More Universal Operators Commuting with a Compact Operator

Carl C. Cowen

IUPUI

Celebrating Jim Jamison’s contributions to our part of mathematics!

Memphis, 18 October 2015
More Universal Operators Commuting with a Compact Operator

Carl C. Cowen

IUPUI

joint work with

Eva Gallardo Gutiérrez

Universidad Complutense de Madrid

Memphis, 18 October 2015
This talk is about work started about 3 years ago while on sabbatical from IUPUI at Depto. Análisis Matemático, Univ. Complutense de Madrid.

Most of the results in the talk are from our paper

Consequences of Universality Among Toeplitz Operators


Early work supported by Plan Nacional I+D grant no. MTM2010-16679.
Some terminology:

If $A$ is a bounded linear operator mapping a Banach space $\mathcal{X}$ into itself,

a closed subspace $M$ of $\mathcal{X}$ is an invariant subspace for $A$

if for each $v$ in $M$, the vector $Av$ is also in $M$.

The subspaces $M = (0)$ and $M = \mathcal{X}$ are trivial invariant subspaces and we are not interested in these.

The Invariant Subspace Question is:

• Does every bounded operator on a Banach space have a non-trivial invariant subspace?
We will only consider vector spaces over the complex numbers.

If the dimension of the space $\mathcal{X}$ is finite and at least 2, then any linear transformation has eigenvectors and each eigenvector generates a one dimensional (non-trivial) invariant subspace.

Knowledge of these invariant subspaces provides the foundation for constructing the Jordan Canonical Form for the transformation.

Once constructed, Jordan Canonical Form Theorem provides the information to find all of the invariant subspaces of a operator in finite dimensions.

Thus, knowledge of Invariant Subspaces is key in understanding the structures of operators!!
If $A$ is an operator on $\mathcal{X}$ and $x$ is a vector in $\mathcal{X}$,

then the *cyclic subspace generated by* $x$ *is the closure of*

\[
\{ p(A)x : p \text{ is a polynomial} \}
\]

Clearly, the cyclic subspace generated by $x$ is an invariant subspace for $A$.

If the cyclic subspace generated by the vector $x$ is all of $\mathcal{X}$,

we say *$x$ is a cyclic vector for* $A$.

Every cyclic subspace is separable, in the sense of topology, so if $\mathcal{X}$ is

*NOT* separable, every operator on $\mathcal{X}$ has non-trivial invariant subspaces.

**Therefore**, in thinking about the Invariant Subspace Question,

we restrict attention to infinite dimensional, separable Banach spaces.
Some history:

- Spectral Theorem: yes, self-adjoint and normal operators
- Beurling (1949): characterized invariant subspaces of isometric shift
- von Neumann ('51), Aronszajn & Smith ('54): yes, compact operators
- Lomonosov ('73): If $S$ is operator that commutes with operator $T \neq \lambda I$, and $T$ commutes with a non-zero compact operator then $S$ has a non-trivial invariant subspace.
- Lomonosov did not solve ISP: Hadwin, Nordgren, Radjavi, Rosenthal ('80)
- Enflo ('75/'87), Read ('85): No!! For some (Read: $\ell^1$) Banach spaces!
Some history:

- Spectral Theorem: yes, self-adjoint and normal operators
- Beurling (1949): characterized invariant subspaces of isometric shift
- von Neumann ('51), Aronszajn & Smith ('54): yes, compact operators
- Lomonosov ('73): If $S$ is operator that commutes with operator $T \neq \lambda I$, and $T$ commutes with a non-zero compact operator, then $S$ has a non-trivial invariant subspace.
- Lomonosov did not solve ISP: Hadwin, Nordgren, Radjavi, Rosenthal ('80)
- Enflo ('75/'87), Read ('85): No!! For some (Read: $\ell^1$) Banach spaces!

