Commutators of composition operators with adjoints of composition operators on weighted Bergman spaces

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Composition Operators

- The weighted Bergman space $A^2_\alpha(\mathbb{D})$: $f$ analytic in $\mathbb{D}$,
  \[\|f\|_\alpha := \left(\int_{\mathbb{D}} |f(z)|^2 w_\alpha(z) \, dA(z)\right)^{1/2} < \infty,\]
  where $w_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha$.

- The Hardy space $H^2(\mathbb{D})$: $f$ analytic in $\mathbb{D}$,
  \[\|f\| := \left(\sup_{0<r<1} \int_{\partial \mathbb{D}} |f(r\zeta)|^2 \, d\sigma(z)\right)^{1/2} < \infty.\]

- If $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is analytic, we define $C_\varphi$ by
  \[C_\varphi(f) = f \circ \varphi,\]
  where $f$ is analytic in $\mathbb{D}$.
A self-map \( \varphi \) of \( \mathbb{D} \) is **linear-fractional** if

\[
\varphi(z) = \frac{az + b}{cz + d},
\]

for some \( a, b, c, d \in \mathbb{C} \) with \( ad - bc \neq 0 \).

The **Krein adjoint** of \( \varphi \) is the linear-fractional map

\[
\sigma_{\varphi}(z) = \frac{\bar{a}z - \bar{c}}{-bz + \bar{d}}.
\]

- If \( \varphi \) is a self-map of \( \mathbb{D} \), then so is \( \sigma_{\varphi} \).
- If \( \varphi \) has a fixed point at \( a \), then \( \sigma_{\varphi} \) has a fixed point at \( 1/\bar{a} \).
- If \( \varphi(\zeta) = \eta \) for some \( \zeta, \eta \in \partial \mathbb{D} \), then \( \sigma_{\varphi}(\eta) = \zeta \).
Essential Normality

- $C_\varphi$ is **normal** if

$$[C_{\varphi}^*, C_\varphi] := C_{\varphi}^* C_\varphi - C_\varphi C_{\varphi}^* = 0.$$

- $C_\varphi$ is normal in $H^2$ or $A^2_\alpha$ iff $\varphi(z) = az$, where $|a| \leq 1$.

- $C_\varphi$ is **essentially normal** if $[C_{\varphi}^*, C_\varphi]$ is compact.
If $\varphi$ is linear-fractional, then $C_\varphi$ is compact on $H^2$ or $A^2_\alpha$ iff $\|\varphi\|_\infty < 1$.

Thus, if $\|\varphi\|_\infty < 1$, then $C_\varphi$ is trivially essentially normal.

If $\varphi$ is linear-fractional and if $\|\varphi\|_\infty = 1$, then $\varphi$ must be one of the following:

- an automorphism of $\mathbb{D}$
- a hyperbolic non-automorphism (a fixed point in $\partial \mathbb{D}$ and a fixed point outside of $\partial \mathbb{D}$)
- a parabolic non-automorphism (a fixed point in $\partial \mathbb{D}$ of multiplicity two)
- a non-automorphism with no fixed point in $\partial \mathbb{D}$
Theorem (Bourdon, Levi, Narayan, Shapiro)

Let \( \varphi \) be a linear-fractional self-map of \( \mathbb{D} \) with \( \| \varphi \|_\infty = 1 \). Then \( C_\varphi \) is essentially normal on \( H^2(\mathbb{D}) \) iff \( \varphi \) is a rotation or \( \varphi \) is a parabolic non-automorphism.

Result was extended to \( A^2_\alpha \) by MacCluer, Narayan, Weir.
Overview of Proof

Theorem

If \( \varphi \) is an automorphism of \( \mathbb{D} \) which is not a rotation, then \( C_\varphi \) is not essentially normal on \( A^2(\mathbb{D}) \).

This gives the result in the automorphism case.

Theorem

If \( \varphi \) is a non-automorphic linear fractional self-map of \( \mathbb{D} \) with no fixed point on \( \partial \mathbb{D} \) and \( \| \varphi \|_\infty = 1 \), then \( C_\varphi \) is not essentially normal on \( A^2(\mathbb{D}) \).

