Math 276: Solution to Problem 34b, page 160

**Problem:** Find a one-to-one correspondence between the natural numbers and the set of (positive) integers that are divisible by 5 but not divisible by 7.

**Construction:**

The first observation is that an integer $m$ is a multiple of 5 if $m = 5n$ for some integer $n$, and it is also a multiple of 7 if and only if $n$ is a multiple of 7. More specifically, $n$ is a multiple of 7, say $n = 7k$, then $m = 5n = 5(7k) = 7(5k)$ so that $m = 5n$ is also a multiple of 7. On the other hand, if $n$ is not a multiple of 7, then $m = 5n$ is not a multiple of 7:

For each of the cases in which $n$ is not divisible by 7, we see that $m = 5n$ is also not divisible by 7 because the above shows that $m$ is 7 times an integer plus one of the remainders 1, 2, 3, 4, 5, or 6.

Thus, the integers we want to identify come from the 6 different remainders that give $n$ not divisible by 7. We can pair these integers with the set of natural numbers by thinking about whether they are, or are not divisible by 6:

For $p$ a positive integer,

$$f(p) = \begin{cases} 
5(7k + 1) & \text{if } p = 6k + 1 \\
5(7k + 2) & \text{if } p = 6k + 2 \\
5(7k + 3) & \text{if } p = 6k + 3 \\
5(7k + 4) & \text{if } p = 6k + 4 \\
5(7k + 5) & \text{if } p = 6k + 5 \\
5(7k - 1) & \text{if } p = 6k
\end{cases}$$

To check this, we can see $f(1) = 5$ (because $p = 1 = 6 \cdot 0 + 1$), $f(2) = 10$ (because $p = 2 = 6 \cdot 0 + 2$), $f(3) = 15$ (because $p = 3 = 6 \cdot 0 + 3$), $f(4) = 20$ (because $p = 4 = 6 \cdot 0 + 4$), $f(5) = 25$ (because $p = 5 = 6 \cdot 0 + 5$), $f(6) = 30$ (because $p = 6 = 6 \cdot 1 + 0$), $f(7) = 40$ (because $p = 7 = 6 \cdot 1 + 1$), $f(8) = 45$ (because $p = 8 = 6 \cdot 1 + 2$), $f(9) = 50$ (because $p = 9 = 6 \cdot 1 + 3$), $f(10) = 55$ (because $p = 10 = 6 \cdot 1 + 4$), $f(11) = 60$ (because $p = 11 = 6 \cdot 1 + 5$), $f(12) = 65$ (because $p = 12 = 6 \cdot 2 + 0$), $f(13) = 75$ (because $p = 13 = 6 \cdot 2 + 1$), etc.