1. Let $Q_0$ be the set of rational numbers in $[0, 1]$ and suppose \( \{I_n\}_{n=1}^N \) is a finite collection of open intervals covering $Q_0$. Prove that $\sum_{n=1}^N \ell(I_n) > 1$. (That is, the definition of outer measure critically depends on using a countably infinite cover.)

2. Suppose $A$ and $B$ are subsets of $\mathbb{R}$. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

3. A set $E$ is called a $G_\delta$ set if it is the intersection of a countable number of open sets. Note that all $G_\delta$ sets are in the Borel $\sigma$-algebra. Prove that every compact set in $\mathbb{R}$ is a $G_\delta$ set. (It is also true that every closed set in $\mathbb{R}$ is a $G_\delta$ set, so you may choose to do that if you find it more convenient.)

4. Show that if $A$ is a subset of $\mathbb{R}$, there is a $G_\delta$ set $E$ so that $A \subset E$ and $m^*(A) = m^*(E)$.

5. (a) Prove directly from the definition of outer measure that $m^*(C) = 0$, where $C$ is the Cantor set. (That is, do not use the fact that the Cantor set is measurable.)

(b) Using measurability, find an easier proof that $m^*(C) = 0$.

6. Find a sequence of measurable sets \( (E_n)_{n=1}^\infty \) so that $E_1 \supset E_2 \supset E_3 \supset \cdots$, and $m(E_n) = \infty$ for all $n$, but that $\cap_{n=1}^\infty E_n = \emptyset$. 