u \le u_c - \varepsilon$, hence, for any $\varepsilon > 0$, $\tilde{F}_N(t)$ admits a uniform asymptotic expansion as well, for

$t_c + \varepsilon \le t < \infty$:

$$\tilde{F}_N(t) \sim \sum_{k=0}^{\infty} \frac{\tilde{F}^{(2k)}(t)}{N^{2g}}, \tag{5.8}$$

where

$$\tilde{F}^{(0)}(t) = \frac{1}{108u^2} + \frac{\ln(3u)}{3} + F^{(0)}(u) = \frac{2\,t^{3/2}}{3} - \frac{\ln(4t)}{4} + F^{(0)}(u) \tag{5.9}$$

and for $k \ge 1$,

$$\tilde{F}^{(2k)}(t) = F^{(2k)}(u), \quad u = \frac{1}{3(4t)^{3/4}}. \tag{5.10}$$

From (1.17) and (5.3) we obtain that the new critical value is

$$t_c = \frac{1}{4(3u_c)^{4/3}} = 3 \cdot 2^{-2/3}. \tag{5.11}$$

Note that with the change of variable (5.1), the family of orthogonal polynomials is scaled and shifted. If we write $z = c\zeta + d$, then:

$$P_n(z) = P_n(c\zeta + d) = c^n \tilde{P}_n(\zeta), \tag{5.12}$$

so that $\tilde{P}_n(\zeta)$ is again monic. The coefficients of the recurrence relation are modified in the following way:

$$\tilde{\beta}_n = \frac{\beta_n - d}{c}, \quad \tilde{\gamma}_n^2 = \frac{\gamma_n^2}{c^2}. \tag{5.13}$$

In our case, according to (4.40), $c = (3u)^{-1/3}$ and $d = \frac{1}{6u}$, so

$$\begin{aligned} \tilde{\gamma}_n^2 &= (3u)^{2/3}\gamma_n^2 = \frac{\gamma_n^2}{2\sqrt{t}}, \\ \tilde{\beta}_n &= (3u)^{1/3}\left(\beta_n - \frac{1}{6u}\right) = \frac{1}{(4t)^{1/4}}\left(\beta_n - \frac{1}{6u}\right). \end{aligned} \tag{5.14}$$

An important fact in our analysis will be that with respect to the variable t we have the following Toda equation (see, e.g., [5]):

$$\frac{\mathrm{d}^2 \tilde{F}_N}{\mathrm{d}t^2} = \tilde{\gamma}_N^2. \tag{5.15}$$

The usual derivation of the Toda equation assumes the existence of the orthogonal polynomials $\tilde{P}_n(\zeta)$ for $n = 0, 1, \ldots, N$. In our case, we know the existence of the orthogonal polynomials $P_n(z)$, and hence $\tilde{P}_n(\zeta)$ only for $n \in I_N$. We will prove that, nevertheless, the Toda equation is valid in our case. Namely, we will prove the following proposition:

Proposition 5.1. Toda equation (5.15) holds for any $t > t_c$. □

Proof. Let N be fixed. Then, as $u \to 0$, the complex-valued measure

$$e^{-N(\frac{z^2}{2} - uz^3)}\, dz$$

on $\Gamma = \alpha\Gamma_0 + (1-\alpha)\Gamma_1$ converges to the Gaussian measure, and the moments of the measure converge to the moments of the Gaussian measure. This implies that there exists $u(N) > 0$ such that for $u \in [0, u(N)]$ the orthogonal polynomials $P_n(z)$ exist for $n = 0, 1, \ldots, N$. Hence the orthogonal polynomials $\tilde{P}_n(\zeta)$ exist for $n = 0, 1, \ldots, N$, if $t \geq t(N)$, where

$$t(N) = \frac{1}{4[3u(N)]^{4/3}} \tag{5.16}$$

cf. (5.10). This implies, by the usual argument (see, e.g., [5]), that Toda equation (5.15) is valid for $t \geq t(N)$. Since both the free energy $\tilde{F}_N$ and the recurrence coefficient γ_N^2 are analytic in t for $t > t_c$, we obtain, by the analytic continuation, that Toda equation (5.15) is valid for $t > t_c$. ■

The idea now is to integrate twice equation (5.15) in the variable t, in order to obtain $\tilde{F}_N$, and then compute F_N using Equation (5.7). From the asymptotic expansion (4.23), we have

$$\tilde{\gamma}_N^2(t) \sim \sum_{k=0}^{\infty} \frac{\tilde{g}_{2k}(t)}{N^{2k}} = \frac{1}{2\sqrt{t}} \sum_{k=0}^{\infty} \frac{g_{2k}(u)}{N^{2k}}, \quad u = \frac{1}{3(4t)^{3/4}}. \tag{5.17}$$

Integrating this expression twice should give an expansion for $\tilde{F}_N$, and then we can compute F_N using (5.7). However, the problem now is that we have to justify that the term-by-term integration of the large N asymptotic expansion of $\tilde{\gamma}_N$ over an unbounded interval in the variable t is permissible. This is the content of the following proposition.

Proposition 5.2. We have that

$$\tilde{F}^{(0)}(t) = \frac{2\,t^{3/2}}{3} - \frac{\ln(4t)}{4} + \int_{\infty}^{t}\int_{\infty}^{\tau} \left(\tilde{g}_0(\sigma) - \frac{1}{2\sqrt{\sigma}} - \frac{1}{4\sigma^2}\right) d\sigma\, d\tau, \tag{5.18}$$

and, for any $k \geq 1$,

$$\tilde{F}^{(2k)}(t) = \int_{\infty}^{t} \int_{\infty}^{\tau} \tilde{g}_{2k}(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau. \tag{5.19}$$

□

Proof. Let us prove first that, for any $k \geq 0$,

$$\frac{\mathrm{d}^2 \tilde{F}^{(2k)}(t)}{\mathrm{d}t^2} = \tilde{g}_{2k}(t). \tag{5.20}$$

Consider the difference operator

$$\Delta_h f(t) = f(t+h) - 2f(t) + f(t-h), \tag{5.21}$$

where $h > 0$ and t are fixed. From the Taylor theorem with an integral remainder, we have

$$\Delta_h f(t) = \int_{t-h}^{t+h} f''(\tau) m_h(\tau, t)\, \mathrm{d}\tau, \tag{5.22}$$

where $m_h(\tau, t)$ is the hat-shaped function

$$m_h(\tau, t) = \begin{cases} h - |\tau - t|, & |\tau - t| \leq h, \\ 0, & \text{otherwise}. \end{cases} \tag{5.23}$$

It is clear that $m_h(\tau, t) \geq 0$ for $\tau \in [t-h, t+h]$. Now, for any $K > 0$, $\varepsilon > 0$, $1 \geq h > 0$, and $t \geq t_c + 1 + \varepsilon$,

$$\begin{aligned} \Delta_h \tilde{F}_N(t) &= \int_{t-h}^{t+h} \tilde{F}_N''(\tau) m_h(\tau, t)\, \mathrm{d}\tau \\ &= \int_{t-h}^{t+h} \tilde{\gamma}_N(\tau) m_h(\tau, t)\, \mathrm{d}\tau \\ &= \sum_{k=0}^{K} \int_{t-h}^{t+h} \frac{\tilde{g}_{2k}(\tau)}{N^{2k}} m_h(\tau, t)\, \mathrm{d}\tau + \mathcal{O}(N^{-2K-2}), \end{aligned} \tag{5.24}$$

where the error term is uniform in $t \geq t_c + 1 + \varepsilon$, since the asymptotic expansion (5.17) is uniform in t. On the other hand,

$$\Delta_h \tilde{F}_N(t) = \sum_{k=0}^{K} \frac{\Delta_h \tilde{F}^{(2k)}(t)}{N^{2k}} + \mathcal{O}(N^{-2K-2}), \tag{5.25}$$

where the error term is uniform in $t \geq t_c + 1 + \varepsilon$, because the asymptotic expansion (5.8) is uniform in t. Since the coefficients of an asymptotic series in powers of N^{-2} for the function $\Delta_h \tilde{F}_N(t)$ are uniquely determined, we obtain from (5.24), (5.25) that, for any $k \geq 0$,

$$\Delta_h \tilde{F}^{(2k)}(t) = \int_{t-h}^{t+h} \tilde{g}_{2k}(\tau) m_h(\tau, t)\, \mathrm{d}\tau. \tag{5.26}$$

