EXISTENCE OF LIMIT CYCLES IN THE REPRESSILATOR EQUATIONS

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The Repressilator is a genetic regulatory network used to model oscillatory behavior of more complex regulatory networks like the circadian clock. We prove that the Repressilator equations undergo a supercritical Hopf bifurcation as the maximal rate of protein synthesis increases, and find a large range of parameters for which there is a cycle.

Keywords: Limit cycles; oscillatory regulatory networks; repressilator.

1. Introduction

Regulatory networks are collections of interacting molecules in a cell. One particular kind, oscillatory networks, has been discovered in many biological processes. Well-known examples are the circadian clock [Dunlap, 1999] and the cell cycle [Nurse, 2000], where the oscillatory nature of the process plays a central role.

In recent years, researchers have been able to implement artificial regulatory networks in the laboratory. Studying these simplified models is an important step to understand real life networks.

Modeling studies suggest several designs for artificial oscillatory networks. There are many implementations of hysteresis-based oscillators, [Atkinson et al., 2003; Barkai & Leibler, 2000; Hasty et al., 2001; Kuznetsov et al., 2004].

Another artificial oscillatory network called the Repressilator [Elowitz & Leibler, 2000] borrows the idea of a ring oscillator from engineering. The mechanism is based on connecting an odd number of inverters (negative control elements) in a ring. Its genetic implementation uses three proteins that cyclically repress the synthesis of one another. Our computational study [Yang et al., 2009] suggests that the oscillatory mechanism of the repressilator is qualitatively different from that in other genetic oscillators. Moreover, the mathematical analysis of such a system has not been done. In this paper, we study the behavior of the following parametric family of differential equations:

**Definition 1.** Fix \( n > 0 \). For \( \alpha > 0 \) the Repressilator equations are

\[
\begin{align*}
dx/dt &= \frac{\alpha}{1 + y^n} - x \\
\frac{dy}{dt} &= \frac{\alpha}{1 + z^n} - y \\
\frac{dz}{dt} &= \frac{\alpha}{1 + x^n} - z \quad (x, y, z \geq 0).
\end{align*}
\]

This is a reduced version of the original model in which three of six equations are assumed to be in quasi-equilibrium and substituted by algebraic relations. The reduction assumes that the three excluded variables evolve an order of magnitude faster than the other three. Note that when any of the three variables is 0, the corresponding derivative...
is positive, so the orbit of any initial point with non-negative coordinates moves in the positive octant \( \{x, y, z > 0\} \) and remains there for all \( t > 0 \).

Our analysis is split in two parts. First, we describe the bifurcation behavior of the family. It is not hard to see that there is a unique fixed point \( R \), which sits on the main diagonal.

**Proposition 1.** The linearization of system (1) at \( R \) has one negative eigenvalue and a pair of complex conjugate eigenvalues.

(a) If \( n \leq 2 \) the complex eigenvalues have negative real part, so \( R \) is always attracting.

(b) If \( n > 2 \) there is a single supercritical Hopf bifurcation when \( \alpha = \alpha_{\text{bif}} = r_0 + r_0, \) where \( r_0 = \sqrt{n-2} \).

In the second part of the paper, we study the topological behavior of orbits. We begin by showing that in all cases the interesting behavior is restricted to a bounded region.

**Lemma 1.** The orbit of any point \( P \) with non-negative coordinates will eventually enter the cube \( \mathcal{C} = \{0 \leq x, y, z \leq \alpha\} \) and stay in the interior of \( \mathcal{C} \) thereafter.

The fixed point \( R \) is always inside \( \mathcal{C} \). When \( n > 2 \) and \( \alpha > \alpha_{\text{bif}} \), \( R \) is a saddle, so standard index theory implies the existence of one or more attracting sets to "absorb" the orbits repelled by \( R \).