The (revised) Invariant Subspace Question is:

- Does every bounded operator on a Banach space have a non-trivial invariant subspace?
Rota’s Universal Operators:

**Defn:** Let $\mathcal{X}$ be a Banach space, let $U$ be a bounded operator on $\mathcal{X}$. We say $U$ is *universal for* $\mathcal{X}$ if for each bounded operator $A$ on $\mathcal{X}$, there is an invariant subspace $M$ for $U$ and a non-zero number $\lambda$ such that $\lambda A$ is similar to $U|_M$. 
Rota’s Universal Operators:

**Defn:** Let $\mathcal{X}$ be a Banach space, let $U$ be a bounded operator on $\mathcal{X}$. We say $U$ is *universal for* $\mathcal{X}$ if for each bounded operator $A$ on $\mathcal{X}$, there is an invariant subspace $M$ for $U$ and a non-zero number $\lambda$ such that $\lambda A$ is similar to $U|_M$

In other words, a universal operator on $\mathcal{X}$ has, inside of it, a miniature copy of *every* bounded operator on $\mathcal{X}$!!
**Rota’s Universal Operators:**

**Defn:** Let $\mathcal{X}$ be a Banach space, let $U$ be a bounded operator on $\mathcal{X}$. We say $U$ is *universal for* $\mathcal{X}$ if for each bounded operator $A$ on $\mathcal{X}$, there is an invariant subspace $M$ for $U$ and a non-zero number $\lambda$ such that $\lambda A$ is similar to $U|_M$

Rota proved in 1960 that if $\mathcal{X}$ is a separable, infinite dimensional Hilbert space, there are universal operators on $\mathcal{X}$!
Theorem (Caradus (1969))

If $\mathcal{H}$ is separable Hilbert space and $U$ is bounded operator on $\mathcal{H}$ such that:

- The null space of $U$ is infinite dimensional.
- The range of $U$ is $\mathcal{H}$.

then $U$ is universal for $\mathcal{H}$. 
The Hardy Hilbert space on the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is:

$$H^2 = \{h \text{ analytic in } \mathbb{D} : h(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|h\|^2 = \sum |a_n|^2 < \infty\}$$

Isometry $z^n \leftrightarrow e^{in\theta}$ shows $H^2$ ‘is’ subspace $\{h \in L^2(\partial \mathbb{D}) : h \sim \sum_{n=0}^{\infty} a_n e^{in\theta}\}$

$H^2$ is a *Hilbert space of analytic functions on* $\mathbb{D}$ *in the sense that*

for each $\alpha$, the linear functional on $H^2$ given by $h \mapsto h(\alpha)$ is continuous.

Indeed, the inner product on $H^2$ gives $h(\alpha) = \langle h, K_\alpha \rangle$

where $K_\alpha(z) = (1 - \bar{\alpha}z)^{-1}$ for $\alpha$ in $\mathbb{D}$. 
Consider four types of operators on $H^2$:

For $f$ in $L^\infty(\partial \mathbb{D})$, *Toeplitz operator* $T_f$ is operator given by $T_f h = P_+ f h$

where $P_+$ is the orthogonal projection from $L^2(\partial \mathbb{D})$ onto $H^2$

For $\psi$ a bounded analytic map of $\mathbb{D}$ into the complex plane,

the *analytic Toeplitz operator* $T_\psi$ is

$$(T_\psi h)(z) = \psi(z)h(z) \quad \text{for } h \text{ in } H^2$$

Note: for $\psi$ in $H^\infty$, $P_+ \psi h = \psi h$

For $\varphi$ an analytic map of $\mathbb{D}$ into itself, the *composition operator* $C_\varphi$ is

$$(C_\varphi h)(z) = h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

and for $\psi$ in $H^\infty$ and $\varphi$ an analytic map of $\mathbb{D}$ into itself,

the *weighted composition operator* $W_{\psi,\varphi} = T_\psi C_\varphi$ is

$$(W_{\psi,\varphi} h)(z) = \psi(z)h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$
Lemma.

If $f$ is a function in $H^\infty(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^\infty(\partial \mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator $T_f$. 
Lemma.

If \( f \) is a function in \( H^\infty(\mathbb{D}) \) and there is \( \ell > 0 \) so that \( |f(e^{i\theta})| \geq \ell \) almost everywhere on the unit circle, then \( 1/f \) is in \( L^\infty(\partial \mathbb{D}) \) and the (non-analytic) Toeplitz operator \( T_{1/f} \) is a left inverse for the analytic Toeplitz operator \( T_f \).