But by (5.22),

$$\Delta_h \tilde{F}^{(2k)}(t) = \int_{t-h}^{t+h} \frac{\mathrm{d}^2 \tilde{F}^{(2k)}(\tau)}{\mathrm{d}\tau^2}\, m_h(\tau, t)\, \mathrm{d}\tau, \tag{5.27}$$

hence

$$\int_{t-h}^{t+h} \frac{\mathrm{d}^2 \tilde{F}^{(2k)}(\tau)}{\mathrm{d}\tau^2}\, m_h(\tau, t)\, \mathrm{d}\tau = \int_{t-h}^{t+h} \tilde{g}_{2k}(\tau) m_h(\tau, t)\, \mathrm{d}\tau, \tag{5.28}$$

and since h and t are arbitrary and $m_h(\tau, t)$ is a positive function, Equation (5.20) holds.

By Corollary 4.5, the function $g_{2k}(u)$ is an even function of u, analytic at $u = 0$, hence as $u \to 0$,

$$g_{2k}(u) = a_0(k) + a_2(k) u^2 + \mathcal{O}(u^4). \tag{5.29}$$

Because of (5.17), we know that

$$\tilde{g}_{2k}(t) = \frac{g_{2k}(u)}{2\sqrt{t}}, \qquad k \geq 0, \tag{5.30}$$

and this implies that as $t \to \infty$,

$$\tilde{g}_{2k}(t) = \frac{g_{2k}(u)}{2\sqrt{t}} = \frac{a_0(k)}{2\sqrt{t}} + \frac{a_1(k)}{72 t^2} + \mathcal{O}(t^{-7/2}). \tag{5.31}$$

Integrating twice we obtain that

$$\tilde{F}^{(2k)}(t) = \frac{4 a_0(k) t^{3/2}}{3} - \frac{a_1(k) \ln t}{72} + C + Dt + R_{2k}(t), \tag{5.32}$$

where

$$R_{2k}(t) = \mathcal{O}(t^{-3/2}) \tag{5.33}$$

and C and D are some unknown constants. The error term can be written as

$$R_{2k}(t) = \int_\infty^t \int_\infty^\tau \left(\tilde{g}_{2k}(\sigma) - \frac{a_0(k)}{2\sqrt{\sigma}} - \frac{a_1(k)}{72\sigma^2} \right) \mathrm{d}\sigma\, \mathrm{d}\tau. \tag{5.34}$$

Combining (5.32), for $k=0$ with (5.9), we obtain

$$\begin{aligned} \tilde{F}^{(0)}(t) &= \frac{2\,t^{3/2}}{3} - \frac{\ln(4t)}{4} + F^{(0)}(u) \\ &= \frac{4a_0(0)t^{3/2}}{3} - \frac{a_1(0)\ln t}{72} + C + Dt + R_0(t). \end{aligned} \tag{5.35}$$

Since

$$\lim_{u\to 0} F^{(0)}(u) = 0, \quad \lim_{t\to\infty} R_0(t) = 0, \tag{5.36}$$

we conclude from (5.35), by taking $t\to\infty$, that

$$a_0(0) = \frac{1}{2} \quad a_1(0) = 18, \quad C = -\frac{\ln 2}{2}, \quad D = 0, \tag{5.37}$$

and (5.18) follows from (5.34) and (5.35).

Similarly, combining (5.32) for $k \geq 1$ with (5.10) we obtain that

$$\begin{aligned} \tilde{F}^{(2k)}(t) &= F^{(2k)}(u) \\ &= \frac{4a_0(0)t^{3/2}}{3} - \frac{a_1(0)\ln t}{72} + C + Dt + R_{2k}(t). \end{aligned} \tag{5.38}$$

Again, since

$$\lim_{u\to 0} F^{(2k)}(u) = 0, \quad \lim_{t\to\infty} R_{2k}(t) = 0, \tag{5.39}$$

we conclude from (5.38), by taking $t\to\infty$, that

$$a_0(k) = 0 \quad a_1(k) = 0, \quad C = 0, \quad D = 0, \tag{5.40}$$

and (5.19) follows from (5.34) and (5.38). ■

We remark that from (5.29) and (5.40), we obtain that if $k \geq 1$, then

$$g_{2k}(u) = \mathcal{O}(u^4), \quad u \to 0. \tag{5.41}$$

Also, from (5.9), (5.10), (5.18), and (5.19), we have the following corollary of Proposition 5.2.

Corollary 5.3.

$$F^{(0)}(u) = \int_\infty^t \int_\infty^\tau \left(\tilde{g}_0(\sigma) - \frac{1}{2\sqrt{\sigma}} - \frac{1}{4\sigma^2} \right) \mathrm{d}\sigma\, \mathrm{d}\tau \tag{5.42}$$

and

$$F^{(2k)}(u) = \int_\infty^t \int_\infty^\tau \tilde{g}_{2k}(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau, \quad t = \frac{1}{4(3u)^{4/3}}; \quad k \geq 1. \tag{5.43}$$

□

Using this result, we will prove Theorem 1.2, by integrating the explicit expression that we found for the leading coefficient $g_0(u)$. Also, we will prove Theorem 1.3 by using an expression for $\hat{g}_2(w)$ obtained from the string equations.

6 Proof of Theorems 1.2 and 1.3

Observe that, from (5.3), we have

$$72u^2 = t^{-3/2}, \tag{6.1}$$

so (4.26) implies that

$$g_0(u) = \sum_{j=0}^{\infty} \frac{72^j \Gamma(\frac{3j+1}{2}) u^{2j}}{2\Gamma(j+1)\Gamma(\frac{j+3}{2})} = \sum_{j=0}^{\infty} \frac{\Gamma(\frac{3j+1}{2}) t^{-3j/2}}{2\Gamma(j+1)\Gamma(\frac{j+3}{2})}. \tag{6.2}$$

Multiplying by the factor $(3u)^{2/3} = \frac{1}{2\sqrt{t}}$, we obtain

$$\tilde{g}_0(t) = \frac{1}{2\sqrt{t}} + \frac{1}{4t^2} + \sum_{j=2}^{\infty} \frac{\Gamma(\frac{3j+1}{2}) t^{-\frac{3j+1}{2}}}{4\Gamma(j+1)\Gamma(\frac{j+3}{2})}. \tag{6.3}$$

Integrating twice in t, according to formula (5.42), we obtain

$$F^{(0)}(u) = \sum_{j=2}^{\infty} \frac{\Gamma(\frac{3j+1}{2}) t^{-\frac{3j-3}{2}}}{(3j-1)(3j-3)\Gamma(j+1)\Gamma(\frac{j+3}{2})}, \tag{6.4}$$

and making the change of variable $t=\frac{1}{4(3u)^{4/3}}$, we have

$$F^{(0)}(u)=\sum_{j=2}^{\infty}\frac{3^{2(j-1)}4^{\frac{3j-3}{2}}\Gamma(\frac{3j+1}{2})u^{2(j-1)}}{(3j-1)(3j-3)\Gamma(j+1)\Gamma(\frac{j+3}{2})}$$

$$=\sum_{j=1}^{\infty}\frac{72^{j}\Gamma(\frac{3j}{2})u^{2j}}{2\Gamma(j+3)\Gamma(\frac{j}{2}+1)}. \tag{6.5}$$

This proves formula (1.18).