Note that Proposition 1 already guarantees an attracting cycle near \( R \) when \( 0 < \alpha - \alpha_{\text{bif}} \ll 1 \), because the bifurcation is supercritical. In fact there is always a cycle:

**Theorem 1.** For \( n > 2 \) and all \( \alpha > \alpha_{\text{bif}} \), system (1) has a limit cycle.

Our proof is quasi-constructive and it yields more about the behavior of the orbits inside \( \mathcal{C} \). We will describe a toroidal region \( T \subset \mathcal{C} \) that is partitioned into a chain of simply connected pieces, and show that the orbit of any point \( P \in T \) travels in these six regions in a specific order. This proves the existence of a Poincaré map \( \theta : \mathcal{K} \to \mathcal{K} \) where \( \mathcal{K} \) is a section of \( T \) by a nullcline. Theorem (1) then follows from Brower’s Fixed Point Theorem [Munkres, 2000, p. 351].

One advantage of our method is that the construction gives a heuristic reason for the existence of the cycle. This mechanism is qualitatively different from classical systems like the relaxation oscillator. An exploration of distinguishing features will appear in [Buge et al., 2009].

This paper is organized as follows: In Sec. 2 we prove Proposition 1 after describing the linearization of system (1) at \( R \). The computation showing that the Hopf bifurcation is supercritical is delayed to an appendix at the end of the paper. In Sec. 3, we prove Lemma 1 and give a decomposition of \( \mathcal{C} \) into pieces on which orbit behavior is simple to describe. Section 4 uses this decomposition to prove Theorem 1.

### 2. Bifurcation Analysis

From the description of nullclines in Sec. 3, it will follow that there is a unique equilibrium point \( R \). Because of the symmetry between coordinates, \( R \) is of the form \((r, r, r)\), where \( r \) satisfies

\[
\frac{\alpha}{1 + r^2} = r; \tag{2}
\]

i.e. \( r^{n+1} + r - \alpha = 0 \). This section is devoted to the analysis of the linearization of system (1) near \( R \), which yields the proof of Proposition 1.

The jacobian matrix of (1) is:

\[
J = \begin{pmatrix}
-1 & -n\alpha z^{n-1} & 0 \\
0 & -1 & -n\alpha z^{n-1} \\
-\frac{n\alpha}{1 + z^2} & 0 & -1
\end{pmatrix}, \tag{3}
\]

so using (2), the linearization of system (1) at \( R \) is

\[
\begin{pmatrix}
\frac{du}{dt} \\
\frac{dv}{dt} \\
\frac{dw}{dt}
\end{pmatrix} = \begin{pmatrix}
-1 & -\frac{n\alpha}{1 + r^2} & 0 \\
0 & -1 & -\frac{n\alpha}{1 + r^2} \\
-\frac{n\alpha}{1 + r^2} & 0 & -1
\end{pmatrix} \begin{pmatrix}
u \\
v \\
w
\end{pmatrix}. \tag{4}
\]

The eigenvalues of (4) are

\[
\lambda = \omega \frac{n\alpha}{1 + r^2} - 1, \quad (\omega^3 = -1). \tag{5}
\]

The eigenvalue corresponding to \( \omega = -1 \) is negative, while the other two are complex-valued. Note that along \( \text{diag} \) the vector field points in the diagonal direction. Moreover, the function
(α/(1 + r^n)) − u is positive at u = 0, negative at u = α, and monotone. This implies that diag constitutes the global stable manifold of R.

By symmetry, the plane of remaining eigendirections of R is orthogonal to diag. To determine the type of R, it remains to study the real part of the complex eigenvalues:

\[ \text{Reλ} = \cos \left( \frac{π}{2} n \right) \frac{m^n}{1 + r^n} - 1 = \frac{n}{2} \frac{r^n}{1 + r^n} - 1. \]

As r varies from 0 to ∞, this quantity changes monotonically between −1 and (n/2) − 1, and therefore it is always negative when n ≤ 2. However, for n > 2 the value

\[ r_0 = \sqrt{\frac{2}{n - 2}} \]

is such that Reλ is negative for r < r_0 and positive for r > r_0. In particular, for the unique eigenvalue \( \alpha_{\text{viol}} = r_0^{n+1} + r_0 \)

(dependent on n), the linearization of (1) at R has one real and two purely imaginary eigenvalues. This is the setting for a Hopf bifurcation. In an appendix we compute the first Lyapunov coefficient \( c_1(R) \) of (1) near R, and find that \( c_1(R) = -(n^2 + 5n - 14)/18\sqrt{3}r_0^n \). The polynomial \( -(n^2 + 5n - 14) = -(n + 7)(n - 2) \) is negative for all n > 2. This shows that the bifurcation is supercritical, as a consequence, the cycles are attracting, and appear when \( \alpha > \alpha_{\text{viol}} \).

