Dfn. The Cantor set, $C$, is the set of real numbers in $[0, 1]$ that can be represented in base 3 by a ternary expansion using only the digits 0 and 2. For example, $\frac{1}{7} = (.1000\cdots)_3 = (.0222\cdots)_3$ and $\frac{2}{3} = (.2000\cdots)_3 = (.1222\cdots)_3$ are both in $C$, but $\frac{7}{12} = (.120202\cdots)_3$ is not.

Dfn. The Cantor function, $c(x)$, is the function on $[0, 1]$ defined by

\[
c(x) = \begin{cases} 
  (\frac{d_1 d_2 d_3 \cdots}{2})_2 & x \in C, \ x = (d_1 d_2 d_3 \cdots)_3, \text{where each } d_j \neq 1 \\
  c(x_\ell) = c(x_r) & x_\ell = \max \{y \in C : y < x\} \ \text{and} \ x_r = \min \{y \in C : x < y\}
\end{cases}
\]

That is, for $x$ in the Cantor set, $c(x)$ has binary expansion obtained by dividing each digit in the ternary expansion of $x$ by 2. For example, $c(\frac{1}{3}) = c(\frac{7}{12}) = c(\frac{2}{3}) = \frac{1}{2}$.

1. Show that the Cantor function is continuous.
   (Hint: Use ternary expansions with $\delta$ and binary expansions with $\epsilon$.)

2. Suppose $s_n$, for $n = 1, 2, 3, \cdots$ is a sequence of real numbers and $\lim_{n \to \infty} s_n = S$. For each positive integer $n$, let $\sigma_n$ be defined by

\[
\sigma_n = \frac{1}{n}(s_1 + s_2 + s_3 + \cdots + s_n)
\]

Prove that $\lim_{n \to \infty} \sigma_n = S$ also.

3. Decide whether each of the following series converges or diverges, and prove your answer.

   (a) \[\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\]

   (b) \[\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}\]

   (c) \[\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}\]

   (d) \[\sum_{n=1}^{\infty} \frac{n!}{n^n}\]

4. Show that the series

\[\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots\]

converges, but the Ratio and Root tests do not apply.