From the explicit expression of the coefficients $f_{2j}^{(0)}$ in formula (1.18) we obtain the asymptotic formula,

$$f_{2j}^{(0)}=\frac{(2j)!}{\sqrt{6\pi}\,u_c^{2j}\,j^{7/2}}\left(1-\frac{115}{36j}+\frac{19705}{2592j^2}+\mathcal{O}(j^{-3})\right),\quad j\to\infty, \tag{6.6}$$

which gives (1.19), together with some higher-order corrections. Theorem 1.2 is proved.

When $k=1$, system (4.44)–(4.45) reads

$$\begin{pmatrix} 6 & 6\hat{b}_0(w)-1 \\ 1-6\hat{b}_0(w) & -6\hat{g}_0(w)\end{pmatrix}\begin{pmatrix}\hat{g}_2(w)\\ \hat{b}_2(w)\end{pmatrix}=\frac{3}{4}\begin{pmatrix}-\hat{g}_0''(w)\\ \hat{g}_0(w)\hat{b}_0''(w)\end{pmatrix}. \tag{6.7}$$

This system can be solved (and simplified, using the string equations) to give

$$\hat{g}_2(w)=\frac{162\hat{g}_0(w)(5-324\hat{g}_0(w))}{(1-108\hat{g}_0(w))^4}, \tag{6.8}$$

$$\hat{b}_2(w)=\frac{54w}{\hat{g}_0(w)(1-108\hat{g}_0(w))^4}. \tag{6.9}$$

An explicit expression (1.21) for the coefficients $f_{2j}^{(2)}$ can now be obtained from Equation (6.8). Namely, $\hat{g}_2(w)$ can be expanded in powers of w around the origin,

$$\hat{g}_2(w)=\sum_{j=2}^{\infty}c_j w^j, \tag{6.10}$$

starting with a w^2 term. From cubic equation (4.39), we obtain

$$w=\hat{g}_0(w)\sqrt{1-72\hat{g}_0(w)}, \tag{6.11}$$

so writing $v = \hat{g}_0(w)$ and applying the Cauchy integral formula, we find

$$
\begin{aligned}
c_j &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{(5-324v)v^{-j}(1-72v)^{\frac{-j-1}{2}}}{162(1-108v)^4} \left(\sqrt{1-72v} - \frac{36v}{\sqrt{1-72v}}\right) \mathrm{d}v \\
&= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{(5-324v)v^{-j}(1-72v)^{-\frac{j}{2}}}{162(1-108v)^3} \,\mathrm{d}v,
\end{aligned}
\tag{6.12}
$$

where γ is a smooth, closed contour around the origin in the v plane. Now we expand the binomial series and pick up the residue at $v = 0$, to obtain

$$
c_j = 162 \cdot 72^{j-1} \sum_{m=0}^{j-1} \binom{\frac{3j}{2} - m - 1}{j - m - 1} (m+1)(m+5) \left(\frac{3}{2}\right)^m. \tag{6.13}
$$

If we write this in terms of the standard Pochhammer symbol, see [12], we see that

$$
(m+1)(m+5) = 5\,\frac{(2)_m (6)_m}{(1)_m (5)_m}, \tag{6.14}
$$

and also

$$
\binom{\frac{3j}{2} - m - 1}{j - m - 1} = \binom{\frac{3j}{2} - 1}{j - 1} \frac{(-j+1)_m}{(-\frac{3j}{2}+1)_m}, \tag{6.15}
$$

so we can identify c_j with the following hypergeometric function:

$$
c_j = \frac{45 \cdot 72^j}{4} \binom{\frac{3j}{2} - 1}{j - 1} {}_3F_2 \left(\begin{matrix} -j+1, 2, 6 \\ 5, -\frac{3j}{2}+1 \end{matrix} ; \frac{3}{2} \right). \tag{6.16}
$$

Since one of the parameters in the numerator equals another one in the denominator plus one, we can simplify this ${}_3F_2$ function in terms of Gauss hypergeometric functions using the following straightforward identity:

$$
{}_3F_2 \left(\begin{matrix} a, b, c+1 \\ c, d \end{matrix} ; z \right) = {}_2F_1 \left(\begin{matrix} a, b \\ d \end{matrix} ; z \right) + \frac{abz}{cd} \, {}_2F_1 \left(\begin{matrix} a+1, b+1 \\ d+1 \end{matrix} ; z \right), \tag{6.17}
$$

and thus

$$c_j = \frac{45}{4} 72^j \binom{\frac{3j}{2}-1}{j-1} \times \left[{}_2F_1\left(\begin{matrix} -j+1, 2 \\ -\frac{3j}{2}+1 \end{matrix}; \frac{3}{2} \right) + \frac{6(j-1)}{5(3j-2)} {}_2F_1\left(\begin{matrix} -j+2, 3 \\ -\frac{3j}{2}+2 \end{matrix}; \frac{3}{2} \right) \right]. \tag{6.18}$$

Next, observe from (4.35) and (4.36) that

$$\tilde{g}_2(t) = \frac{1}{2\sqrt{t}} g_2(u), \quad g_2(u) = u^2 \hat{g}_2(w), \tag{6.19}$$

so

$$\begin{aligned} \tilde{g}_2(t) &= \frac{1}{2\sqrt{t}} g_2(u) = \frac{1}{144t^2} \hat{g}_2(w) = \frac{1}{144t^2} \hat{g}_2\left(\frac{1}{72t^{3/2}}\right) \\ &= \frac{1}{144t^2} \sum_{j=1}^{\infty} c_j \left(\frac{1}{72t^{3/2}}\right)^j. \end{aligned} \tag{6.20}$$

Using formula (5.43), we integrate twice in t, and going back to the variable u, we find

$$F^{(2)}(u) = \frac{1}{36} \sum_{j=1}^{\infty} \frac{c_j u^{2j}}{3j(3j+2)}, \tag{6.21}$$

so the coefficients are

$$\begin{aligned} f_{2j}^{(2)} &= \frac{c_j (2j)!}{108j(3j+2)} \\ &= \frac{45 \cdot 72^j}{4} \binom{\frac{3j}{2}-1}{j-1} {}_3F_2\left(\begin{matrix} -j+1, 2, 6 \\ 5, -\frac{3j}{2}+1 \end{matrix}; \frac{3}{2} \right) \frac{(2j)!}{108j(3j+2)} \\ &= \frac{5 \cdot 72^j \Gamma(\frac{3j}{2})(2j)!}{48(3j+2)\Gamma(j+1)\Gamma(\frac{j}{2}+1)} {}_3F_2\left(\begin{matrix} -j+1, 2, 6 \\ 5, -\frac{3j}{2}+1 \end{matrix}; \frac{3}{2} \right). \end{aligned} \tag{6.22}$$

This proves formula (1.21). The large j asymptotic behavior of these coefficients is given in Appendix 2, proving formula (1.22).

7 Case of Genus $g>1$

The expressions for the terms in the topological expansion become more and more complicated as k grows. Fortunately, as we pointed out before, in order to understand the large j behavior of the terms $f_{2j}^{(2g)}$, it is enough to consider the leading order terms near the critical point $w=w_c$ only. We recall that $w_c=\frac{\sqrt{3}}{324}$, as given in (4.40).