3. Orbit Dynamics

3.1. The attracting cube C

In this section we prove Lemma 1; then we decompose C into pieces where orbit behavior is simple to describe. The proof of Lemma 1 is a consequence of the following two claims.

Claim 1. If 0 ≤ x(0) ≤ α, then 0 < x(t) < α for all t > 0.

Claim 2. If x(0) ≥ α, then there is a time \( t_1 \) such that \( x(t_1) < α \).

Indeed, by the symmetry of the system, y and z satisfy analogous inequalities, so Claims 1 and 2 imply:

1. The cube C is a trapping region: i.e. the vector field is transversal to ∂C, pointing to the interior.
2. The orbit of any point \( P \notin C \) moves toward, and eventually enters C.

These two assertions are equivalent to the statement of Lemma 1. Now, the proof of Claim 1 is immediate:

(i) If x(0) = 0, then \( (dx/dt)(0) = \alpha/(1 + y^n) > 0 \).
(ii) If x(0) = α, then \( (dx/dt)(0) = \alpha/(1 + y^n) \alpha = \alpha((1 + y^n)) - 1 < 0 \).

Proof of Claim 2. Generalizing the above inequality, it is obvious that \( dx/dt < 0 \) when \( x > \alpha \). However, this is not enough to conclude Claim 2. Instead, we need to show that \( dx/dt \) is bounded away from 0, and this requires the chain of estimates (7)–(10) on x, y, z, and back to x. Suppose then for a contradiction that

\[ x(t) = \alpha + ε_t \geq \alpha \quad \text{for all } t \geq 0. \quad (7) \]

Notice that (7) implies \( dy/dt \leq (α/(1 + α^n)) − z \) for all t. As long as \( z(t) \) remains larger than \( α \), the derivative \( dz/dt \leq α((1/(1 + α^n)) − 1) < 0 \) is bounded away from 0, so \( z \) decreases at a steady rate. It follows that there is a time \( t_0 > 0 \) such that

\[ z(t) < \alpha \quad \text{for all } t > t_0. \quad (8) \]

Definition 2. The quantity \( α/(1 + α^n) \) appears frequently enough in estimates that it will be convenient to represent it with the letter A.

By Claim 1, \( y(t_0) > 0 \). Now, if \( y(t_0) \leq A \), inequality (8) gives \( dy/dt \geq A - y > 0 \) for \( t > t_0 \), so the y coordinate cannot become less than \( y(t_0) \). On the other hand, if \( y(t_0) > A \), the y coordinate cannot become A since \( dy/dt|_{t_0} \geq A - y > 0 \). In either case, if \( Y = \min\{y(t_0), A\} \) (a positive quantity), then

\[ y(t) \geq Y > 0 \quad \text{for all } t > t_0. \quad (9) \]

Using (7) and (9), we find that \( dx/dt \) is bounded away from 0:

\[ \frac{dx}{dt} = \alpha \left( \frac{1}{1 + y^n} - 1 \right) - ε_t \leq \alpha \left( \frac{1}{1 + y^n} - 1 \right) - ε_t < 0, \quad (10) \]

so x becomes arbitrarily small, contradicting (7). This proves Claim 2 and the Lemma.

3.2. Nullcline description

We have just seen that any interesting dynamical behavior is bound to happen in C. For the rest of the paper, x, y, z are restricted to C. As usual, the shape and relative position of the nullcline surfaces
for the three variables is an important indicator of how the system behaves. In this section we describe the configuration of nullclines.

