Composition Operators on
Spaces of Analytic Functions, III

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www.math.iupui.edu/~ccowen/EPAF09.html
Today, we want to consider more recent developments, some specific unsolved problems, and future directions in the subject.

**Circular Symmetry**

Wednesday we discussed spectra of composition operators in the various cases. In many of the non-compact cases, there is considerable circular symmetry in the spectra, in fact, all cases except the plane translation case.

- In the non-compact cases with a fixed point inside, the spectra were circularly symmetric except for finitely many points. When the Denjoy-Wolff point is on the boundary, spectra exhibit circular symmetry, except in the plane translation case.
• In one case, (half-plane dilation) the operator is similar to rotates of itself. In the other half-plane case, some parts of spectra have circular symmetry.

• If \( \alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots \) are points of disk satisfying \( \varphi(\alpha_k) = \alpha_{k+1} \), then

\[
C_\varphi^* \left( \frac{1}{\|K_{\alpha_k}\|} K_{\alpha_k} \right) = \frac{1}{\|K_{\alpha_k}\|} K_{\alpha_{k+1}} = \left( \frac{\|K_{\alpha_{k+1}}\|}{\|K_{\alpha_k}\|} \right) \left( \frac{1}{\|K_{\alpha_{k+1}}\|} K_{\alpha_{k+1}} \right)
\]

so \( \text{span}\{K_{\alpha_k}\} \) is an invariant subspace for \( C_\varphi^* \) and it looks like \( C_\varphi^* \) is a weighted shift on this subspace

In half-plane cases, this can be made into a proof that the restriction of \( C_\varphi \) to an invariant subspace is similar to every rotate of itself.
Conjecture (C. & MacCluer, 1998)

If \( \varphi \) has a fixed point in \( D \) and the essential spectral radius of \( C_\varphi \) is positive, there is an invariant subspace for \( C_\varphi \) on which the restriction of \( C_\varphi \) is similar to rotates of itself.


Other Problems

- Hyponormality and subnormality of \( C_\varphi \) or \( C^*_\varphi \)
- Equivalence (unitary equivalence or similarity)
- Commutant of \( C_\varphi \)
**Weighted Composition Operators**

For \( \varphi \) an analytic map of the disk into itself and \( \psi \) a analytic function of the disk into \( \mathbb{C} \) the *weighted composition operator* \( W_{\psi, \varphi} \) is the operator on \( H^2(\mathbb{D}) \) given by

\[
(W_{\psi, \varphi} f)(z) = \psi(z) f(\varphi(z))
\]

for \( z \) in \( \mathbb{D} \).

Weighted composition operators are an extension of the idea of composition operator, but also have a claim to be more basic because they arose before composition operators in some natural contexts.
The \textit{weighted composition operator} $W_{\psi,\varphi}$ is the operator on $H^2$ given by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$$

for $z$ in $\mathbb{D}$.

Clearly, if $\psi$ is in $H^\infty(\mathbb{D})$ and $\varphi$ maps the disk into the disk, then $W_{\psi,\varphi}$ is bounded and if $\psi$ is in $H^\infty$ and $C_\varphi$ is compact, then $W_{\psi,\varphi}$ is compact also.

Perhaps, surprisingly, these conditions are not necessary!

Because $W_{\psi,\varphi}1 = \psi$, the multiplier $\psi$ must be in $H^2$, but it is possible for $\psi$ to be unbounded in $H^2$ and $C_\varphi$ non-compact and have $W_{\psi,\varphi}$ bounded or compact.
Considered: conditions for $W_{ψ,φ}$ bounded, compact, or self-adjoint, 

spectra of compact or self-adjoint weighted composition operators, \cdots

Study of weighted composition operators just begun in an organized way \cdots

**Problems**

- Characterizations of boundedness and compactness are not yet in an easily applicable form

- Fredholm weighted composition operators not characterized; same for other classes except self adjoint

- Spectra of compact and self adjoint weighted composition operators completely understood, but no other spectra have been analyzed except special cases.
Adjoints

Descriptions of adjoints of operators are standard parts of the general
description of operators.

While the relation $C_\varphi^*(K_\alpha) = K_{\varphi(\alpha)}$ is very useful,
it does not extend easily to a formula for $C_\varphi^*$

Theorem (C, 1988).

If $\varphi(z) = \frac{az+b}{cz+d}$ is a non-constant linear fractional map of the unit disk
into itself, then

\[ C_\varphi^* = T_g C_\sigma T_h^* \]

where $\sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$, $g(z) = \frac{1}{-\bar{b}z + \bar{d}}$, and $h(z) = cz + d$. 
Inner functions can be handled

\( C_\phi^* \) can be explicitly described as an integral operator, it is a “folk theorem”

For \( z \) in \( \mathbb{D} \),

\[
(C_\phi^* f)(z) = \langle C_\phi^* f, K_z \rangle = \langle f, C_\phi K_z \rangle = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})}z} \frac{d\theta}{2\pi}
\]
Inner functions can be handled

\( C_{\phi}^* \) can be described as an integral operator, explicitly,

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For \( z \) in \( \mathbb{D} \),

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(C_{\phi}^* f)(z) = \langle C_{\phi}^* f, K_z \rangle = \langle f, C_{\phi} K_z \rangle = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \phi(e^{i\theta})z} \frac{d\theta}{2\pi}
\]

This “formula” is very difficult to actually use \( \cdots \)
$C^*_\varphi$ can be described as an integral operator, explicitly, it is a “folk theorem”

For $z$ in $\mathbb{D}$,

$$(C^*_\varphi f)(z) = \langle C^*_\varphi f, K_z \rangle = \langle f, C\varphi K_z \rangle = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \varphi(e^{i\theta})z} \frac{d\theta}{2\pi}$$

This “formula” is very difficult to actually use \ldots but it was this that allowed me to discover the formula for the linear fractional case.