When $w=w_c$, the cubic equation $72\hat{g}_0^3(w)-\hat{g}_0^2(w)+w^2=0$ given in (4.39) has three roots,

$$r_1=r_2=\frac{1}{108},\quad r_3=-\frac{1}{216}. \tag{7.1}$$

Substituting

$$w=w_c-\Delta w,\quad \hat{g}_0(w)=\frac{1}{108}+\Delta\hat{g}_0(w) \tag{7.2}$$

into the cubic equation, we have

$$72(\Delta\hat{g}_0(w))^2\left(\hat{g}_0(w)+\frac{1}{216}\right)={w_c}^2-w^2=\Delta w(w_c+w). \tag{7.3}$$

Therefore,

$$\begin{aligned}\Delta\hat{g}_0&=\pm\sqrt{\frac{(w+w_c)\Delta w}{72(\hat{g}_0(w)+\frac{1}{216})}}=\pm\sqrt{\frac{2w_c\Delta w}{72(\hat{g}_0(w_c)+\frac{1}{216})}}+\mathcal{O}(\Delta w)\\&=\pm\frac{2^{1/2}3^{1/4}}{18}(\Delta w)^{1/2}+\mathcal{O}(\Delta w).\end{aligned} \tag{7.4}$$

So, near the critical value $w=w_c$, our solution satisfies

$$\hat{g}_0(w)=\frac{1}{108}-\frac{2^{1/2}3^{1/4}}{18}(\Delta w)^{1/2}+\mathcal{O}(\Delta w) \tag{7.5}$$

taking the minus sign since the function $\hat{g}_0(w)$ is increasing with w and therefore $\hat{g}_0(w)<\hat{g}(w_c)$.

Since we have $\hat{b}_0(w)$ explicitly in terms of $\hat{g}_0(w)$, see formula (4.38), we can obtain the asymptotic behavior of $\hat{b}_0(w)$ as well:

$$\hat{b}_0(w)=\frac{3-\sqrt{3}}{18}-2^{1/2}3^{-1/4}(\Delta w)^{1/2}+\mathcal{O}(\Delta w). \tag{7.6}$$

We also note that near the critical point $w=w_c$, the determinant $D(w)$ in (4.47) behaves as follows:

$$D(w)=2^{3/2}3^{5/4}(\Delta w)^{1/2}+\mathcal{O}(\Delta w)=D'(\Delta w)^{1/2}+\mathcal{O}(\Delta w). \tag{7.7}$$

We can write formulae (7.5) and (7.6) as

$$\begin{aligned}\hat{g}_0(w)&=\hat{g}_0(w_c)+C_0(\Delta w)^{1/2}+\mathcal{O}(\Delta w),\\ \hat{b}_0(w)&=\hat{b}_0(w_c)+D_0(\Delta w)^{1/2}+\mathcal{O}(\Delta w),\end{aligned} \tag{7.8}$$

where

$$\hat{g}_0(w_c)=\frac{1}{108}\quad \hat{b}_0(w_c)=\frac{3-\sqrt{3}}{18} \tag{7.9}$$

and

$$C_0=-\frac{2^{1/2}3^{1/4}}{18}\quad D_0=-2^{1/2}3^{-1/4}. \tag{7.10}$$

Observe that

$$D_0=6\sqrt{3}\,C_0. \tag{7.11}$$

For higher-order terms, $k\geq 1$, we make the following Ansatz:

$$\begin{aligned}\hat{g}_{2k}(w)&=C_{2k}(\Delta w)^{\frac{1}{2}-\frac{5k}{2}}+\mathcal{O}((\Delta w)^{1-\frac{5k}{2}}),\\ \hat{b}_{2k}(w)&=D_{2k}(\Delta w)^{\frac{1}{2}-\frac{5k}{2}}+\mathcal{O}((\Delta w)^{1-\frac{5k}{2}}),\end{aligned} \tag{7.12}$$

where the functions $\hat{g}_{2k}(w)$ and $\hat{b}_{2k}(w)$ have a square root singularity in the complex plane at the point $w=w_c$. We will prove this Ansatz by induction with respect to k, by using the system of Equations (4.44)–(4.45). Simultaneously, we will derive recurrence equations for C_{2k} and D_{2k}.

Let us analyze the first sum on the right-hand side of (4.44). By differentiating the first formula in (7.12), we obtain

$$\hat{g}_{2m}^{(2j)}(w)=C_{2m}^{(2j)}(\Delta w)^{\frac{1}{2}-\frac{5m}{2}-2j}+\cdots,\qquad m\leq k-1, \tag{7.13}$$

where the dots indicate higher-order terms with respect to Δw and

$$C_{2m}^{(2j)}=C_{2m}\left(\frac{5m}{2}-\frac{1}{2}\right)\cdots\left(\frac{5m}{2}+(2j-1)-\frac{1}{2}\right). \tag{7.14}$$

Since $m+j=k$, we can write formula (7.13) as

$$\hat{g}_{2m}^{(2j)}(w)=C_{2m}^{(2j)}(\Delta w)^{\frac{1}{2}-\frac{5k}{2}+\frac{j}{2}}+\cdots, \qquad j\geq 1. \tag{7.15}$$

Hence the leading term in the first sum on the right-hand side of (4.44) corresponds to $j=1$, $m=k-1$, so

$$-6\sum_{m+j=k;\ m\leq k-1}\frac{\hat{g}_{2m}^{(2j)}(w)}{(2j)!2^{2j}}=-\frac{3C_{2k-2}}{4}\left(\frac{5(k-1)}{2}-\frac{1}{2}\right)\left(\frac{5(k-1)}{2}+\frac{1}{2}\right)(\Delta w)^{1-\frac{5k}{2}}+\cdots$$
$$=-\frac{3C_{2k-2}(5k-6)(5k-4)}{16}(\Delta w)^{1-\frac{5k}{2}}+\cdots. \tag{7.16}$$

In the second sum, we have

$$-3\sum_{\substack{m+m'=k\\ m,m'\leq k-1}}\hat{b}_{2m}(w)\hat{b}_{2m'}(w)=-3\sum_{\substack{m+m'=k\\ m,m'\leq k-1}}D_{2m}D_{2m'}(\Delta w)^{1-\frac{5k}{2}}+\cdots. \tag{7.17}$$

For $k=1$ this sum is equal to 0. Substituting formulae (7.16), (7.17) into (4.44), we obtain

$$6\hat{g}_{2k}(w)-(1-6\hat{b}_0(w))\hat{b}_{2k}(w)=A_{2k}(\Delta w)^{1-\frac{5k}{2}}+\cdots, \tag{7.18}$$

where

$$A_{2k}=-\frac{3C_{2k-2}(5k-6)(5k-4)}{16}-3\sum_{m+m'=k;\ m,m'\leq k-1}D_{2m}D_{2m'}. \tag{7.19}$$

Similarly, in equation (4.45) the leading singular term on the right-hand side, with $(\Delta w)^{1-\frac{5k}{2}}$, corresponds to $j=1,\ m=0,\ m'=k-1$ and to $j=0,\ m+m'=k$, so that

$$(1-6\hat{b}_0(w))\hat{g}_{2k}(w)-6\hat{g}_0(w)\hat{b}_{2k}(w)=B_{2k}(\Delta w)^{1-\frac{5k}{2}}+\cdots, \tag{7.20}$$

where

$$B_{2k}=\frac{D_{2k-2}(5k-6)(5k-4)}{576}+6\sum_{m+m'=k;\ m,m'\leq k-1}C_{2m}D_{2m'}. \tag{7.21}$$