**Definition 3.** The nullclines \( N_x, N_y, N_z \) are the loci of points where each of the derivatives \( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \) vanish. They are three surfaces given implicitly by

\[
\begin{align*}
N_x &= \left\{ \frac{\alpha}{1 + y^n} = x \right\} \cup C, \\
N_y &= \left\{ \frac{\alpha}{1 + z^n} = y \right\} \cup C, \\
N_z &= \left\{ \frac{\alpha}{1 + x^n} = z \right\} \cup C.
\end{align*}
\]

Let \( \eta \) be the curve \( \{ (\alpha/(1 + y^n)) - x, z = 0 \} \), which connects \((\alpha, 0, 0)\) with \((A, \alpha, 0)\). Then, \( N_x \) is formed by translating \( \eta \) along the \( z \)-direction. The nullclines \( N_y \) and \( N_z \) are formed in a similar fashion and their union has rotational symmetry about the diagonal; see Fig. 1.

Fig. 1. \((\alpha = 4, n = 3)\). The nullclines \( N_x \) and \( N_y \) intersect monotonically along an open-ended curve. One of the straight sides of each nullcline ends at an edge of \( C \), while the opposite side intersects \( C \) on a face, very close to the opposite edge.

Note that \( N_x \) divides \( C \) in exactly two pieces, one where \( dx/dt < 0 \) and one where \( dx/dt > 0 \). Since \( (\alpha/(1 + y^n))' = -n\alpha y^{n-1}/(1 + y^n)^2 < 0, \) the curve \( \eta \) is monotonically decreasing in the \( x \)-direction.

The intersection of \( N_x \) with \( N_y \) is a curve \( \beta_{xy} \) with parametric equations

\[
\begin{align*}
x(t) &= \frac{\alpha}{1 + \left( \frac{\alpha}{1 + t^n} \right)^n}, \\
y(t) &= \frac{\alpha}{1 + t^n}, \\
z(t) &= t,
\end{align*}
\]

with \( 0 \leq t \leq \alpha \). This curve joins the points \((A, \alpha, 0)\) and \((\alpha/(1 + A^n), A, \alpha)\). Similar descriptions hold for the nullcline intersection curves \( \beta_{yz}, \beta_{zx} \). These three curves intersect at the single point \( R \). It follows that the union \( N_x \cup N_y \cup N_z \) divides \( C \) in eight regions in a manner qualitatively similar to the union of three orthogonal planes. Extending this analogy, we refer to the eight components of \( \text{int} C \setminus (N_x \cup N_y \cup N_z) \) as octant regions; see Fig. 2.

**3.3. Breaking \( C \) into pieces**

In this section we partition \( C \) in several pieces where the behavior of system (1) is easy to describe. Then, we use this decomposition to prove Theorem 1.

**Definition 4.** The signature of a point \( P \in C \) is a triple of symbols from the alphabet \( \{-, +, o\} \)
which describe the signs of the derivatives of $x, y$ and $z$ at $P$. We extend this notation to refer to any subset of $C$ where the signs remain constant. Thus, for instance, $(+++) \text{ or } (-++)$ represents the quadrant region near the origin where $dx/dt, dy/dt, dz/dt > 0$, so $\text{diag} C = (+++) \cup (0 \circ 0) \cup (--)$. We extend this notation further, including the symbol $*$ to represent an undefined sign. For example, $N_2 = (0 * * *)$.

Definition 5. The six octant regions $(++-), (+-+), (-+), (+-), (-+), (--)$, which are disjoint from $\text{diag}$, are called lateral regions.

3.4. The Spindle $S$

In this section we construct a small neighborhood $S$ of $R$ that blends well with the orbits and with the layout of the nullclines $N_2$ of system (1). First, we give an analogous construction in the linearization domain, and then pull it back by homeomorphism.

The nullclines of the linear system (4) are the planes

$\{u = Qv\}, \{v = Qu\}, \{w = Qu\}$, (12)

where $Q = -m r^3/(1 + r^3)$. As with system (1), these planes determine eight regions with different sign patterns for the derivatives $du/dt, dv/dt, dw/dt$. We adopt the alphabet $\{+, -, 0, *\}$.

