LINEAR FRACTIONAL
COMPOSITION OPERATORS ON $H^2$

CARL C. COWEN

ABSTRACT. If $\varphi$ is an analytic function mapping the unit disk $D$ into itself, the composition operator $C_\varphi$ is the operator on $H^2$ given by $C_\varphi f = f \circ \varphi$. The structure of the composition operator $C_\varphi$ is usually complex, even if the function $\varphi$ is fairly simple. In this paper, we consider composition operators whose symbol $\varphi$ is a linear fractional transformation mapping the disk into itself. That is, we will assume throughout that

$$\varphi(z) = \frac{az + b}{cz + d}$$

for some complex numbers $a, b, c, d$ such that $\varphi$ maps the unit disk $D$ into itself. For this restricted class of examples, we address some of the basic questions of interest to operator theorists, including the computation of the adjoint.

For any $\varphi$ that maps the disk into itself, it is known that $C_\varphi$ is a bounded operator, and some general properties of $C_\varphi$ have been established (see for example, [15], [12], [17], [13], [10], [3], [11], [14], and [16]). However, not all questions that would be considered basic by operator theorists are understood. For example, for general $\varphi$, no convenient description of $C_\varphi^*$ is known and it is not known how to compute $\|C_\varphi\|$ (although order of magnitude estimates are available [3]).

J. S. Shapiro (see [16]) has completely answered the question “When is $C_\varphi$ compact?” Although the general answer is complicated, if $\varphi$ is a linear fractional transformation $C_\varphi$ is compact if and only if $\varphi$ maps the closed unit disk into the open disk. It follows from this that for a linear fractional $\varphi$, all powers of $C_\varphi$ are non-compact if and only if $\varphi$ has a fixed point on the unit circle.

The first section illustrates the diversity of this class of examples by showing there are eight distinct classes on the basis of spectral information alone. Much of the spectral information depends on the behavior of $\varphi$ near the Denjoy-Wolff point, the unique fixed point $\hat{\alpha}$ of $\varphi$ in the closed disk such that $|\varphi'(\hat{\alpha})| \leq 1$.

In the second section of the paper, we find that in the linear fractional case $C_\varphi^*$ is the product of Toeplitz operators and another composition operator. From this computation, we derive $\|C_\varphi\|$ in certain cases and give a short proof of the subnormality of $C_\varphi^*$ when $\varphi$ is a hyperbolic inner linear fractional transformation (see also [14, 5]). Finally, the class of linear fractional transformations for which $C_\varphi$ is hyponormal or subnormal is identified.

The class of composition operators is related to other areas of operator theory in somewhat surprising ways. For example, Deddens [6] established a connection between the discrete Cesaro operator and $C_\varphi$ where $\varphi(z) = sz + 1 - s$ for $0 < s < 1$ and showed that therefore $C_\varphi^*$ is subnormal for these $\varphi$. In addition, commutants of many analytic Toeplitz operators are generated by composition and multiplication operators.

Although this paper makes progress in answering some basic questions about linear fractional composition operators, there are still problems to be considered. For example, computing the norm is still unsolved except in special cases and exact conditions for unitary equivalence and similarity are not known. It is hoped that the results here will point the way toward results about more general composition operators, both on $H^2$ and on related Hilbert spaces of analytic functions.

Supported in part by National Science Foundation Grant DMS 8300883.
Eight Examples

In spite of their apparent simplicity, composition operators on \( H^2 \) with linear fractional symbol exhibit great diversity. In the following table, we collect some examples that show that linear fractional transformations give rise to most of the major spectral types. The Denjoy-Wolff point will be denoted \( \hat{\alpha} \).