Solving systems (7.18) and (7.20) and using asymptotic formula (7.7) for its determinant, we obtain that

$$\hat{g}_{2k}(w) = C_{2k}(\Delta w)^{\frac{1}{2}-\frac{5k}{2}} + \mathcal{O}((\Delta w)^{1-\frac{5k}{2}}),$$
$$\hat{b}_{2k}(w) = D_{2k}(\Delta w)^{\frac{1}{2}-\frac{5k}{2}} + \mathcal{O}((\Delta w)^{1-\frac{5k}{2}}), \tag{7.22}$$

where

$$C_{2k} = \frac{1}{2^{3/2}3^{5/4}}[-6\hat{g}_0(w_c)A_{2k} + (1-6\hat{b}_0(w_c))B_{2k}],$$
$$D_{2k} = \frac{1}{2^{3/2}3^{5/4}}[-(1-6\hat{b}_0(w_c))A_{2k} + 6B_{2k}]. \tag{7.23}$$

This proves Ansatz (7.12) and gives the recurrence formula for the coefficients C_{2k} and D_{2k}. Moreover, substituting formulae (7.9), we obtain that

$$C_{2k} = \frac{1}{2^{3/2}3^{5/4}}\left(-\frac{A_{2k}}{18} + \frac{\sqrt{3}\,B_{2k}}{3}\right),$$
$$D_{2k} = \frac{1}{2^{3/2}3^{5/4}}\left(-\frac{\sqrt{3}\,A_{2k}}{3} + 6B_{2k}\right). \tag{7.24}$$

Hence

$$D_{2k} = 6\sqrt{3}\,C_{2k}, \tag{7.25}$$

which extends relation (7.11) to $k \geq 1$. Using this relation, we can eliminate D_{2m} from formulae (7.19), (7.21):

$$A_{2k} = -\frac{3C_{2k-2}(5k-6)(5k-4)}{16} - 324 \sum_{m+m'=k;\, m,m'\leq k-1} C_{2m}C_{2m'},$$
$$B_{2k} = \frac{\sqrt{3}C_{2k-2}(5k-6)(5k-4)}{96} + 36\sqrt{3} \sum_{m+m'=k;\, m,m'\leq k-1} C_{2m}C_{2m'}, \tag{7.26}$$

and hence by the first formula in (7.24),

$$C_{2k} = \frac{1}{2^{3/2}3^{5/4}}\left(\frac{(5k-6)(5k-4)C_{2k-2}}{48} + 54 \sum_{\substack{m+m'=k\\ m,m'\leq k-1}} C_{2m}C_{2m'}\right). \tag{7.27}$$

This proves recursive formula (1.27).

From (7.10) and (1.27), we find the values of the constant C_{2k} for $k=0,1,2$ as

$$C_0 = -2^{-1/2}3^{-7/4}, \quad C_2 = \frac{1}{48\cdot 108}, \quad C_4 = \frac{49\cdot 2^{1/2}3^{3/4}}{17,915,904}. \tag{7.28}$$

Let us apply formula (7.12) to find the asymptotic behavior of $F^{(2k)}(u)$ at $u=u_c$ for $k\geq 1$. From (4.36) and (5.30), we know that

$$\tilde{g}_{2k}(t) = \frac{g_{2k}(u)}{2\sqrt{t}} = \frac{u^{4k-2}\hat{g}_{2k}(w)}{2\sqrt{t}} = \frac{36t\hat{g}_{2k}(w)}{(72^2t^3)^k}, \tag{7.29}$$

and then we obtain from (7.22) that

$$\tilde{g}_{2k}(t) = \tilde{C}_{2k}(\Delta t)^{\frac{1}{2}-\frac{5k}{2}} + \cdots, \tag{7.30}$$

where

$$\tilde{C}_{2k} = \frac{36t_c C_{2k}}{(72^2t_c^3)^k}\left(-\frac{dw}{dt}\bigg|_{t=t_c}\right)^{\frac{1}{2}-\frac{5k}{2}}. \tag{7.31}$$

If we integrate (7.30) twice with respect to Δt, we obtain

$$\tilde{F}^{(2k)}(t) = \frac{4\tilde{C}_{2k}}{(5k-5)(5k-3)}(\Delta t)^{\frac{5}{2}-\frac{5}{2}k} + \cdots, \tag{7.32}$$

if $k\geq 2$. By (5.10) we obtain now that

$$F^{(2k)}(u) = A_{2k}\left(1-\frac{u^2}{u_c^2}\right)^{\frac{5}{2}-\frac{5}{2}k} + \cdots, \quad k\geq 2, \tag{7.33}$$

where

$$A_{2k} = \frac{4\tilde{C}_{2k}u_c^{5-5k}}{(5k-5)(5k-3)}\left(-\frac{dt}{dw}\bigg|_{w=w_c}\right)^{\frac{5}{2}-\frac{5}{2}k}. \tag{7.34}$$

Observe that by (6.1),

$$w = \frac{1}{72t^{3/2}}, \quad t_c = 3\cdot 2^{-2/3}, \quad -\frac{dw}{dt}\bigg|_{t=t_c} = \frac{1}{48t_c^{5/2}}, \tag{7.35}$$

hence

$$A_{2k} = \frac{144\cdot 48^2 t_c^6 u_c^5}{(5k-5)(5k-3)}\left(\frac{1}{72^2t_c^3u_c^5}\right)^k C_{2k}. \tag{7.36}$$

Substituting expression (7.31) and simplifying, we obtain

$$A_{2k} = \frac{24 \cdot 3^{1/4} C_{2k}}{(5k-5)(5k-3)u_c^k}. \tag{7.37}$$

Using the binomial expansion, we have

$$\left(1 - \frac{u^2}{u_c^2}\right)^{\frac{5}{2}-\frac{5}{2}k} = \sum_{j=0}^{\infty} \frac{c_j u^{2j}}{u_c^{2j}}, \tag{7.38}$$

where c_j has the following asymptotic behavior as $j \to \infty$:

$$c_j = \frac{j^{(5k-7)/2}}{\Gamma(\frac{5k-5}{2})}(1 + \mathcal{O}(j^{-1})). \tag{7.39}$$

Combining (7.33), (7.37)–(7.39), we find

$$F^{(2k)}(u) = \sum_{j=1}^{\infty} d_{2j}^{(2k)} \frac{u^{2j}}{u_c^{2j}}, \tag{7.40}$$

where as $j \to \infty$,

$$d_{2j}^{(2k)} = K_{2k} j^{(5k-7)/2}(1 + \mathcal{O}(j^{-1/2})) \tag{7.41}$$

with

$$K_{2k} = \frac{6 \cdot 3^{1/4} C_{2k}}{\Gamma(\frac{5k-1}{2})u_c^k}. \tag{7.42}$$

This proves Theorem 1.4.

