Homework 2

**Definition** If $a$ and $b$ are integers, $a \neq 0$, we say $b$ is divisible by $a$ or $a$ divides $b$, and write \(a|b\), if there is an integer $x$ so that $b = ax$.

1. In the following statements, suppose $a$, $b$, $c$, $x$, and $y$ are integers.
   (a) Show that if $a|b$, then $a|(bc)$.
   (b) Prove that if $a|b$ and $b|c$, then $a|c$.
   (c) Show: If $a|b$ and $a|c$, then $a|(bx + cy)$ for any integers $x$ and $y$.
   (d) Prove: If $a|b$ and $b|a$, then $a = \pm b$.

2. Use the fact that every integer is either even or it is odd to show that for all integers, $n$, the number $n^2 - n$ is divisible by 2.

3. Show that for each integer $n$, either $n - 1$ is divisible by 3 or $n$ is divisible by 3 or $(n + 1)$ is divisible by 3.

4. (a) Show that for each integer $n$, the number $n^3 - n$ is divisible by 3.
   (b) Prove that for each integer $n$, the number $n^3 - n$ is divisible by 6.

**Definition** If $b$ and $c$ are integers, not 0, such that $a|b$ and $a|c$, we say $a$ is a common divisor of $b$ and $c$. Of course, 1 is divisor every integer, so for any integers $b$ and $c$, 1 is a common divisor of $b$ and $c$. Since every positive divisor of $b$ is less than or equal to $|b|$, there are only finitely many divisors of $b$, and every pair of integers has only finitely many common divisors. The greatest common divisor of $b$ and $c$ is the largest of the positive, common divisors of $b$ and $c$.

For example, the common divisors of 63 and 147 are $\pm 1$, $\pm 3$, $\pm 7$, and $\pm 21$, so the greatest common divisor of 63 and 147 is 21.

5. Find the greatest common divisor of each of given pairs of integers:

   \[
   \begin{align*}
   (a) & \quad 24 \text{ and } 84 & \quad (b) & \quad 525 \text{ and } 315 \\
   (c) & \quad 3003 \text{ and } 2805 & \quad (d) & \quad 11433 \text{ and } 23051
   \end{align*}
   \]