<table>
<thead>
<tr>
<th>Example</th>
<th>Properties</th>
<th>Spectrum ( C_\varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi(z) = \zeta z ) where (</td>
<td>\zeta</td>
<td>= 1)</td>
</tr>
<tr>
<td>( \varphi(z) = \frac{3z+1}{z+3} )</td>
<td>( \hat{\alpha} = 1, \varphi'(1) = \frac{1}{2} ) hyperbolic, inner</td>
<td>{( \lambda : \frac{1}{\sqrt{2}} \leq</td>
</tr>
<tr>
<td>( \varphi(z) = \frac{(1+2iz-1}{z+1+2} )</td>
<td>( \hat{\alpha} = 1, \varphi'(1) = 1 ) parabolic, inner</td>
<td>{( \lambda :</td>
</tr>
<tr>
<td>( \varphi(z) = sz + 1 - s ) where ( 0 &lt; s &lt; 1 )</td>
<td>( \hat{\alpha} = 1, \varphi'(1) = s )</td>
<td>{( \lambda :</td>
</tr>
<tr>
<td>( \varphi(z) = \frac{2i+2t+1}{-t+2z+1} ) where ( \text{Re}(t) &gt; 0 )</td>
<td>( \hat{\alpha} = 1, \varphi'(1) = 1 ) ( \varphi''(1) = t ) {( e^{\beta t} : \beta \leq 0 } \cup {0} ) (a spiral) [3, Cor. 6.2]</td>
<td></td>
</tr>
<tr>
<td>( \varphi(z) = \frac{r^2}{r(1-r)z} ) where ( 0 &lt; r &lt; 1 )</td>
<td>( \hat{\alpha} = 0, \varphi'(0) = r ) ( \varphi(1) = 1 )</td>
<td>{( \lambda :</td>
</tr>
<tr>
<td>( \varphi(z) = -\frac{1}{2}z + \frac{1}{2} )</td>
<td>( \hat{\alpha} = \frac{1}{2}, \varphi'(\frac{1}{2}) = -\frac{1}{2} ) ( C_\varphi^2 ) compact, ( C_\varphi ) not</td>
<td>{( (-\frac{1}{2})^k : k = 0, 1, \ldots } \cup {0} } [1]</td>
</tr>
<tr>
<td>( \varphi(z) = -\frac{1}{2}z )</td>
<td>( \hat{\alpha} = 0, \varphi'(0) = -\frac{1}{2} ) ( C_\varphi ) compact</td>
<td>{( (-\frac{1}{2})^k : k = 0, 1, \ldots } \cup {0} } [1]</td>
</tr>
</tbody>
</table>

Adjoint

If \( \varphi \) is an inner function in \( H^\infty \) so that it may be considered to be a mapping of the unit circle into itself, then a satisfactory formula can easily be obtained for \( C_\varphi^* \) by changing variables in the integral giving the inner product. The formula is more or less simple depending on the multiplicity of \( \varphi \), but in any case, the operator can be described as a weighted expectation operator. If \( \varphi \) is a general analytic function mapping \( D \) into itself, no satisfactory formula for \( C_\varphi^* \) is known. In this section, we obtain a simple formula for \( C_\varphi^* \) when \( \varphi \) is a linear fractional transformation. The adjoint is a product of Toeplitz operators and a composition operator.

**Lemma 1.** If \( \varphi(z) = (az + b)(cz + d)^{-1} \) is a linear fractional transformation mapping \( D \) into itself, where \( ad - bc = 1 \), then \( \sigma(z) = (\bar{a}z - \bar{c})(\bar{b}z + \bar{d})^{-1} \) maps \( D \) into itself.

**Proof.** Linear fractional transformations may be regarded as one-to-one mappings of the Riemann sphere onto itself. Let \( \hat{D} \) denote the open set

\[
\hat{D} = \{ z : |z| > 1 \} \cup \{ \infty \}.
\]
Now, \( \varphi \) maps \( D \) into itself, so \( \gamma(z) = \overline{\varphi(z)} \) also maps \( D \) into itself. It follows that \( \gamma^{-1}(z) \) maps \( \overline{D} \) into itself. An easy calculation shows that
\[
\sigma(z) = \frac{1}{\gamma^{-1}(\frac{z}{2})}
\]
which implies \( \sigma(z) \) maps \( D \) into \( D \).

Recall that for \( g \) in \( L^\infty(\partial D) \), the Toeplitz operator \( T_g \) is the operator on \( H^2 \) given by \( T_g(f) = Pf g \) for \( f \) in \( H^2 \) and \( P \) the orthogonal projection of \( L^2 \) onto \( H^2 \). (For general properties of Toeplitz operators, see [7, Chapter 7].)