The stable direction is along the diagonal $\{u = v = w\}$, which is contained in the regions $(+++) \text{ and } (---)$. The unstable manifold is the plane $\Pi = \{u + v + w = 0\}$, orthogonal to the diagonal. This plane sits inside the union of the six lateral regions.

For any $\varepsilon > 0$, consider the infinite cylinder of radius $\varepsilon$ in the direction of the diagonal $\{u = v = w\}$. Let $m > 0$ be a fixed number (for all $\varepsilon$) large enough that the planes $\Pi_+, \Pi_-$ at distance $m \varepsilon$ from $\Pi$ intersect the cylinder in two disks $D_+, D_-$ that are completely contained in $(+++) \text{ and } (---)$, respectively. Denote by $C$, the compact portion of the cylinder bounded between $D_+$ and $D_-$.

The behavior of the orbit $\gamma$ of any point $P \in C \cap \Pi$ is as follows. In negative time, $\gamma$ spirals toward the diagonal, while monotonically moving away from $\Pi$, so that it exits $C$, in finite time through either $D_+$ or $D_-$. In positive time, $\gamma$ approaches $\Pi$ monotonically while spiraling away from the diagonal (since the complex eigenvalues have positive real part). Thus, $\gamma$ stays between $\Pi_+$ and $\Pi_-$. We claim that $\gamma$ escapes into a lateral region at a distance $d_\gamma$ from $\text{diag}$, where $d_\gamma$ becomes arbitrarily small as $\varepsilon$ goes to 0. Indeed, the planes $\Pi_+, \Pi_-$ intersect the nullcline planes in two triangles $\Delta_+, \Delta_-$ contained within distance $d_\gamma$ from the diagonal. The rectangular faces of the prism between $\Delta_+ \text{ and } \Delta_-$ are contained inside the lateral regions. Since the orbit $\gamma$ intersects one of these prism faces in positive finite time, the claim follows.

Now consider the ball $B_\varepsilon$ of radius $\varepsilon$ centered at the origin. For any $P \in B_\varepsilon \cap \Pi$, let $s_P$ be the segment of the full orbit $\gamma_P$ that starts in $D_+ \cup D_-$, and ends at the first point that is both inside a lateral region and outside $C$.

Definition 6. The linear spindle of size $\varepsilon$ is the union $S_\varepsilon = B_\varepsilon \cup \bigcup_{P \in B_\varepsilon \cap \Pi} s_P$.

It is a compact set such that

(a) $S_\varepsilon$ contains $B_\varepsilon$ and is simply connected.

(b) The intersection $S_\varepsilon \cap D_+$ is compactly contained in $(+++ \cup \text{ quadrants})$.

Fig. 3. The ball $B_\varepsilon$ around the origin is encased in the cylinder $C$. The disks $D_+$ at both ends of $C$ are inside the region $(+++ \cup \text{ quadrants})$, so backward orbits from $B_\varepsilon$ exit $C$ into one of these regions. The forward orbit approaches the plane $\Pi$ through the equator of $B_\varepsilon$, so it eventually enters a lateral region. The linear spindle $S_\varepsilon$ consists of $B_\varepsilon$ and all the orbits that start at $D_\varepsilon$, enter $B_\varepsilon$, and exit $C$ into a lateral region.
(c) Any orbit that enters $S_1$ does so through $S_1 \cap D_3$.
(d) All orbits in $S_3 \setminus \text{diag}$ leave $S_3$ through a lateral region, never to return to $S_3$ (nor, in fact, to $(+++))$. 
(e) $\lim_{t \to 0} \text{diam } S_3 = 0$.

Since the linearization of $R$ is a saddle, there is a radius $\delta > 0$, and a homeomorphism

\[ \phi : B_3(R) \to N \]

that conjugates the nonlinear system (1) within the ball $B_3(R)$, to the linearization (4) in a neighborhood $N$ of $0$. Let $\varepsilon > 0$ be small enough that the linear spindle $S_3$ is compactly contained in $N$.