It worked because linear fractional maps are rational functions and univalent \ldots
In the past decade or so, with contributions from several mathematicians, published and not published, we now have a formula for the adjoints of composition operators with symbol a rational function.

Wahl, 1997
Gallardo-Gutiérrez & Montes-Rodríguez, 2003
Martín & Vukotić, 2006
Hammond, Morehouse, & Robbins, 2008
Bourdon & Shapiro, preprint 2008
We were able to find a formula for $C^*_\varphi$ for $\varphi(z) = (z + z^2)/2$:

$$(C^*_\varphi f)(z) = \frac{z + \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} f \left( \frac{z + \sqrt{z^2 + 8z}}{4} \right) - \frac{z - \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} f \left( \frac{z - \sqrt{z^2 + 8z}}{4} \right)$$

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We were able to find a formula for \( C^*_\varphi \) for \( \varphi(z) = (z + z^2)/2 \):

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BUT, this does not make sense because \( \sqrt{z^2 + 8z} \) has a singularity at \( z = 0 \).

On the other hand, the formula as a whole DOES make sense for every \( f \) in \( H^2 \) and defines \( C^*_\varphi f \) as a single-valued analytic function!
Definition. (C. & Gallardo-Gutiérrez, 2006)

Let $K$ be a finite subset of $\mathbb{D}$. Suppose $\sigma$ is an $n$-valued function that is arbitrarily continuable in $\mathbb{D} \setminus K$ and takes values in $\mathbb{D}$. Suppose $\psi$ is an $m$-valued analytic function defined and arbitrarily continuable in $\mathbb{D} \setminus K$ with values in $\mathbb{C}$ and bounded.


The multiple valued weighted composition operator $W_{\psi, \sigma}$ is defined by

$$(W_{\psi, \sigma} f)(z) = \sum_{\text{all branches}} \psi(z) f(\sigma(z))$$

for $z$ in $\mathbb{D} \setminus K$ and extended to $\mathbb{D}$. 
Definition. (C. & Gallardo-Gutiérrez, 2006)

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*Plus compatibility condition.* *Plus finiteness condition.*

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for $z$ in $\mathbb{D} \setminus K$ and extended to $\mathbb{D}$.

In the example, $\psi(z) = \frac{z \pm \sqrt{z^2 + 8z}}{\pm 2\sqrt{z^2 + 8z}}$ and $\sigma(z) = \frac{z \pm \sqrt{z^2 + 8z}}{4}$
These conditions ensure that \((W_{\psi,\sigma}f)(z)\) is bounded on every compact subset of \(\mathbb{D}\) and therefore that all the singularities at points of \(K\) are removable so that \(W_{\psi,\sigma}f\) is analytic in \(\mathbb{D}\).

Moreover, if \(\psi\) is bounded, that is, there is \(M < \infty\) so that

\[
\limsup_{|z| \to 1^-} |\psi_j(z)| \leq M
\]

for each branch of \(\psi\), then \(W_{\psi,\sigma}\) is bounded on \(H^2\) and

\[
\|W_{\psi,\sigma}\| \leq M \sqrt{n \sum \frac{1 + |\sigma(0)|}{1 - |\sigma(0)|}}
\]
Theorem. (C. 1978)

Suppose $\varphi$ is a finite Blaschke product and let

$$K = \{ \beta \in \mathbb{D} : \varphi(\beta) = \varphi(\alpha) \text{ and } \varphi'(\alpha) = 0 \}$$

If $S$ is a bounded operator on $H^2$ such that $ST\varphi = T\varphi S$, then $S = W_{\psi,\eta}$ where $\eta = \varphi^{-1} \circ \varphi$ and $\psi = G\eta'/\varphi'$ for some $G$ multiple valued, bounded, analytic function on $\mathbb{D} \setminus K$.