Theorem.

If \( f \) is a function in \( H^\infty(\mathbb{D}) \) for which there is \( \ell > 0 \) so that \( |f(e^{i\theta})| \geq \ell \) almost everywhere on the unit circle and \( Z_f = \{ \alpha \in \mathbb{D} : f(\alpha) = 0 \} \) is an infinite set, then the Toeplitz operator \( T_f^* \) is universal in the sense of Rota.
Lemma.

If \( f \) is a function in \( H^\infty(\mathbb{D}) \) and there is \( \ell > 0 \) so that \( |f(e^{i\theta})| \geq \ell \) almost everywhere on the unit circle, then \( 1/f \) is in \( L^\infty(\partial\mathbb{D}) \) and the (non-analytic) Toeplitz operator \( T_{1/f} \) is a left inverse for the analytic Toeplitz operator \( T_f \).

Theorem.

If \( f \) is a function in \( H^\infty(\mathbb{D}) \) for which there is \( \ell > 0 \) so that \( |f(e^{i\theta})| \geq \ell \) almost everywhere on the unit circle and \( Z_f = \{ \alpha \in \mathbb{D} : f(\alpha) = 0 \} \) is an infinite set, then the Toeplitz operator \( T_f^* \) is universal in the sense of Rota.

Proof:

By the Lemma, the analytic Toeplitz operator \( T_f \) has a left inverse, so the Toeplitz operator \( T_f^* \) has a right inverse and \( T_f^* \) maps \( H^2(\mathbb{D}) \) onto itself. Since \( T_f^*(K_\alpha) = \overline{f(\alpha)}K_\alpha = 0 \) for \( \alpha \) in \( Z_f \), the kernel of \( T_f^* \) is infinite dimensional. Thus, Caradus’ Theorem implies \( T_f^* \) is universal.
Some previously known Universal Operators (in sense of Rota):

Best Known: adjoint of a unilateral shift of infinite multiplicity:

If $S$ is analytic Toeplitz operator whose symbol is an inner function that is \textit{not} a finite Blaschke product, then $S^*$ is a universal operator.

Also well known: (Nordgren, Rosenthal, Wintrobe (’84,’87)):

If $\varphi$ is an automorphism of $\mathbb{D}$ with fixed points $\pm 1$ and Denjoy-Wolff point 1, that is, $\varphi(z) = \frac{z + s}{1 + sz}$ for $0 < s < 1$,

then a translate of the composition operator $C_\varphi$ is a universal operator.

(C. and Gallardo Gutiérrez showed (2011) that this translate, restricted to a co-dimension one invariant subspace on which it is universal, is unitarily equivalent to the adjoint of an analytic Toeplitz operator $T_\psi$.)
**Some previously known Universal Operators** (in sense of Rota):

Best Known: adjoint of a unilateral shift of infinite multiplicity:

If $S$ is analytic Toeplitz operator whose symbol is an inner function that is *not* a finite Blaschke product, then $S^*$ is a universal operator.

Also well known: (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If $\varphi$ is an automorphism of $\mathbb{D}$ with fixed points $\pm 1$ and Denjoy-Wolff point $1$, that is, $\varphi(z) = \frac{z + s}{1 + s z}$ for $0 < s < 1$,

then a translate of the composition operator $C_\varphi$ is a universal operator.

In C.’s thesis (’76): The analytic Toeplitz operators $S$ and $T_\psi$ (hence $C_\varphi$) *DO NOT* commute with non-trivial compact operators.

Also proved: *IF* an analytic Toeplitz operator commutes with a non-trivial compact, then the compact operator is quasi-nilpotent.
Some previously known Universal Operators (in sense of Rota):

Main Theorem of 2013 paper (C. and Gallardo Gutiérrez):

There are bounded analytic functions $\varphi$ and $\psi$ on the unit disk
and an analytic map $J$ of the unit disk into itself
so that the Toeplitz operator $T_{\varphi}^*$ is a universal operator in the sense of Rota
and the weighted composition operator $W_{\psi,J}^*$
is an injective, compact operator with dense range
that commutes with the universal operator $T_{\varphi}^*$. 
Observations:

- The best known operators that are universal in the sense of Rota are, or are unitarily equivalent to, adjoints of analytic Toeplitz operators.