**Definition 7.** The preimage $\bar{S} = \phi^{-1}(S_1) \subset B_3$ will be called the spindle of (1). Note that $\phi$ takes local nullclines $N_v \cap B_3(R)$ to the planes (12) inside $N$. By a slight abuse of notation, we denote by $D_3$, the preimages $\phi^{-1}(D_3)$ in $(+++), (---) \subset C$. With these considerations in mind, $\bar{S}$ satisfies

(a) $\bar{S}$ is simply connected.
(b) The intersection $\bar{S} \cap D_3$ is compactly contained in $(+++), (---)$.
(c) Any orbit that enters $\bar{S}$ does so through $\bar{S} \cap D_3$.
(d) All orbits in $\bar{S} \setminus \text{diag}$ leave $\bar{S}$ through a lateral region and escape $B_3$ without re-entering $\bar{S}$.

In particular, if an orbit that leaves $\bar{S}$ were to return to $\bar{S}$, it would have to follow a homoclinic connection. One consequence of Proposition 3 below will be that this does not happen.

### 3.5. The trapping torus $T$

**Definition 8.** The region $T = \text{cl}(---) \cup \text{cl}(---) \cup \text{cl}(---) \cup \text{cl}(---) \cup \text{cl}(---) \cup \text{cl}(---)$ (where $\text{cl}$ stands for closure) has the topological type of a filled torus. It is divided into six pieces whose interiors are just lateral octant regions truncated by $\bar{S}$. We call these truncated regions 

\[ T \]

### 4. Poincaré Map

Our first lemma discusses the way in which the vector fields cross the nullclines.

**Lemma 2.** The orbit of $P \in N_v \setminus R$ moves into an octant region according to the following table.

\[
\begin{array}{ccc}
P \in (a++) & \text{moves inside} & (---) \\
(a+) & \text{---} & (---) \\
(a-) & (---) & (---) \\
(o+) & (---) & (---) \\
(o-) & (---) & (---) \\
(o-o) & (---) & (---) \\
(o-o) & (---) & (---) \\
(o-a) & (---) & (---) \\
\end{array}
\]

The behavior for a point $P$ in $N_v$ or $N_o$ is analogous, and the corresponding table is obtained from the above one by permuting the coordinates.

**Proof.** Since the tangent plane of $N_v$ at any point contains the vector $(0,0,1)$, the vector field is parallel to $N_v$ exactly on the intersection $N_v \cap N_o = (oo\ast)$. Let $n$ be a normal vector to $N_v$ at $P$ pointing away from the $z$-axis. Recall that $f(u) = \alpha/(1 + u^2)$ has negative derivative. Then $n$ is of the form $(1, -f'(y), 0)$. If $P \in (o++)$, the dot product of $n$ with a flow vector of the form $(o++)$ must be positive, while if $P \in (o--)$, the dot product is negative.

Orbit behavior for points on the other two nullclines can be deduced from this table by permuting the three coordinates.

Next we show that orbits must cross between octant regions.

**Definition 9.** A point $P \in C \setminus R$ is close to diag if its forward orbit enters $\bar{S}$.

**Proposition 2.** Let $P = P(0) \in C \setminus \text{diag}$ lie in the interior of any octant region. Then the orbit of $P$ will cross in finite time to a different lateral region and outside the spindle.

**Proof.** If $P$ is close to diag, its orbit enters $\bar{S}$ in finite time by the construction of $\bar{S}$. Then, since it is not in diag, it crosses over to a lateral region. This proves the claim in this case.