Thus, we may assume that \( \varphi \) has a fixed point \( \zeta \) on \( \partial \mathbb{D} \).
Theorem (Cowen: $H^2$; Hurst: $A^2_\alpha$)

If $\varphi(z) = \frac{az + b}{cz + d}$, then

$$C^*_\varphi = T_g C_{\sigma_\varphi} T^*_h,$$

where

$$g(z) = (\overline{-bz + d})^{-\gamma} \text{ and } h(z) = (cz + d)^\gamma$$

with $\gamma = 1$ on $H^2(\mathbb{D})$ and $\gamma = \alpha + 2$ on $A^2_\alpha(\mathbb{D})$. 
Overview of Proof

This leads to the commutator formula

\[
[C^*_\varphi, C_\varphi] = T_g[C_\sigma, C_\varphi]T^*_h + T_g C_\sigma[T^*_h, C_\varphi] + T_{g - g \circ \varphi} C_{\sigma \circ \varphi} T^*_h
\]

Theorem

Let \( \varphi \) be a linear-fractional non-automorphic self-map of \( \mathbb{D} \) with \( \varphi(\zeta) = \eta \) for some \( \zeta, \eta \in \partial \mathbb{D} \). If \( b \) is continuous on \( \overline{\mathbb{D}} \) with \( b(\zeta) = 0 \), then the operator \( T_b C_\varphi \) is compact on \( A^2_\alpha(\mathbb{D}) \) for \( \alpha > -1 \).

- In our case, \( \varphi(\zeta) = \sigma(\zeta) = \zeta \), where \( \zeta \in \partial \mathbb{D} \).
- Therefore, \( (\sigma \circ \varphi)(\zeta) = \zeta \) and \( (g - g \circ \varphi)(\zeta) = 0 \).
- Thus, the third term is always compact in this setting.
Overview of Proof

Theorem

Let $\varphi$ be a linear-fractional non-automorphic self-map of $\mathbb{D}$ with $\varphi(\zeta) = \zeta$ for some $\zeta \in \partial \mathbb{D}$. Then $[T_{z_0}^*, C_\varphi]$ is compact on $A_\alpha^2(\mathbb{D})$ for every positive integer $n$.

- It follows that $[T_h^*, C_\varphi]$ is compact on $A_\alpha^2(\mathbb{D})$.
- Thus, the second term is always compact in this setting.
The first term will be compact precisely when $[C_\sigma, C_\varphi]$ is compact.

But $[C_\sigma, C_\varphi] = C_{\varphi \circ \sigma} - C_{\sigma \circ \varphi}$, which is compact iff $\sigma \circ \varphi = \varphi \circ \sigma$ (Moorhouse, 2005).

It can be shown that $\sigma \circ \varphi = \varphi \circ \sigma$ iff $\varphi$ is parabolic.
Compact Commutators

Next question: When is $[C^*_\psi, C_\varphi]$ compact?

The commutator $[C^*_\psi, C_\varphi]$ is **non-trivially compact** if

- $[C^*_\psi, C_\varphi]$ is compact,
- $[C^*_\psi, C_\varphi]$ is nonzero, and
- $C^*_\psi C_\varphi$ and $C_\varphi C^*_\psi$ are not compact.
Theorem (MacCluer, Narayan, Weir)

Let $\varphi$ and $\psi$ be linear-fractional self-maps of $\mathbb{D}$, at least one of which is a non-automorphism. The commutator $[C_\psi^*, C_{\varphi}]$ is non-trivially compact on $A^2_\alpha(\mathbb{D})$ if and only if either

1. $\varphi$ and $\psi$ are both parabolic with the same boundary fixed point, or

2. $\varphi$ and $\psi$ are both hyperbolic with the same boundary fixed point and with non-boundary fixed points which are conjugate reciprocals.

Result was proved in $H^2$ by Clifford, Levi, and Narayan.
Main Result: Automorphism Case

Theorem (MacCluer, Narayan, Weir)

Let $\varphi$ and $\psi$ be automorphisms of $\mathbb{D}$, neither of which equal to the identity map. The commutator $[C_\psi^*, C_\varphi]$ is compact on $A_\alpha^2(\mathbb{D})$ if and only if $\varphi$ and $\psi$ are both rotations.

Result was proved in $H^2$ by Clifford, Levi, and Narayan.
Theorem (MacCluer, Narayan, Weir)

Suppose \( \psi(z) = (az + b)/(cz + d) \) is a non-automorphic self-map of \( \mathbb{D} \) and that \( \psi(\zeta) = \eta \) for some \( \zeta, \eta \in \partial \mathbb{D} \). Let \( \sigma_\psi \) be the Krein adjoint of \( \psi \) and let

\[
s = \frac{(\overline{c}\zeta + \overline{d})^{\alpha+2}}{(-\overline{b}\eta + \overline{d})^{\alpha+2}}.
\]

Then there exists an operator \( K \), compact on \( A^2_\alpha(\mathbb{D}) \), such that

\[
C^*_\psi = sC_{\sigma_\psi} + K.
\]