Conversely, if $G$ is a multiple valued, bounded, analytic function on $\mathbb{D} \setminus K$, $\eta = \varphi^{-1} \circ \varphi$, $\psi = G\eta'/\varphi'$, and $S = W_{\psi,\eta}$, then $ST\varphi = T\varphi S$. 
Theorem. (version of Hammond, Moorhouse, Robbins)

Let \( \varphi \) be a rational map of \( \mathbb{D} \) into itself. Then for any \( f \) in \( H^2 \)

\[
(C_\varphi^* f)(z) = (W_{\psi, \sigma} f)(z) + \frac{f(0)}{1 + \varphi(\infty)z}
\]

where \( W_{\psi, \sigma} \) is multiple valued weighted composition operator induced by

\[
\sigma(z) = \left( \frac{\varphi^{-1}(1/\bar{z})}{\varphi^{-1}(1/\bar{z})} \right)^{-1} \quad \text{and} \quad \psi(z) = \frac{z\sigma'(z)}{\sigma(z)}
\]

and \( \varphi(\infty) = \lim_{|w| \to \infty} \varphi(w) \)
Study of multiple valued weighted composition operators has not begun in an organized way · · ·

Nothing is known beyond these two examples, nor are other examples related to other parts of functional analysis known.
Composition Operators in Several Variables

Hardy Hilbert space in the ball, $H^2(B_N)$ where

$$B_N = \{ z \in \mathbb{C}^N : |z| < 1 \}$$

Let $\sigma$ denote normalized surface measure on the sphere, $\{ z \in \mathbb{C}^N : |z| = 1 \}$.

$$H^2(B_N) = \{ f \text{ analytic in } B_N : \sup_{0<r<1} \int |f(r\zeta)|^2 \, d\sigma(\zeta) < \infty \}$$

If $\varphi$ is a map of $B_N$ into $B_N$, define $C_\varphi$ on $H^2(B_N)$ by

$$(C_\varphi f)(z) = f(\varphi(z))$$
Not all maps $\varphi$ of $B_N$ into $B_N$ give bounded composition operators!

There are even polynomial maps $\varphi$ for which $C_\varphi$ is not bounded!

For example, for

$$\varphi(z_1, z_2) = (2z_1z_2, 0)$$

$C_\varphi$ is unbounded on $H^2(B_N)$.

While there are Carleson conditions for boundedness and compactness, these questions are not satisfactorily resolved.

If $C_\varphi$ is compact on $B_N$, there is an attracting fixed point $a$ for $\varphi$ in $B_N$ and

$$\sigma(C_\varphi) = \{0, 1\} \cup \{\lambda : \lambda = \text{a product of eigenvalues of } \varphi'(a)\}$$

Biggest problem is that we don’t know enough functions in several variables.
Theorem. (C. & MacCluer 1994)

Suppose $\varphi$ is a holomorphic map of $B_N$ into $B_N$ that is univalent, $\varphi(0) = 0$, and $\varphi$ is not unitary on any slice of $B_N$. If $\Omega < \infty$, then the spectrum of $C_\varphi$ as an operator on $H^2(B_N)$ includes the disk

$$\left\{ \lambda : |\lambda| \leq \frac{\tilde{\rho}^N}{\Omega^{N/2}} \right\}$$

where $\tilde{\rho}$ is the essential spectral radius of $C_\varphi$ on $H^2(B_N)$ and

$$\Omega = \sup\left\{ \frac{\|\varphi'(z)\|^2}{|J_\varphi(z)|^2} : z \in B_N \right\}$$

Theorem.

Suppose $\varphi$ is a holomorphic map of $B_N$ into $B_N$, $\varphi(0) = 0$, and $\varphi$ is not unitary on any slice of $B_N$. If $C_\varphi$ is bounded on $H^2(B_N)$ then 1 and each number $\overline{\lambda}$ for $\lambda$ a product of eigenvalues of $\varphi'(0)$ is an eigenvalue of $C_\varphi^*$. 
Definition.

A map \( \varphi \) will be called a linear fractional map if

\[
\varphi(z) = (Az + B)(\langle z, C \rangle + D)^{-1}
\]

where \( A \) is an \( N \times N \) matrix, \( B \) and \( C \) are (column) vectors in \( \mathbb{C}^N \), and \( D \) is a complex number. We will regard \( z \) as a column vector also and \( \langle \cdot, \cdot \rangle \) denotes the usual Euclidean inner product on \( \mathbb{C}^N \).

Clearly, this is analytic except for \( z \) in the hyperplane \( \langle z, C \rangle = -D \).

Every automorphism of the ball is a linear fractional map with this definition.
For a linear fractional map \( \varphi(z) = (az + b)/(cz + d) \), associate

\[
m_{\varphi} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

In several variables, for \( \varphi(z) = (Az + B)(\langle C, z \rangle + D)^{-1} \), associate

\[
m_{\varphi} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}
\]

In both cases, \( m_{\varphi}m_{\psi} = m_{\varphi \circ \psi} \), so we use this as a tool.
If $\varphi$ is a linear fractional map of $B_N$ into itself, then $C_\varphi$ is bounded operator on $H^2(B_N)$ and compact iff $\overline{\varphi(B_N)} \subset B_N$.

The theorem on adjoints of $C_\varphi$ for $\varphi$ a linear fractional map of the ball carries over exactly as in one variable.

**Problems**

- Can every map of the ball to the ball be modeled by a linear fractional map? If not, which can be? Are they special in any way?
- Find spectra, even in special cases.
- Little known about composition operators on other spaces, for example, $H^2(\mathbb{D} \times \mathbb{D})$, or the analogue of the Bergman space.
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Slides posted on webpage:

www.math.iupui.edu/~ccowen/EPAF09.html