The initial values of K_{2k} are

$$K_0 = \frac{1}{\sqrt{6\pi}}, \quad K_2 = \frac{1}{48}, \quad K_4 = \frac{7}{1440\sqrt{6\pi}}. \tag{7.43}$$

When $k=1$, formula (7.33) becomes

$$F^{(2)}(u) = -\frac{1}{48} \ln\left(1 - \frac{u^2}{u_c^2}\right) + \cdots = \sum_{j=0}^{\infty} \frac{c_j u^{2j}}{u_c^{2j}}, \tag{7.44}$$

where

$$c_j = \frac{1}{48j}(1 + \mathcal{O}(j^{-1/2})). \tag{7.45}$$

8 Counting 3-Valent Graphs on a Riemannian Surface

In this final section, we prove that the coefficient $f_p^{(2g)}$ in series (1.16) is equal to the number of 3-valent graphs with p vertices on a closed Riemannian surface of genus g. By differentiating formula (1.5) p times with respect to u and evaluating at the point $u=0$, we obtain

$$\frac{Z_N^{(p)}(0)}{Z_N(0)} = N^p \left\langle \left(\sum_{i=1}^{N} z_i^3 \right)^p \right\rangle_0 , \tag{8.1}$$

where

$$\langle f(z_1, \dots, z_N) \rangle_0 = \frac{1}{Z_N(0)} \int_\Gamma \cdots \int_\Gamma f(z_1, \dots, z_N) \times \prod_{1\le j<k\le N} (z_j - z_k)^2 \prod_{j=1}^{N} \mathrm{e}^{-\frac{N z_j^2}{2}} \, \mathrm{d}z_1 \cdots \mathrm{d}z_N \tag{8.2}$$

is the mathematical expectation with respect to the Gaussian ensemble. By the Cauchy theorem, we can deform the contour of integration Γ in the integral on the right in (8.1) to the real axis, and then we can return to the matrix integral:

$$\left\langle \left(\sum_{i=1}^{N} z_i^3 \right)^p \right\rangle_0 = \langle (\mathrm{Tr}\, M^3)^p \rangle_0 = \frac{1}{\tilde{Z}_N(0)} \int_{\mathcal{H}_N} (\mathrm{Tr}\, M^3)^p \, \mathrm{e}^{-N \mathrm{Tr} \frac{M^2}{2}} \, \mathrm{d}M. \tag{8.3}$$

For odd p the latter integral is equal to 0. To evaluate the integral for even $p=2q$, we apply the Wick theorem.

We have

$$\mathrm{Tr}\, M^3 = \sum_{i,j,k=1}^{N} M_{ij} M_{jk} M_{ki} \tag{8.4}$$

and

$$(\mathrm{Tr}\, M^3)^p = \sum_{i_1, j_1, k_1, \dots, i_p, j_p, k_p = 1}^{N} M_{i_1 j_1} M_{j_1 k_1} M_{k_1 i_1} \cdots M_{i_p j_p} M_{j_p k_p} M_{k_p i_p}. \tag{8.5}$$

The covariance matrix of M_{ij}'s is

$$\langle M_{ij} M_{kl} \rangle_0 = \frac{\delta_{il} \delta_{jk}}{N}. \tag{8.6}$$

By the Wick theorem, the expectation of the product of matrix entries with respect to the Gaussian measure can be expressed in terms of the product of expectations of all possible pairings of the matrix entries (see for instance [21, Section 1.6] for a more detailed exposition). In this case,

$$\langle M_{i_1 j_1} M_{j_1 k_1} M_{k_1 i_1} \cdots M_{i_p j_p} M_{j_p k_p} M_{k_p i_p} \rangle_0$$
$$= \sum_{\pi} \prod_{s=1}^{3p/2} \langle M_{i^{(s)} j^{(s)}} M_{k^{(s)} l^{(s)}} \rangle_0 = \sum_{\pi} N^{-3p/2} \prod_{s=1}^{3p/2} \delta_{i^{(s)} l^{(s)}} \delta_{j^{(s)} k^{(s)}}, \tag{8.7}$$

where the sum is taken over all partitions

$$\pi : \{M_{i_1 j_1}, M_{j_1 k_1}, M_{k_1 i_1}, \ldots, M_{i_p j_p}, M_{j_p k_p}, M_{k_p i_p}\} = \bigsqcup_{s=1}^{3p/2} \{M_{i^{(s)} j^{(s)}}, M_{k^{(s)} l^{(s)}}\} \tag{8.8}$$

of the set $\{M_{i_1 j_1}, M_{j_1 k_1}, M_{k_1 i_1}, \ldots, M_{i_p j_p}, M_{j_p k_p}, M_{k_p i_p}\}$ into disjoint pairs $\{M_{i^{(s)} j^{(s)}}, M_{k^{(s)} l^{(s)}}\}$.

From (8.5) and (8.7) we have that

$$\langle (\mathrm{Tr}\, M^3)^p \rangle = \sum_{\pi} \sum_{i_1, j_1, k_1, \ldots, i_p, j_p, k_p = 1}^{N} N^{-3p/2} \prod_{s=1}^{3p/2} \delta_{i^{(s)} l^{(s)}} \delta_{j^{(s)} k^{(s)}}$$
$$= \sum_{\pi} N^{f - 3p/2}, \tag{8.9}$$

where f is the number of cycles in the set

$$\{M_{i_1 j_1}, M_{j_1 k_1}, M_{k_1 i_1}, \ldots, M_{i_p j_p}, M_{j_p k_p}, M_{k_p i_p}\}$$

generated by the equalities

$$i^{(s)} = l^{(s)}, \quad j^{(s)} = k^{(s)}, \quad s = 1, \ldots, 3p/2. \tag{8.10}$$

Thus,

$$\frac{Z_N^{(p)}(0)}{Z_N(0)} = \sum_{\pi} N^{p+f-l}, \quad l = \frac{3p}{2}. \tag{8.11}$$

By the second Wick theorem, we now obtain that

$$F_N^{(p)}(0) = \frac{1}{N^2} \ln \frac{Z_N^{(p)}(0)}{Z_N(0)} = \sum_{\pi}^{c} N^{p+f-l-2}, \tag{8.12}$$

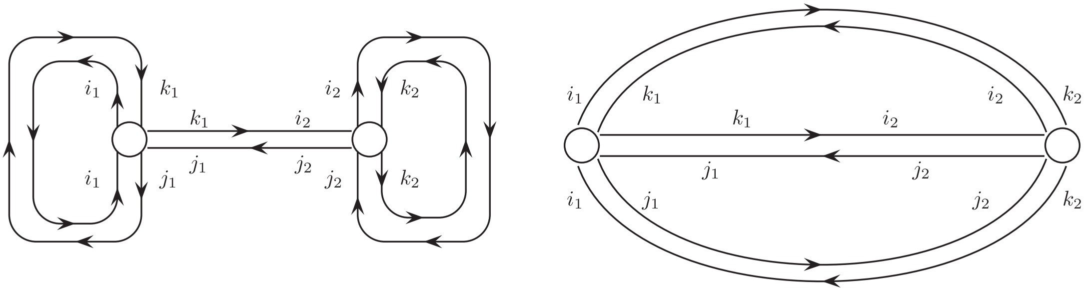

Fig. 3. 3-valent connected ribbon graphs with two vertices, genus 0.

where the sum is taken over partitions π such that the graph $\Gamma(\pi)$ is connected (for a general formulation of this result in combinatorics see for instance [25, Vol. 2]).

From (1.15) and (1.16) we have

$$F_N^{(p)}(0) \sim \sum_{g \geq 0} \frac{f_p^{(2g)}}{N^{2g}}. \tag{8.13}$$

Comparing this with (8.12), we obtain that, for each partition π,

$$p + f - l - 2 = -2g \tag{8.14}$$

and

$$f_p^{(2g)} = \sum_{\pi: p+f-l-2=-2g}^{c} 1. \tag{8.15}$$

As was noticed in [4], by the Euler formula, the number g in (8.14) is equal to the minimal genus of a closed oriented Riemannian manifold on which the graph $\Gamma(\pi)$ can be realized without self-intersections. See the work [22] for a rigorous proof. This proves that $f_p^{(2g)}$ counts the number of 3-valent graphs with p vertices on a closed oriented Riemannian manifold of genus g.