It follows that

\[
[C^*_\psi, C_\varphi] \equiv s(C_{\varphi \circ \sigma_\psi} - C_{\sigma_\psi \circ \varphi})(\text{mod } K).
\]
Corollary

Let $\varphi$ and $\psi$ be linear-fractional self-maps of $\mathbb{D}$, at least one of which is a non-automorphism. If $[C_\psi^*, C_\varphi]$ is non-trivially compact on $A^2_\alpha(\mathbb{D})$, then

- $\varphi \circ \sigma_\psi = \sigma_\psi \circ \varphi$,
- $\varphi$ and $\sigma_\psi$ have a common set of fixed points, and
- $\varphi$ and $\psi$ have a common boundary fixed point.
Theorem (MacCluer, Narayan, Weir)

Let \( \varphi \) and \( \psi \) be linear-fractional self-maps of \( \mathbb{D} \), at least one of which is a non-automorphism. The commutator \([C^*_\psi, C_\varphi]\) is non-trivially compact on \( A^2_\alpha(\mathbb{D}) \) if and only if either

1. \( \varphi \) and \( \psi \) are both parabolic with the same boundary fixed point, or
2. \( \varphi \) and \( \psi \) are both hyperbolic with the same boundary fixed point and with non-boundary fixed points which are conjugate reciprocals.
Overview of Proof

($\Rightarrow$)

- Assume that $[C^*_\psi, C_\varphi]$ is non-trivially compact on $A^2_\alpha(\mathbb{D})$.
- By the Corollary, $\varphi \circ \sigma_\psi = \sigma_\psi \circ \varphi$, and $\varphi$ and $\sigma_\psi$ have a common set of fixed points.
- Case 1: Assume that $\psi$ is parabolic with fixed point $\zeta \in \partial \mathbb{D}$. Then
  - $\sigma_\psi$ is parabolic with fixed point $\zeta$, so
  - $\varphi$ is parabolic with fixed point $\zeta$.
- Case 2: Assume that $\psi$ is hyperbolic with fixed points $\zeta$ and $a$. Then
  - $\sigma_\psi$ is hyperbolic with fixed points $\zeta$ and $1/\bar{a}$, so
  - $\varphi$ is hyperbolic with fixed points $\zeta$ and $1/\bar{a}$.
Overview of Proof

\(\Leftarrow\)

- Assume that one of the two fixed point conditions hold.
- Then \(\varphi\) and \(\sigma_\psi\) have the same fixed point set, so

\[\varphi \circ \sigma_\psi = \sigma_\psi \circ \varphi.\]

- Therefore, \([C^*_\psi, C_\varphi]\) is compact on \(A^2_\alpha(D)\).
- The remainder of the proof involves checking that \([C^*_\psi, C_\varphi]\) is \textit{non-trivially} compact.
An Expression for $C^*_\psi$

Theorem (Bourdon, MacCluer: $H^2$, $A^2$; MacCluer, Narayan, Weir: $A^2_\alpha$)

Suppose that

$$\psi(z) = \lambda \frac{a - z}{1 - \bar{a}z},$$

where $a \in \mathbb{D}$ and $|\lambda| = 1$. Then

$$C^*_\psi = T_f C_{\psi^{-1}},$$

where $T_f$ is the Toeplitz operator with symbol

$$f(z) = \left( \frac{1 - |a|^2}{|1 - \lambda \bar{a}z|^2} \right)^\gamma$$

with $\gamma = 1$ on $H^2(\mathbb{D})$, and $\gamma = \alpha + 2$ on $A^2_\alpha(\mathbb{D})$. 
Automorphism Case

**Theorem**

If \( \varphi \) and \( \psi \) are automorphisms of \( \mathbb{D} \) and if \( [C_\psi^*, C_\varphi] \) is compact on \( A^2_{\alpha}(\mathbb{D}) \), then \( \varphi \circ \psi = \psi \circ \varphi \).

Why?

- Let \( \omega \in \partial \mathbb{D} \) and assume that \( [C_\psi^*, C_\varphi] \) is compact.
- Since \( \varphi \) and \( \psi \) are automorphisms, there exist points \( \zeta_1, \zeta_2 \in \partial \mathbb{D} \) with \( \omega = \varphi(\zeta_1) = \psi(\zeta_2) \).
- The normalized kernel functions \( k_{r\zeta_1} \) and \( k_{r\zeta_2} \) tend weakly to 0 as \( r \to 1^- \). Therefore, \( \langle [C_\psi^*, C_\varphi]k_{r\zeta_2}, k_{r\zeta_1} \rangle \to 0 \) as \( r \to 1^- \).
- Now,

\[
\langle [C_\psi^*, C_\varphi]k_{r\zeta_2}, k_{r\zeta_1} \rangle = \langle C_\varphi k_{r\zeta_2}, C_\psi k_{r\zeta_1} \rangle - \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle.
\]
Lemma