- Some universal operators commute with compact operators and some do not.
Our Goals Today:

- There are *VERY MANY* analytic Toeplitz operators
  whose adjoints are universal operators in the sense of Rota
  
  and *VERY MANY* of them commute with non-trivial compact operators!
Our Goals Today:

- There are \textit{VERY MANY} analytic Toeplitz operators whose adjoints are universal operators in the sense of Rota and \textit{VERY MANY} of them commute with non-trivial compact operators!

- Describe some properties of such operators

- Raise two questions about invariant subspaces of ‘the’ Shift Operator that we haven’t been able to answer.
Let $\mathcal{U}_0$ be the set of adjoints of analytic Toeplitz operators that the Lemma above implies are left invertible, that is

$$\mathcal{U}_0 = \{ T_f^* : f \in H^\infty \text{ and } 1/f \in L^\infty(\partial\mathbb{D}) \}$$

and let

$$\mathcal{U} = \{ T_f^* \in \mathcal{U}_0 : \text{kernel}(T_f^*) \text{ is infinite dimensional} \}$$
Let $\mathcal{U}_0$ be the set of adjoints of analytic Toeplitz operators that the Lemma above implies are left invertible, that is

$$\mathcal{U}_0 = \{ T_f^* : f \in H^\infty \text{ and } 1/f \in L^\infty(\partial \mathbb{D}) \}$$

and let

$$\mathcal{U} = \{ T_f^* \in \mathcal{U}_0 : \text{kernel}(T_f^*) \text{ is infinite dimensional} \}$$

**Theorem.**

If $f$ is in $H^\infty$ and $T_f^*$ is in $\mathcal{U}$, the Toeplitz operator $T_f^*$ is universal for $H^2$. 

Let $\mathcal{U}_0$ be the set of adjoints of analytic Toeplitz operators that the Lemma above implies are left invertible, that is

$$\mathcal{U}_0 = \{ T_f^* : f \in H^\infty \text{ and } 1/f \in L^\infty(\partial\mathbb{D}) \}$$

and let

$$\mathcal{U} = \{ T_f^* \in \mathcal{U}_0 : \text{kernel}(T_f^*) \text{ is infinite dimensional} \}$$

**Theorem.**

If $f$ is in $H^\infty$ and $T_f^*$ is in $\mathcal{U}$, the Toeplitz operator $T_f^*$ is universal for $H^2$.

**Corollary.**

If $f$ and $g$ are in $H^\infty$ with $T_f^*$ in $\mathcal{U}$ and $T_g^*$ in $\mathcal{U}_0$,

then $T_f^* T_g^* = T_{fg}^*$ is also in $\mathcal{U}$ and is a universal operator for $H^2$. 
For $F$ bounded on $H^2$, the commutant of $F$ is the closed algebra in $\mathcal{B}(H^2)$

$$\{F\}' = \{G \in \mathcal{B}(H^2) : GF = FG\}$$

For $f$ in $H^\infty$, clearly $\{T_f^*\}'$ includes $T_g^*$ for all $g$ in $H^\infty$.

**Definition.** For $T_f^*$ in $\mathcal{U}$, let $\mathcal{C}_f$ be the set of compact operators in $\{T_f^*\}'$:

$$\mathcal{C}_f = \{G \in \mathcal{B}(H^2) : G \text{ is compact, and } T_f^*G = GT_f^*\}$$
For $F$ bounded on $H^2$, the commutant of $F$ is the closed algebra in $\mathcal{B}(H^2)$

$$\{F\}' = \{G \in \mathcal{B}(H^2) : GF = FG\}$$

For $f$ in $H^\infty$, clearly $\{T_f^*\}'$ includes $T_g^*$ for all $g$ in $H^\infty$.

