Problems

59. A necessary and sufficient condition for an operator to be an isometry is that its spectral radius be less than or equal to 1.

60. Let $H$ be a Hilbert space and $A$ be a continuous linear operator on $H$. Then $A$ is bounded if and only if its spectrum is contained in the closed unit disk.

61. Every bounded linear operator on a Hilbert space is the limit of a sequence of finite-rank operators.

62. If $A$ is a bounded linear operator on a Hilbert space and $x$ is a vector in the Hilbert space, then $\|Ax\| \leq \|A\|\|x\|$.

63. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is compact if and only if its spectrum is contained in the closed unit disk.

64. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is normal if and only if its spectrum is symmetric.

65. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is self-adjoint if and only if its spectrum is real.

66. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is unitary if and only if its spectrum is contained in the unit circle.

67. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is invertible if and only if its spectrum is contained in the open unit disk.

68. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is positive if and only if its spectrum is contained in the non-negative real numbers.

69. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is compact if and only if its spectrum is a countable union of closed intervals.

70. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is trace-class if and only if its spectrum is absolutely summable.

71. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is nuclear if and only if its spectrum is a compact discrete set.

72. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is Fredholm if and only if its spectrum is compact and its essential spectrum is empty.

73. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is Fredholm of index zero if and only if its spectrum is contained in the unit disk and the unit circle.

74. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is Fredholm of index one if and only if its spectrum is contained in the unit circle and the open unit disk.

75. Let $A$ be a bounded linear operator on a Hilbert space. Then $A$ is Fredholm of index $n$ if and only if its spectrum is contained in the unit circle and the open unit disk, and $A$ has $n$ eigenvalues on the unit circle.

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Suppose that the set \( \mathbb{N} \) of natural numbers is the field of real numbers, and let \( \mathbb{Q} \) be any other field. Then, for any \( \alpha \in \mathbb{Q} \), the expression \( \alpha x \) is defined for all \( x \in \mathbb{R} \).

Then, if \( \alpha \) is a real number, we have:

\[
\alpha x = x_1 + \alpha x_2
\]

and for \( \alpha = 0 \), we have:

\[
\alpha x = 0
\]

This implies that \( \alpha x \) is defined for all \( x \in \mathbb{R} \).

Also, the expression \( x^\alpha \) is defined for all \( x \in \mathbb{R} \) and all \( \alpha \in \mathbb{Q} \).

Finally, the expression \( \sqrt[\alpha]{x} \) is defined for all \( x \in \mathbb{R} \) and all \( \alpha \in \mathbb{Q} \), provided \( x \geq 0 \) when \( \alpha \) is even.

Solution 69. The problem of finding the solutions of the equation \( x^2 + 1 = 0 \) in \( \mathbb{C} \) is not straightforward. In order to solve this problem, we first need to find the roots of the equation:

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x^2 + 1 = 0
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This equation has no real solutions, but it does have complex solutions. In fact, the solutions are:

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x = \pm i
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where \( i \) is the imaginary unit, defined by \( i^2 = -1 \).

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