**Theorem 2.** Let \( \varphi(z) = (az + b)(cz + d)^{-1} \) be a linear fractional transformation mapping \( D \) into itself, where \( ad - bc = 1 \).
Then \( \sigma(z) = (\overline{az - c})(-\overline{bz + d})^{-1} \) maps \( D \) into itself, \( g(z) = (-\overline{b}z + d)^{-1} \) and \( h(z) = cz + d \) are in \( H^\infty \), and
\[
C_\varphi^* = T_g C_\sigma T_h^*.
\]

**Proof.** The function \( h \) is clearly in \( H^\infty \). By Lemma 1, \( \sigma \) maps \( D \) into itself and since the denominators of \( \sigma \) and \( g \) are the same, \( g \) is in \( H^\infty \). This means the formula makes sense.

Now, for \( \alpha \) in \( D \), let \( K_\alpha(z) = (1-\overline{\alpha}z)^{-1} \). This function is the reproducing kernel at \( \alpha \), that is, \( \langle f, K_\alpha \rangle = f(\alpha) \) for \( f \) in \( H^2 \). It is easily proved that \( T_h^* K_\alpha = \overline{h(\alpha)} K_\alpha \) and \( C_\varphi^* K_\alpha = K_{\varphi(\alpha)} \).
Calculation gives
\[
T_g C_\sigma T_h^*(K_\alpha)(z) = \overline{h(\alpha)} T_g C_\sigma (K_\alpha)(z)
\]
\[
= \overline{c\alpha + d} \left( \frac{1}{-\overline{b}z + d} \right) \left( \frac{1}{1 - \overline{\alpha} \frac{z - \overline{c}}{-\overline{b}z + d}} \right)
\]
\[
= \frac{c\alpha + d}{-\overline{b}z + d - \overline{\alpha}z + \overline{\alpha}c}
\]
\[
= \frac{1}{1 - \overline{\varphi(\alpha)}z} = K_{\varphi(\alpha)}(z) = C_\varphi^*(K_\alpha)(z).
\]
Since the \( K_\alpha \) span a dense set of \( H^2 \), the desired equality holds.

**A Norm Calculation**

The best general estimate of the norm of \( C_\varphi \) is
\[
\frac{1}{\sqrt{1 - |\varphi(0)|^2}} \leq \|C_\varphi\| \leq \frac{1 + |\varphi(0)|}{\sqrt{1 - |\varphi(0)|^2}}
\]
and both inequalities can be achieved by linear fractional transformations [3, page 81].

In this section, we use the adjoint calculation of the previous section to find the norm of the composition operators with affine symbol. It will become clear that the norms of composition operators depend in a rather complex way on the parameters of the symbol.

**Theorem 3.** If \( \varphi(z) = sz + t \) for \( |s| + |t| \leq 1 \), then
\[
\|C_\varphi\| = \sqrt{\frac{2}{1 + |s|^2 + |t|^2 + \sqrt{(1 - |s|^2 + |t|^2)^2 - 4|t|^2}}}
\]

**Proof.** In the trivial cases \( s = 0 \) and \( t = 0 \), the formula gives the correct norm \( \|C_\varphi\| = (1 - |t|^2)^{-\frac{1}{2}} \), [3, page 81]. We therefore assume \( s \) and \( t \) are non-zero.
The function $\varphi$ has not been presented in a way that we may directly apply the adjoint calculation; choosing $a^2 = s$ and $b = t/a$, a normalized expression for $\varphi$ is $\varphi(z) = (az + b)(0z + a^{-1})^{-1}$. In the notation of theorem 2, $C^*_\varphi = T_g C_G T^*_h$ where
\[
g(z) = \left(-\bar{b}z + \alpha^{-1}\right)^{-1},
\]
\[
\sigma(z) = \frac{-\bar{a}z}{-\bar{b}z + \alpha^{-1}},
\]
and $h(z) = a^{-1}$.