Now assume that $P$ is in a given octant region, but not close to diag. By definition the forward orbit does not enter $\bar{S}$, so there is an $\varepsilon$ such that for all $t > 0$, $P(t)$ is at least $\varepsilon$ away from $R$. Since $R$ is the only point with vanishing derivative, there is a $\delta > 0$ such that $|P(t)| > \delta$ for all $t > 0$. Thus, at any given time $t$ one of the three coordinates has speed larger than $\delta$. After $t > 3\delta/\delta$, one of the three variables had speed larger than $\delta$ for a union of disjoint time
intervals of total length at least $\alpha/\delta$. Assuming that the orbit remains in the same octant region for all time, this coordinate changes monotonically. By the Mean Value Theorem it follows that this coordinate changes by at least $\alpha$, hence $P$ would escape $C$.

This contradiction proves that the orbit of $P$ must cross through a nullcline to a different octant region. This new region cannot be $(++)$ or $(---)$ because of the transition pattern from Lemma 2. ■

We continue by showing that the torus $T$ is a trapping region, and we describe the behavior of the flow inside $T$.

**Proposition 3.** The following hold

1. All orbits in $C \setminus \text{diag}$ are trapped in finite time in the interior of $T$.
2. Inside $T$ every orbit moves from torus piece to torus piece, crossing nullclines in the following pattern:

   - $(0-0) \rightarrow (++) \rightarrow (-+) \rightarrow (--) \rightarrow (0-0)$
   - $(+0-) \rightarrow (++-) \rightarrow (+0+) \rightarrow (-+0) \rightarrow (+-0) \rightarrow (+0-) \rightarrow (-+0) \rightarrow (--0) \rightarrow (-+0) \rightarrow (-o+) \rightarrow (-++-)$

**Proof.** Let $P \in C \setminus \text{diag}$. By Lemma 1 the flow points toward the interior of $T$. By Proposition 2, if $P$ is in an octant region, its forward orbit enters int $T$ in finite time. If $P$ is anywhere on a nullcline, then Lemma 2 and the construction of the spindle prove the first claim.

Each torus piece is bounded by five surfaces; a portion of $\partial C$, a portion of $\partial S$, and three nullcline components (one from each of $N_x, N_y, N_z$). Let us call the latter three faces of the torus piece. Now consider a torus piece, say $(++)$. Its three faces are truncations of $(0-0), (+0+)$ and $(+0-).$ Again, by Lemma 2, the flow enters $(++)$ through the first two faces, and exits $L$ toward $(+-0)$ only through the third face. The same argument applied to the remaining five torus pieces proves that the only possible transitions between torus pieces are given by the pattern (13).

**Definition 10.** Denote by $K_s = c(0-0) \cap T$.

**Corollary 1.** There is a well defined Poincaré return map $\theta: K_s \rightarrow K_s$.

The proof is immediate from Proposition 3. Theorem 1 follows from Brower’s Fixed point Theorem applied to the map $\theta$. Moreover, any such cycle must move through lateral regions in the pattern (13).

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**References**


Let us first rewrite the vector field in the format

\[ F(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z)) \]

\[
\begin{pmatrix}
\frac{\alpha}{1+\alpha} - x - \frac{\gamma}{1+\alpha} - \frac{\beta}{1+\gamma} - \frac{\alpha}{1+\alpha} - y
\end{pmatrix}
\]

The third order Taylor approximation of \( F \) around \( R \) has a particularly simple expression because only partial derivatives of higher order that do not vanish are \( f_{y, y}, g_{y, z}, \) and \( h_{z, x} \) (compare (3)). Indeed,

\[
F(x, y, z) = A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2} B((x, y, z), (x, y, z))
\]

\[
+ \frac{1}{3!} C((x, y, z), (x, y, z), (x, y, z)) + O(((x, y, z))^3),
\]

where \( A \) is the jacobian matrix

\[
\begin{pmatrix}
-1 & -\frac{n r_0^2}{1 + r_0^2} & 0 \\
0 & -1 & -\frac{n r_0^2}{1 + r_0^2} \\
-\frac{n r_0^2}{1 + r_0^2} & 0 & -1
\end{pmatrix}
\]

with eigenvalues \(-3, \pm \sqrt{3}\) (regardless of \( n \)), and \( B \) and \( C \) are bi- and trilinear functions respectively, with components