**Definition.** For $T_f^*$ in $\mathcal{U}$, let $\mathcal{C}_f$ be the set of compact operators in $\{T_f^*\}'$:

$$\mathcal{C}_f = \{G \in \mathcal{B}(H^2) : G \text{ is compact, and } T_f^* G = GT_f^*\}$$

**Theorem.**

Let $T_f^*$ be in $\mathcal{U}$. The set $\mathcal{C}_f$ is a closed ideal in $\{T_f^*\}'$ and, in particular,

$g$ and $h$ in $H^\infty$ and $G$ in $\mathcal{C}_f$ implies $T_g^* G$, $GT_h^*$, and $T_g^* GT_h^*$ are all in $\mathcal{C}_f$. Moreover, *every* operator in $\mathcal{C}_f$ is quasi-nilpotent.
For some $T_f^*$ in $\mathcal{U}$, including all the classical universal operators noted above, the algebra $C_f$ is $\{0\}$.

On the other hand, for many operators $T_f^*$ in $\mathcal{U}$, including the example $T_\varphi^*$ from our earlier paper, the algebra $C_f$ is quite large!!
For some $T_f^*$ in $\mathcal{U}$, including all the classical universal operators noted above, the algebra $\mathcal{C}_f$ is $\{0\}$.

On the other hand, for many operators $T_f^*$ in $\mathcal{U}$, including the example $T_\varphi^*$ from our earlier paper, the algebra $\mathcal{C}_f$ is quite large!!

Following is a trivial, but surprising, application of Lomonosov’s theorem:

**Theorem. (!!)**

If $f$ is a non-constant function in $H^\infty$ for which $\mathcal{C}_f \neq \{0\}$, there is a backward shift invariant subspace, $L = (\eta H^2)^\perp$ for some inner function $\eta$, that is invariant for every operator in $\{T_f^*\}'$. 
For some $T_f^*$ in $\mathcal{U}$, including all the classical universal operators noted above, the algebra $\mathcal{C}_f$ is $\{0\}$.

On the other hand, for many operators $T_f^*$ in $\mathcal{U}$, including the example $T_\varphi^*$ from our earlier paper, the algebra $\mathcal{C}_f$ is quite large!!

Following is a trivial, but surprising, application of Lomonosov’s theorem:

**Theorem.** (!!)

If $f$ is a non-constant function in $H^\infty$ for which $\mathcal{C}_f \neq \{0\}$, there is a backward shift invariant subspace,

$$L = (\eta H^2)^\perp$$

for some inner function $\eta$, that is invariant for every operator in $\{T_f^*\}'$.

**Proof:** $T_{z^*}$ commutes with $T_f^*$. 
Theorem. (!!)  

If \( f \) is a non-constant function in \( H^\infty \) for which \( C_f \neq \{0\} \),  
there is a backward shift invariant subspace,  
\[ L = (\eta H^2)^\perp \]  
for some inner function \( \eta \),  
that is invariant for every operator in \( \{T_f^*\}' \).  

In the case of the \( T_\varphi^* \) and the compact operator \( W_\psi^*:J \) noted above,  
the commutant \( \{T_\varphi^*\}' \) is known!
Theorem. (!!)

If $f$ is a non-constant function in $H^\infty$ for which $C_f \neq \{0\}$,

there is a backward shift invariant subspace,

$L = (\eta H^2)^\perp$ for some inner function $\eta$,

that is invariant for every operator in $\{T_f^*\}'$.

In the case of the $T^*_\varphi$ and the compact operator $W^*_\psi$, noted above,

the commutant $\{T^*_\varphi\}'$ is known!

It is the algebra generated by $T^*_z$ and $C^*_J$!
To prove the Invariant Subspace Theorem, need to show that every bounded operator, $A$, on $H^2$ has an invariant subspace. But the universality of $T_f^*$ in $\mathcal{U}$ means that we are interested only in restrictions of $T_f^*$ to its infinite dimensional invariant subspaces, $M$.

This means the Invariant Subspace Theorem will be proved if every infinite dimensional invariant subspace, $M$, for $T_f^*$ contains a smaller subspace that is also invariant for $T_f^*$. 