As an illustration, Figure 2 depicts possible 3-valent (ribbon) graphs with two vertices on the sphere, and Figure 3 on the torus. Observe that by counting all possible enumerations of the edges, we obtain the multiplicity factor of the first graph in Figure 2 to be equal to 9, and of the second graph to be equal to 3. The total number of the graphs with multiplicities is equal to 12, which fits well to the first coefficient $6 = \frac{12}{2}$ in the expansion of $F^{(0)}(u)$ given in (1.20). Similarly, we obtain that the multiplicity factor

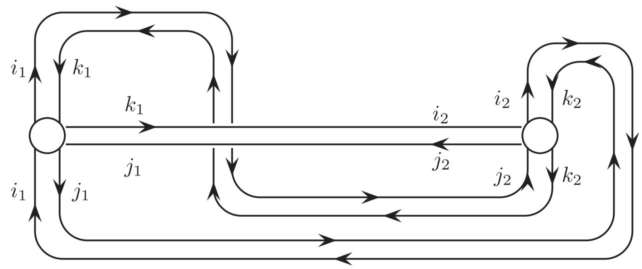

Fig. 4. 3-valent connected fat graph with two vertices, genus 1.

of the graph in Figure 4 on the torus is equal to 3, which corresponds with the first coefficient $\frac{3}{2}$ in the expansion of $F^{(2)}(u)$ presented in (1.24).

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Appendix 1. RHP and Properties of the Orthogonal Polynomials $P_n(z)$

In this appendix, we prove various propositions from Section 3.

Proof of Proposition 3.1. Applying Equations (3.3) to the last column in the determinant for D_n, we obtain that

$$D_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & c_{2n-1} \\ c_n & c_{n+1} & \cdots & c_{2n-1} & c_{2n} \end{vmatrix} = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & 0 \\ c_1 & c_2 & \cdots & c_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & 0 \\ c_n & c_{n+1} & \cdots & c_{2n-1} & h_n \end{vmatrix} = h_n D_{n-1}, \qquad \text{(A.1)}$$

where

$$h_n = \int_\Gamma z^n P_n(z)\, \mathrm{e}^{-NV(z)}\, \mathrm{d}z = \int_\Gamma P_n(z)^2\, \mathrm{e}^{-NV(z)}\, \mathrm{d}z, \tag{A.2}$$

which proves (3.8). ■

Proof of Proposition 3.2. Let $y_n(z) = \det Y_n(z)$. Then by (2.28), $y_{n+}(z) = y_{n-}(z)$, $z \in \Gamma_0 \cup \Gamma_1$, hence $y_n(z)$ is an entire function. By (2.29), $y_n(z) \to 1$ as $z \to \infty$. Hence, by the Liouville theorem, $y_n(z) \equiv 1$.

Next, we claim that $Y_n(z)$ is the unique solution of the RHP. Indeed, suppose that $Y_n^{1,2}(z)$ are two solutions. Then $X(z) = Y_n^1(z) Y_n^2(z)^{-1}$ has no jump on $\Gamma_1 \cup \Gamma_2$ and $X(z) \to I$ as $z \to \infty$. By the Liouville theorem, $X(z) \equiv 1$, hence $Y_n^1(z) = Y_n^2(z)$, so the solution is unique.

Consider now the (11)-element $Y_{n11}(z)$ of the matrix $Y_n(z)$. From (2.28) we obtain that $Y_{n11+}(z) = Y_{n11-}(z)$, hence $Y_{n11}(z)$ is an entire function. By (2.30), $Y_{n11}(z) = z^n + \mathcal{O}(z^{n-1})$, hence $Y_{n11}(z)$ is a monic polynomial of degree n.

Consider then the (12)-element $Y_{n12}(z)$ of the matrix $Y_n(z)$. From (2.28) we obtain that for $z \in \Gamma_0 \cup \Gamma_1$,

$$Y_{n12+}(z) - Y_{n12-}(z) = \alpha(z) \mathrm{e}^{-NV(z)} Y_{n11}(z), \tag{A.3}$$

and from (2.30), that as $z \to \infty$,

$$Y_{n12}(z) = \mathcal{O}(z^{-n-1}). \tag{A.4}$$

This implies that

$$Y_{n12}(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{-NV(s)} Y_{n11}(s)}{s - z}\, \mathrm{d}s, \tag{A.5}$$

where again $\Gamma = \Gamma_0 \cup \Gamma_1$. By expanding $1/(s - z)$ into a geometric series, we obtain that

$$Y_{n12}(z) = -\frac{1}{2\pi \mathrm{i}} \sum_{k=0}^{n-1} \int_\Gamma \frac{s^k\, \mathrm{e}^{-NV(s)} Y_{n11}(s)}{z^{k+1}}\, \mathrm{d}s + \mathcal{O}(z^{-n-1}). \tag{A.6}$$

Comparing this with (A.4), we obtain that

$$\int_\Gamma s^k\, \mathrm{e}^{-NV(s)} Y_{n11}(s)\, \mathrm{d}s = 0, \quad k = 0, 1, \ldots, n-1, \tag{A.7}$$

hence $Y_{n11}(z)$ is a monic orthogonal polynomial of degree n, $Y_{n11}(z) = P_n(z)$. Since the solution $Y_n(z)$ to the RHP (2.28)–(2.30) is unique, the orthogonal polynomial $P_n(z)$ is unique as well. ■

Proof of Proposition 3.3. Consider the matrix-valued function

$$U_n(z) = Y_{n+1}(z)Y_n(z)^{-1}. \tag{A.8}$$

It has no jump, hence it is an entire function. As $z \to \infty$, $U_n(z) = \mathcal{O}(z)$, hence $U_n(z)$ is a linear function. Moreover, (2.30) implies that $U_n(z)$ has the form

$$U_n(z) = \begin{pmatrix} z + c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{A.9}$$

Considering the equation $Y_{n+1}(z) = U_n(z)Y_n(z)$ for the (1,1)-entry, we obtain the three-term recurrence relation (3.11) with some coefficients, that we denote β_n and γ_n^2. ■

Proof of Proposition 3.4. Integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_\Gamma P_n(z)P_n'(z)\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z = N\int_\Gamma P_n(z)^2 V'(z)\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z \\ &= N\int_\Gamma P_n(z)^2(z - 3uz^2)\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z. \end{aligned} \tag{A.10}$$

Now, from (3.11), we have that

$$\int_\Gamma P_n(z)^2 z\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z = h_n\beta_n \tag{A.11}$$

and

$$\begin{aligned} \int_\Gamma P_n(z)^2 z^2\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z &= \int_\Gamma [P_{n+1}(z) + \beta_n P_n(z) + \gamma_n^2 P_{n-1}(z)]^2\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z \\ &= h_{n+1} + \beta_n^2 h_n + \gamma_n^4 h_n, \end{aligned} \tag{A.12}$$

hence

$$0 = h_n\beta_n - 3u(h_{n+1} + \beta_n^2 h_n + \gamma_n^4 h_{n-1}). \tag{A.13}$$

This implies the first equation in (3.14).