Thus,
\[
C^*_\varphi C_\varphi = \alpha^{-1} T_g C_G C_\varphi = T_{(1-\alpha^2)} C_G C_\varphi = T_f C_\psi
\]
where $f(z) = (1 - \bar{t}z)^{-1}$ and
\[
\psi(z) = \varphi(\sigma(z)) = \frac{(|s|^2 - |t|^2)z + t}{-\bar{t}z + 1}.
\]

Now
\[
\|C_\varphi\|^2 = \|C^*_\varphi C_\varphi\| = \lim_{n \to \infty} \|(C^*_\varphi C_\varphi)^n\|^{\frac{1}{n}}
\]
\[
= \lim_{n \to \infty} \|T_f T_0 \psi \cdots T_0 \psi_{n-1} C_\psi^n\|^{\frac{1}{n}}
\]
\[
\leq \lim_{n \to \infty} \left(\|f\| \|f \circ \psi\| \cdots \|f \circ \psi_{n-1}\|\right)^{\frac{1}{n}} \lim_{n \to \infty} \|C_\psi^n\|^{\frac{1}{n}}.
\]

(Here $\psi_k$ denotes the $k^{th}$ iterate of $\psi$.) The last quantity in this expression is just the spectral radius of $C_\psi$ which was calculated in [3, Theorem 2.1]. If $|s| + |t| = 1$, then $\psi(t/|t|) = t/|t|$ and $\psi'(t/|t|) = 1$, so the spectral radius of $C_\psi$ is 1. If $|s| + |t| < 1$, then $\psi$ maps the closed disk into the open disk and $C_\psi$ is compact and has spectral radius 1. Thus, the last quantity in this expression is always 1.

In the case $|s| + |t| = 1$, since $0 < |t| < 1$, we find
\[
|\psi(-t/|t|)| = \left|3 - \frac{4}{1+|t|}\right| < 1.
\]

This information, together with the fact from the above paragraph that the Denjoy-Wolff point of $\psi$ is $t/|t|$, implies that $\psi$ maps the closed unit disk onto a proper subdisk internally tangent to the unit circle at $t/|t|$. In particular, this means that $\lim_{n \to \infty} \psi_n(z) = t/|t|$, uniformly, in the closed unit disk (see, for example, [2]). It follows that $\lim_{n \to \infty} f \circ \psi_n = (1 - |t|)^{-1} = |s|^{-1}$ so
\[
\lim_{n \to \infty} \left(\|f\| \|f \circ \psi\| \cdots \|f \circ \psi_{n}\|\right)^{\frac{1}{n}} = |s|^{-1}.
\]

The above inequality now implies that $\|C_\varphi\| \leq |s|^{-\frac{1}{2}}$.

On the other hand, taking $\alpha = rt|s|/(|s| |t|)$, we find
\[
\|C_\varphi\|^2 \geq \lim_{r \to 1} \left(\|\frac{C^*_\varphi K_\alpha}{\|K_\alpha\|^2}\right)^2 = \lim_{r \to 1} \left(\|\frac{K_\alpha}{\|K_\alpha\|^2}\right)^2
\]
\[
= \lim_{r \to 1} \frac{1 - r^2}{1 - (r|s| + |t|)^2} = \frac{1}{|s|}
\]
Thus, $\|C_\varphi\| = |s|^{-\frac{1}{2}}$ when $|s| + |t| = 1$, which agrees with the conclusion for this case.

In the case $|s| + |t| < 1$, then $\psi(z) = (pz + t)(-\bar{t}z + 1)^{-1}$, where $p = |s|^2 - |t|^2$, and the fixed point of $\psi$ in $D$ is the smaller solution of
\[
\bar{t}z^2 + (p - 1)z + t = 0
\]
that is, the smaller of
\[ z = \frac{1 - p \pm \sqrt{(p - 1)^2 - 4|t|^2}}{2t} . \]
Noting that \(-1 < p < 1\) and \((p - 1)^2 \geq (1 - (1 - |t|)^2 + |t|^2)^2 = 4|t|^2\), we see that the numerator is a positive number in either case, so the Denjoy-Wolff point is
\[ \hat{\alpha} = \frac{1 - p - \sqrt{(p - 1)^2 - 4|t|^2}}{2t} . \]
As before, \(\lim_{n \to \infty} f \circ \psi_n = f(\hat{\alpha})\), and \(\|C_{\varphi}f\|^2 \leq |f(\hat{\alpha})|\). On the other hand, since
\[
C_{\varphi}^* C_{\varphi}(K_{\hat{\alpha}}) = \left( C_{\varphi}^* C_{\varphi} \right)^*(K_{\hat{\alpha}}) = (T_f C_{\psi})^*(K_{\hat{\alpha}})
\]
\[ = C_{\psi}^* T_f^*(K_{\hat{\alpha}}) = \frac{f(\hat{\alpha})}{K_{\psi}(\hat{\alpha})} K_{\psi}(\hat{\alpha}) = \frac{f(\hat{\alpha})}{K_{\psi}(\hat{\alpha})} K_{\hat{\alpha}}, \]
we see that \(\|C_{\varphi}\|^2 = \|C_{\varphi}^* C_{\varphi}\| \geq |f(\hat{\alpha})|\). Therefore, in this case, \(\|C_{\varphi}\| = \sqrt{|f(\hat{\alpha})|}\) which is the conclusion of the theorem.