Our strategy for applying universal Toeplitz operators to the Invariant Subspace Problem is to also consider operators that commute with the universal operator.

**Theorem.**

Let $T$ be a universal operator on $H^2$ that is in the class $\mathcal{U}$, and let $M$ be an infinite dimensional, proper invariant subspace for $T$.

If $W$ is an operator on $H^2$ that commutes with $T$, then

- either $\text{ker}(W) \cap M = (0)$, or $M \subset \text{ker}(W)$,
- or $\text{ker}(W) \cap M$ is a proper subspace of $M$ that is invariant for $T$. 
**Theorem.**

Let $T$ be a universal operator on $H^2$ that is in the class $\mathcal{U}$,
and let $M$ be an infinite dimensional, proper invariant subspace for $T$.

If $W$ is an operator on $H^2$ that commutes with $T$, then

either $\ker(W) \cap M = (0)$, or $M \subset \ker(W)$,

or $\ker(W) \cap M$ is a proper subspace of $M$ that is invariant for $T$.

**Corollary.**

Let $M$ be an infinite dimensional, proper invariant subspace for $T$,
a universal operator on $H^2$ that is in the class $\mathcal{U}$.

If $M$ contains a vector, $v \neq 0$, that is non-cyclic vector for the backward shift
and $\eta$ is smallest inner function for which $T_{\eta}^*v = 0$, then $M \subset \ker(T_{\eta}^*)$,
or else $\ker(T_{\eta}^*) \cap M$ is a non-trivial invariant subspace for $T$. 
This suggests the question

*Does every closed, infinite dimensional subspace of $H^2$ include a non-zero, non-cyclic vector for the backward shift?*

but Prof. N. Nikolski pointed out that the answer to this question is “No!”.
This suggests the question

*Does every closed, infinite dimensional subspace of $H^2$ include a non-zero, non-cyclic vector for the backward shift?*

but Prof. N. Nikolski pointed out that the answer to this question is “No!”.

On the other hand, we are not interested in arbitrary subspaces of $H^2$ so we specialize our query to address the issue at hand:

**Question 1:** *Is there an operator in the class $\mathcal{U}$ for which each of its closed, infinite dimensional, invariant subspaces includes a non-zero vector that is not cyclic for the backward shift?*
The other alternative in the Corollary above is that \( M \subset \text{kernel}(T_\eta^*) \) and every vector in \( M \) is non-cyclic for the backward shift! Thus, we have

**Corollary.**

If \( M \) is an infinite dimensional, proper invariant subspace for \( T \), a universal operator on \( H^2 \) that is in the class \( \mathcal{U} \) and \( M \) contains a vector, \( v \neq 0 \), that is not cyclic for the backward shift and also a vector \( w \) that is cyclic for the backward shift,

then, for \( \eta \) the smallest inner function for which \( T_\eta^*v = 0 \), the subspace \( \text{kernel}(T_\eta^*) \cap M \) is a proper subspace of \( M \) that is invariant for \( T \).
On the other hand, another possible reduction for this situation leads to the following question:

**Question 2:** Suppose $M$ is an infinite dimensional closed subspace that is invariant for $T$, a universal operator in the class $\mathcal{U}$, and suppose $\eta$ is an inner function for which $M \subset \text{kernel}(T_{\eta}^*)$.

Is there always an inner function $\zeta$ dividing $\eta$ so that

\[(0) \neq M \cap \text{kernel}(T_{\zeta}^*) \neq M?\]
On the other hand, another possible reduction for this situation leads to the following question:

**Question 2:** Suppose $M$ is an infinite dimensional closed subspace that is invariant for $T$, a universal operator in the class $\mathcal{U}$, and suppose $\eta$ is an inner function for which $M \subset \text{kernel}(T_\eta^*)$.

Is there always an inner function $\zeta$ dividing $\eta$ so that

$$(0) \neq M \cap \text{kernel}(T_\zeta^*) \neq M?$$

If the answers to both Question 1 and Question 2 are ‘Yes’, then every bounded operator on a Hilbert space of dimension at least 2 has a non-trivial invariant subspace!
THANK YOU!

Slides available:  http://www.math.iupui.edu/~cowen