Similarly, integrating by parts, we obtain that

$$\begin{aligned} nh_{n-1} &= \int_\Gamma P_{n-1}(z)P_n'(z)\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z \\ &= N\int_\Gamma P_{n-1}(z)P_n(z)V'(z)\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z \\ &= N\int_\Gamma P_{n-1}(z)P_n(z)(z-3uz^2)\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z. \end{aligned} \tag{A.14}$$

Now, from (3.11), we have that

$$\int_\Gamma P_{n-1}(z)P_n(z)z\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z = h_n \tag{A.15}$$

and

$$\begin{aligned} \int_\Gamma P_{n-1}(z)P_n(z)z^2\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z &= \int_\Gamma [P_n(z)+\beta_{n-1}P_{n-1}(z)+\gamma_{n-1}^2P_{n-2}(z)]P_n(z)z\,\mathrm{e}^{-NV(z)}\,\mathrm{d}z \\ &= (\beta_n+\beta_{n-1})h_n, \end{aligned} \tag{A.16}$$

hence

$$nh_{n-1} = N[h_n - 3u(\beta_n+\beta_{n-1})h_n]. \tag{A.17}$$

This implies the second equation in (3.14). ■

Proof of Proposition 3.5. Observe that the moments

$$c_j = \int_\Gamma z^j\,\mathrm{e}^{-N(z^2/2-uz^3)}\,\mathrm{d}z, \tag{A.18}$$

are C^∞-functions of u for $u \geq 0$, and they are analytic for $u > 0$. Hence, the same is true for the determinant D_{n-1} and the polynomial $D_n(z)$, for any n. ■

Appendix 2. Proof of Formula (1.22)

In order to prove (1.22), we will use formula (6.17) and some linear transformations and integral representations of the ${}_2F_1$ functions. Namely, we note that we can use [12, 15.8.7]:

$$ {}_2F_1\left({-m, b \atop c}; z\right) = \frac{(c-b)_m}{(c)_m}\, {}_2F_1\left({-m, b \atop b-c-m+1}; 1-z\right), $$

setting $m = j-1$ and $m = j-2$:

$$ {}_2F_1\left({-j+1, 2 \atop -\frac{3j}{2}+1}; \frac{3}{2}\right) = \frac{\left(\frac{j}{2}+3\right)_{j-1}}{\left(\frac{j}{2}+1\right)_{j-1}}\, {}_2F_1\left({-j+1, 2 \atop \frac{j}{2}+3}; -\frac{1}{2}\right) \tag{A.19} $$

and

$$ {}_2F_1\left({-j+2, 3 \atop -\frac{3j}{2}+2}; \frac{3}{2}\right) = \frac{(\frac{j}{2}+4)_{j-2}}{(\frac{j}{2}+1)_{j-2}}\, {}_2F_1\left({-j+2, 3 \atop \frac{j}{2}+4}; -\frac{1}{2}\right). \tag{A.20} $$

Additionally, we can use

$$ {}_2F_1\left({a, b \atop c}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\, \mathrm{d}t, \tag{A.21} $$

which is valid provided that $\operatorname{Re} c > \operatorname{Re} b > 0$. In our case,

$$ \begin{aligned} {}_2F_1\left({-j+1, 2 \atop \frac{j}{2}+3}; \frac{3}{2}\right) &= \frac{\Gamma(\frac{j}{2}+3)}{\Gamma(\frac{j}{2}+1)} \int_0^1 t(1-t)^{\frac{j}{2}} \left(1+\frac{1}{2}t\right)^{j-1} \mathrm{d}t, \\ {}_2F_1\left({-j+2, 3 \atop \frac{j}{2}+4}; \frac{3}{2}\right) &= \frac{\Gamma(\frac{j}{2}+4)}{2\,\Gamma(\frac{j}{2}+1)} \int_0^1 t^2(1-t)^{\frac{j}{2}} \left(1+\frac{1}{2}t\right)^{j-2} \mathrm{d}t. \end{aligned} \tag{A.22} $$

If we write the previous integrands as follows:

$$ I_1 = \int_0^1 t\left(1+\frac{1}{2}t\right)^{-1} \mathrm{e}^{-j\phi(t)}\, \mathrm{d}t, \quad I_2 = \int_0^1 t^2\left(1+\frac{1}{2}t\right)^{-2} \mathrm{e}^{-j\phi(t)}\, \mathrm{d}t, \tag{A.23} $$

with

$$ \phi(t) = -\tfrac{1}{2}\ln(1-t) - \ln(1+\tfrac{1}{2}t). \tag{A.24} $$

This function has a minimum at $t = 0$, since

$$ \phi'(t) = \frac{3t}{2(1-t)(2+t)}, \tag{A.25} $$

which is positive for $0 < t < 1$. Therefore, for large j the main contribution to both integrals comes from $t = 0$. We make the change of variable

$$-\tfrac{1}{2}\ln(1-t) - \ln(1+\tfrac{1}{2}t) = \tau, \tag{A.26}$$

which maps $t = 0$ to $\tau = 0$. We expand around $t = 0$:

$$-\tfrac{1}{2}\ln(1-t) - \ln(1+\tfrac{1}{2}t) = \tfrac{3}{8}t^2 + \tfrac{1}{8}t^3 + \tfrac{9}{64}t^4 + \mathcal{O}(t^5), \tag{A.27}$$

so inverting this series, we obtain

$$t = \frac{2\sqrt{6}}{3}\tau^{1/2} - \frac{4}{9}\tau - \frac{17\sqrt{6}}{81}\tau^{3/2} + \mathcal{O}(\tau^2). \tag{A.28}$$

Additionally, one can work out the following:

$$\begin{aligned} t\left(1+\frac{t}{2}\right)^{-1}\frac{\mathrm{d}t}{\mathrm{d}\tau} &= \frac{4}{3} - \frac{8\sqrt{6}}{9}\tau^{1/2} + \mathcal{O}(\tau), \\ t^2\left(1+\frac{t}{2}\right)^{-2}\frac{\mathrm{d}t}{\mathrm{d}\tau} &= \frac{8\sqrt{6}}{9}\tau^{1/2} - \frac{160}{27}\tau + \mathcal{O}(\tau^{3/2}). \end{aligned} \tag{A.29}$$

An application of classical Watson's lemma [23], shows now that the integrals in (A.23) are $I_1 = \mathcal{O}(j^{-1})$ and $I_2 = \mathcal{O}(j^{-3/2})$, respectively. The factors that multiply in front are

$$\frac{(\frac{j}{2}+3)_{j-1}}{(\frac{j}{2}+1)_{j-1}}\frac{\Gamma(\frac{j}{2}+3)}{\Gamma(\frac{j}{2}+1)} = \frac{\Gamma(\frac{3j}{2}+2)}{\Gamma(\frac{3j}{2})} \sim \frac{9j^2}{4} \tag{A.30}$$

and

$$\frac{(\frac{j}{2}+4)_{j-2}}{(\frac{j}{2}+1)_{j-2}}\frac{\Gamma(\frac{j}{2}+4)}{2\Gamma(\frac{j}{2}+1)} = \frac{\Gamma(\frac{3j}{2}+2)}{2\Gamma(\frac{3j}{2}-1)} \sim \frac{27j^3}{16}, \tag{A.31}$$

so we only need to consider the second integral I_2, which is dominant for large j. Writing everything together and noting that the binomial number has the following behavior:

$$\binom{\frac{3j}{2}-1}{j-1} = \frac{\Gamma(\frac{3j}{2})}{\Gamma(j)\Gamma(\frac{j}{2}+1)} \sim \frac{2^{-j+\frac{1}{2}}3^{\frac{3j}{2}-\frac{1}{2}}j^{-1/2}}{\sqrt{\pi}}, \quad j \to \infty, \tag{A.32}$$

we obtain

$$\frac{45\cdot 72^j}{4}\frac{\Gamma(\frac{3j}{2})}{\Gamma(j)\Gamma(\frac{j}{2}+1)}\frac{6(j-1)}{5(3j-2)}\,{}_2F_1\left(\begin{matrix}-j+2,3\\ -\frac{3j}{2}+2\end{matrix};\frac{3}{2}\right)\sim 2^{2j-2}3^{\frac{7j}{2}+3}j, \tag{A.33}$$

when $j\to\infty$. We observe that we can write this as

$$2^{-2}3^3 j(3^{\frac{7}{2}}2^2)^j=\frac{27j}{4w_c^j}, \tag{A.34}$$

and recalling (6.21) we obtain

$$f_{2j}^{(2)}\sim\frac{27j(2j)!}{4\cdot 36\cdot 9j^2 w_c^j}=\frac{(2j)!}{48jw_c^j}, \tag{A.35}$$

which proves asymptotic formula (1.22).

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