Except when \(s > 0\) and \(|t| = 1 - s\), the operator \(C_{\varphi}^2\) is compact and the spectral radius is 1. Except when \(t = 0\), the norm of \(C_{\varphi}\) is greater than 1, so in general, we see the spectral radius is less than the norm.

### Co-subnormality for the Inner Hyperbolic Transformations

Let \(\varphi\) be an inner linear fractional transformation with fixed points \(\pm 1\). Nordgren, Rosenthal, and Wintrobe [14] and Cowen and Kriete [5] have given proofs that such \(C_{\varphi}^*\) are subnormal. Nordgren, Rosenthal, and Wintrobe, in addition, study other properties of these operators, and Cowen and Kriete compute the associated measure and study co-subnormality of other composition operators. In this section, we use the adjoint calculation to give a very easy proof of the co-subnormality based on a condition of Embry [8]. This proof does not construct the associated scalar measure, but constructs an associated operator measure.

**THEOREM 4.** For \(0 < r, r \neq 1\), if
\[ \varphi(z) = \frac{(r^{-1} + r)z + (r^{-1} - r)}{(r^{-1} - r)z + (r^{-1} + r)} \]
is the associated inner linear fractional transformation with fixed points \(\pm 1\), then \(C_{\varphi}^*\) is subnormal.

**PROOF.** We will use Embry’s condition [8]:
\(S\) is subnormal if and only if there is a positive operator measure \(Q\) such that
\[ (S^*)^n S^n = \int t^{2n} dQ(t). \]
To apply Embry’s condition to \(C_{\varphi}^*\), we must calculate \(C_{\varphi}^n (C_{\varphi}^*)^n = C_{\varphi}^n C_{\varphi}^n\).

A straightforward calculation gives
\[
\varphi_n(z) = \frac{1}{2} (r^{-n} + r^n) z + \frac{1}{2} (r^{-n} - r^n)
\]
\[ = \frac{1}{2} (r^{-n} - r^n) z + \frac{1}{2} (r^{-n} + r^n). \]
By theorem 2, \(C_{\varphi}^* = T_g C_{\sigma} T_h^*\) where
\[ g(z) = 2 \left( - (r^{-1} - r) z + (r^{-1} + r) \right)^{-1}, \]
\[ \sigma(z) = \frac{(r^{-1} + r) z - (r^{-1} - r)}{- (r^{-1} - r) z + (r^{-1} + r)}. \]
Thus, $Q$ is a positive operator valued measure, and Embry’s theorem implies that $C^*_\varphi$ is subnormal.

COROLLARY For $0 < r, r \neq 1$, let

$$\varphi_t(z) = \frac{(r^{-t} + r^t)z + (r^{-t} - r^t)1}{(r^{-t} - r^t)z + (r^{-t} + r^t)}$$

be the associated representation of the group of inner linear fractional transformations with fixed points $\pm 1$. Then the group of operators $\{C^*_\varphi_t : t \in \mathbb{R}\}$ is a subnormal group.

PROOF. By a theorem of Ito [9], it is sufficient to prove that each operator $C^*_\varphi_t$ is subnormal which is a consequence of theorem 4.

Hyponormality and Subnormality of $C_\varphi$

In [5, theorem 1.2], it is noted that if $C_\varphi$ is hyponormal, then $\varphi(0) = 0$. In this section, using the adjoint formula of theorem 2, we find all hyponormal composition operators with linear fractional symbol. As Nordgren observed in [12], if $\varphi$ is any inner function with $\varphi(0) = 0$, then $C_\varphi$ is an isometry and is subnormal, so we have not found a complete list of all hyponormal composition operators on $H^2$.