Let $\varphi, \psi$ be analytic self-maps of $\mathbb{D}$ with $\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial \mathbb{D}$. If the angular derivatives $\varphi'(\zeta_1)$ and $\psi'(\zeta_2)$ exist. Then
\[
\lim_{r \to 1^-} \left| \langle C_{\psi}^* k_r \zeta_2, C_{\varphi}^* k_r \zeta_1 \rangle \right| > 0 \text{ in } A_{2\alpha}^2(\mathbb{D}).
\]

Proposition

Let $\varphi, \psi$ be automorphisms of $\mathbb{D}$ with $\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial \mathbb{D}$. Then
\[
\lim_{r \to 1^-} \langle C_{\varphi} k_r \zeta_2, C_{\psi} k_r \zeta_1 \rangle = 0
\]
in $A_{2\alpha}^2(\mathbb{D})$ unless $\varphi^{-1}(\psi^{-1}(\omega)) = \psi^{-1}(\varphi^{-1}(\omega))$. 
Automorphism Case

If \( \varphi \) is an automorphism of \( \mathbb{D} \), then \( C_\varphi \) has polar decomposition

\[
C_\varphi = U_\varphi (C_\varphi^* C_\varphi)^{1/2},
\]

where \( U_\varphi \) is unitary.

**Lemma**

Let \( \varphi \) be an automorphism of \( \mathbb{D} \). If \( h \) is continuous on \( \partial \mathbb{D} \) or \( \overline{\mathbb{D}} \), then

\[
T_h U_\varphi \equiv U_\varphi T_{h \circ \varphi^{-1}} \mod \mathcal{K}
\]
on \( A^2_\alpha(\mathbb{D}) \) and \( H^2(\mathbb{D}) \).
Theorem (MacCluer, Narayan, Weir)

Let $\varphi$ and $\psi$ be automorphisms of $\mathbb{D}$, neither of which equal to the identity map. The commutator $[C^*_\psi, C_\varphi]$ is compact on $A^2(\mathbb{D})$ if and only if $\varphi$ and $\psi$ are both rotations.

($\Rightarrow$)

- Assume that $[C^*_\psi, C_\varphi]$ is compact on $A^2(\mathbb{D})$.
- Then
  - $\varphi$ and $\psi$ commute, and
  - $[C^*_\psi, C_\varphi]C_\psi$ is compact.
Overview of Proof

- Write
  
  \[ \psi(z) = \lambda_1 \frac{a_1 - z}{1 - \bar{a}_1 z} \quad \text{and} \quad \varphi(z) = \lambda_2 \frac{a_2 - z}{1 - \bar{a}_2 z}, \]
  
  where \( |\lambda_1| = |\lambda_2| = 1 \) and \( a_1, a_2 \in \mathbb{D} \).

- Then \( C^*_{\psi} = T_f C_{\psi^{-1}} \), where
  
  \[ f(z) = \left( \frac{1 - |a_1|^2}{|1 - \bar{\lambda}_1 a_1 z|^2} \right)^\gamma \]
  
  and \( C^*_{\varphi} = T_g C_{\varphi^{-1}} \), where
  
  \[ g(z) = \left( \frac{1 - |a_2|^2}{|1 - \bar{\lambda}_2 a_2 z|^2} \right)^\gamma. \]
Overview of Proof

Therefore,

\[
[C_{\psi}^*, C_{\varphi}] C_{\psi} = \left( T_f C_{\psi^{-1}} C_{\varphi} - C_{\varphi} T_f C_{\psi^{-1}} \right) C_{\psi} \\
= T_f C_{\psi \circ \varphi \circ \psi^{-1}} - C_{\varphi} T_f \\
= T_f C_{\varphi} - C_{\varphi} T_f
\]

Also,

\[
C_{\varphi} = U_{\varphi} (C_{\varphi}^* C_{\varphi})^{1/2} = U_{\varphi} T_g.
\]

Thus,

\[
T_f C_{\varphi} - C_{\varphi} T_f = T_f U_{\varphi} T_g - U_{\varphi} T_g T_f \\
\equiv U_{\varphi} T_{(f \circ \varphi^{-1} \circ f)} g \mod \mathcal{K}.
\]
Therefore, $T(f \circ \varphi^{-1} - f)g$ is compact.

Using the definitions of $f$ and $g$, we see that this forces $f$ to be constant.

Thus $\psi$ is a rotation.

Since $\varphi \circ \psi = \psi \circ \varphi$, it follows that $\varphi$ is also a rotation.
References