\[ B_1((x_1, y_1, z_1), (x_2, y_2, z_2)) = f_{y_{x_1}, y_{x_2}, y_{x_2}} \cdot y_1 y_2, \]

\[ B_2((x_1, y_1, z_1), (x_2, y_2, z_2)) = g_{x_{y_1}, x_{y_2}, y_{y_2}} \cdot z_1 z_2, \]

\[ B_3((x_1, y_1, z_1), (x_2, y_2, z_2)) = h_{x_{y_1}, x_{y_2}, y_{y_2}} \cdot x_1 x_2, \]

\[ n \left( \frac{1}{1 + r_0^2} - \frac{n - 2}{n^2} \right) = \frac{n}{r_0^2} \]

\[ = \frac{2}{r_0^2} \frac{2(n + 1) - (n - 2)(n - 1)}{n} \]

\[ = \frac{2}{r_0^2} (5 - n). \]

By symmetry, this is also the coefficient of \( B_1 \) and \( B_3 \), so the bilinear function \( B \) is

\[ B((x_1, y_1, z_1), (x_2, y_2, z_2)) = \frac{2}{r_0^2} (5 - n) (y_1 y_2 z_1 z_2 x_1 x_2). \]
We find $C$ in the same manner. From (A.2),

$$f_{yy}(x, y, z) = -n\left(\frac{n^2 + 3n + 2}{1 + y^2}\right)^2,$$

so that

$$C = \frac{n}{r_0} \left(-n^2 + 3n + 2\right) \left(\frac{2}{n - 2}\right)^3 + \frac{n}{r_0} \left(\frac{n}{n - 2}\right)^4,$$

and the trilinear function $C$ becomes

$$C(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)) = \frac{2}{r_0} \left(-n^2 + 15n - 38\right)(x_1x_2x_3, y_1y_2y_3, z_1z_2z_3).$$

Let $\omega = (-1/2) + (\sqrt{3}/2)$ be a cubic root unity. Then the normalized eigenvectors $q$ and $p$ of Lemma 3 are

$$q = p = \left(\begin{array}{c} \omega^2 \\
\omega \\
1 \end{array}\right).$$

Note that formula (A.1) has nested instances of the bilinear form $B$. These values are

$$B(q, q) = \frac{2(5 - n)}{3r_0} (\omega^2, \omega, \omega),$$

and

$$B(q, q) = \frac{2(5 - n)}{3r_0} (1, 1, 1).$$

For the outer expressions involving $B$ we find the inverse matrices

$$A^{-1} = \begin{pmatrix} -\frac{1}{9} & 2 & -\frac{1}{9} \\
-\frac{4}{9} & -1 & 2 \\
2 & -1 & -\frac{1}{9} \end{pmatrix},$$

and

$$B(q, A^{-1} B(q, q)) = \frac{-2\sqrt{3}(5 - n)^2}{27r_0} (\omega^2, \omega, \omega),$$

and

$$B(q, (2\sqrt{3} I - A)^{-1} B(q, q)) = \frac{-2\sqrt{3}(5 - n)^2}{27r_0} \left(1 - \frac{i}{\sqrt{3}}, -1 - \frac{i}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right).$$
Now we can compute the hermitian products
\[ p \cdot C(q, q) = \frac{n^2 - 15n + 38}{3r_0^2} (1 + \sqrt{3}), \]
(A.4)
\[ p \cdot B(q, A^{-1}B(q, q)) = \frac{2(5 - n)^2}{9r_0^2} (1 + \sqrt{3}), \]
(A.5)
and
\[ p \cdot B(q, (2\sqrt{3}I - A)^{-1}B(q, q)) = \frac{4(5 - n)^2}{9r_0^2} i. \]
(A.6)

Putting together (A.4)–(A.6), the Lyapunov coefficient is
\[ \ell_1(R) = \frac{-1}{18\sqrt{3}r_0^2} \text{Re}(n^2 + 5n - 14 + 13\sqrt{3}n^2i - 115\sqrt{3}ni + 286\sqrt{3}) = \frac{-1}{18\sqrt{3}r_0^2}. \]