An easy calculation shows that if $\varphi(z) = z(uz + v)^{-1}$ then $\varphi$ maps $D$ into itself if and only if $|v| > 1 + |u|$. H. J. Schwartz [15] proved that $C_\varphi$ is normal whenever $u = 0$, so the following theorem covers the remaining cases.
THEOREM 5. For \( u \neq 0 \) and \(|v| \geq 1 + |u|\), if \( \varphi(z) = z(uz + v)^{-1} \), the following are equivalent:

(i) \( C_\varphi \) is subnormal.
(ii) \( C_\varphi \) is hyponormal.
(iii) \( v > 1 \) and \(|u| = v - 1\).

PROOF. Subnormality always implies hyponormality, so (i)\(\rightarrow\)(ii) trivially.

We begin by noting that the vector \( K_0 = 1 \) is an eigenvector for both \( C_\varphi \) and \( C_\varphi^* \) since \( (C_\varphi 1)(z) = 1(\varphi(z)) = 1 \) and \( C_\varphi^*(K_0) = K_\varphi(0) = K_0 \). This means that \( zH^2 \), which is the orthogonal complement of the subspace of constants, is a reducing subspace for \( C_\varphi \). Therefore \( C_\varphi \) is hyponormal or subnormal if and only if \( C_\varphi |_{zH^2} \) is hyponormal or subnormal.

Choosing \( \alpha^{-2} = v \) and \( \beta = \alpha u \), a normalized form for \( \varphi \) is

\[
\varphi(z) = \alpha z(\beta z + \alpha^{-1})^{-1}.
\]

By theorem 2, this means \( C_\varphi^* = T_2 C_\sigma T_h^* \) where

\[
g(z) = (\overline{\alpha}^{-1} - 0z)^{-1} = \overline{\alpha},
\]

\[
\sigma(z) = (\overline{\alpha} z - \overline{\beta})(\overline{\alpha}^{-1} - 0z)^{-1} = \overline{\nu}^{-1}z - (u/v),
\]

and \( h(z) = \alpha^{-1} + \beta z \).

Thus

\[
C_\varphi^* = \overline{\alpha} C_\sigma T_{\overline{\alpha}^{-1} + \overline{\beta} z} = C_\sigma T_{1+(u/v)z}.
\]

If \( zF \) is in \( zH^2 \), then this formula shows that

\[
C_\varphi^*(zF) = C_\sigma T_{1+(u/v)z}(zF)
= C_\sigma \left( zF + (u/v)F \right)
= \left( \overline{\nu}^{-1}z - (u/v) \right) F \circ \sigma + (u/v) F \circ \sigma
= \overline{\nu}^{-1}z F \circ \sigma = z(\overline{\nu}^{-1} F \circ \sigma).
\]

That is, if \( U \) is the unitary operator from \( H^2 \) onto \( zH^2 \) given by \( UF = zF \), then

\[
U^*(C_\varphi^* |_{zH^2})U = \overline{\nu}^{-1}C_\sigma.
\]

Taking adjoints in this equation, we see that \( C_\varphi \) is subnormal or hyponormal if and only if \( C_\varphi^* \) is.

If (iii) is true, then \( 0 < v^{-1} < 1 \) and \(|u|/v = 1 - v^{-1}\) so by [5, theorem 2.2] \( C_\sigma^* \) is subnormal and \( C_\varphi \) is also.

On the other hand, if \( C_\varphi \) is hyponormal, then \( C_\varphi^* \) is hyponormal and its spectral radius and norm are equal. By [3, theorem 2.1] the spectral radius of \( C_\sigma \) is 1 if \( \sigma \) has a fixed point in \( D \). Since \( \sigma(0) = -(u/v) \neq 0 \), the norm of \( C_\sigma \) is not 1 and \( \sigma \) does not have a fixed point in \( D \). It follows that \( \sigma \) has a fixed point \( c \) on the unit circle, and by [3, theorem 2.1], we see that \( 0 < \sigma'(c) = \overline{\nu}^{-1} \), and the spectral radius of \( C_\sigma \) is \( \sqrt{\nu} \). Now since \( ||C_\sigma|| = \sqrt{\nu} \), theorem 3 implies that \(|u|/v = 1 - v^{-1}\) so that \(|u| = v - 1 \) and \( v > 1 \) as in (iii). \( \blacksquare \)
References


Department of Mathematics
Purdue University
West Lafayette, Indiana 47907, U.S.A.