Lecture 7

Axioms for Chern Classes and Stiefel-Whitney Classes

The Stiefel-Whitney classes $w_i(V) \in H^i(B;\mathbb{Z}/2)$ are defined for real vector bdles $\begin{smallmatrix} V \\ \downarrow \\ B \end{smallmatrix}$ (or, equivalently, $GL_n\mathbb{R}$ or $O(n)$ bdles).

The Chern classes $c_i(V) \in H^{2i}(B;\mathbb{Z})$ are defined for complex vector bdles $\begin{smallmatrix} V \\ \downarrow \\ B \end{smallmatrix}$ (equiv., $GL_n\mathbb{C}$ or $U(n)$ bdles).

(In both cases, we assume B is paracompact.)

Theorem: There exist unique sequences $c_1, c_2, \cdots$ and $w_1, w_2, \cdots$ of characteristic classes (for $GL_n\mathbb{R}$ / $GL_n\mathbb{C}$ bdles, respectively) with $\dim(w_i) = i$, $\dim(c_i) = 2i$, and coeff. gps $\mathbb{Z}/2$ and $\mathbb{Z}$ satisfying the following axioms:

1) $w_i(V) = 0$ for $i > \dim_{\mathbb{R}}(V)$, $w_0(V) = 1 \in H^0(B;\mathbb{Z}/2)$
$c_i(V) = 0$ for $i > \dim_{\mathbb{C}}(V)$, $c_0(V) = 1 \in H^0(B,\mathbb{Z})$.

"Whitney Sum Formula"
2) If V, W are bdles over B, then
$$w_k(V \oplus W) = \sum_{i=0}^{k} w_i(V) \underset{\substack{\uparrow \\ \text{cup product}}}{\cup} w_{k-i}(W) \quad (V, W \text{ real})$$
$$\text{or} \quad c_k(V \oplus W) = \sum_{i+j=k} c_i(V) \cup c_j(W) \quad (V, W \text{ cplx})$$

3) - $w_1\left(\begin{smallmatrix} V_1(\mathbb{R}^\infty) \\ \downarrow \\ Gr_1(\mathbb{R}^\infty) \end{smallmatrix}\right) \neq 0$ in $H^1(Gr_1\mathbb{R}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2$ (Note: $Gr_1\mathbb{R}^\infty = \mathbb{R}P^\infty$ and this bdle is the canonical bdle over S^1, and $H^1(S^1;\mathbb{Z}/2) = \mathbb{Z}/2$.)

- $c_1\left(\begin{smallmatrix} V_1(\mathbb{C}^\infty) \\ \downarrow \\ Gr_1(\mathbb{C}^\infty) \end{smallmatrix}\right) \in H^2(Gr_1\mathbb{C}^\infty;\mathbb{Z}) = H^2(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$ is the class

Note that the last axiom rules out the possibility $w_i = c_i = 0$ for all i. Also, since all line bdles over CW cplxs are pulled back from $\begin{smallmatrix} \gamma_1(\mathbb{R}^\infty) \\ \downarrow \\ Gr_1(\mathbb{C}^\infty)\end{smallmatrix}$, Axiom 3) determines $c_1(L)$ for all line bdles L. (It's actually enough to just assume $c_1\begin{pmatrix}\gamma_1^1 \\ \downarrow \\ \mathbb{CP}^1\end{pmatrix}$ and $w_1\begin{pmatrix}\gamma_1^1 \\ \downarrow \\ \mathbb{RP}^1\end{pmatrix}$ are the canonical generators, as we'll see). To understand the Whitney Sum Formula, need to define the bdle $\begin{smallmatrix} V\oplus W \\ \downarrow \\ B\end{smallmatrix}$, whose fiber over $b \in B$ is canonically $V_b \oplus W_b$ (the sum of the fibers).

Note that if we define $w(V) = \sum_{i=0}^{\dim V} w_i(V) \in \bigoplus_{i=0}^{\infty} H^i(B;\mathbb{Z}/2)$ and $c(V) = \sum_{i=0}^{\dim V} c_i(V) \in \bigoplus_{i=0}^{\infty} H^i(B;\mathbb{Z})$, then the Whitney Sum Formula takes the form:

$$w(V \oplus W) = w(V) \cdot w(W) \qquad (V, W \text{ real})$$
$$c(V \oplus W) = c(V) \cdot c(W) \qquad (V, W \text{ cplx})$$

where the mult'n takes place in the graded ring

$$H^*(B;\mathbb{Z}/2) = \bigoplus_i H^i(B;\mathbb{Z}/2) \quad \text{or} \quad H^*(B;\mathbb{Z}) = \bigoplus_i H^i(B;\mathbb{Z}).$$

<u>Whitney Sums:</u>

Given two bundles $\begin{smallmatrix} V & & W \\ & \searrow^{\pi_V} \swarrow^{\pi_W} & \\ & B & \end{smallmatrix}$, we define $\begin{smallmatrix} V\oplus W \\ \downarrow \pi \\ B\end{smallmatrix}$ to have total space $V \oplus W = V \times_B W = \{(v,w) \mid \pi_V(v) = \pi_W(w)\}$. The fibers are vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) by component-wise addition and scalar mult'n. If $U \subseteq B$ is an open set over which both V and W are trivial, w/ $\begin{smallmatrix} U \times \mathbb{R}^n \xrightarrow[\varphi]{\cong} V|_U \\ \searrow B \swarrow \end{smallmatrix}$,

$$\begin{array}{c} U\times\mathbb{R}^m \xrightarrow[\psi]{\cong} W|_U \\ \searrow \quad \swarrow \\ U \end{array}, \quad \text{then} \quad \begin{array}{c} (u,x,y) \longmapsto (\varphi(u,x),\psi(u,y)) \\ U\times\mathbb{R}^n\times\mathbb{R}^m \xrightarrow{(\varphi,\psi)} V\times_B W|_U \\ \searrow \quad \swarrow \\ U \end{array}$$ is a homeomorphism,

b/c $(V\times_B W)|_U = V|_U \times_B W|_U \xrightarrow{(\pi,\varphi_2,\psi_2)} U\times\mathbb{R}^n\times\mathbb{R}^m$ gives the inverse.

[Here $\mathbb{R}^n, \mathbb{R}^m$ can of course be replaced by $\mathbb{C}^n, \mathbb{C}^m$].

There are several other ways to view $V\oplus W$.

- There is a bdle $\begin{array}{c} V\times W \\ \downarrow \\ B_1\times B_2 \end{array}$ associated to any $\begin{array}{c} V \\ \downarrow \\ B_1 \end{array}, \begin{array}{c} W \\ \downarrow \\ B_2 \end{array}$.

(The topology on $V\times W$ is the product topology, and the bdle is trivial over $U_1\times U_2$ if $V|_{U_1}\cong U_1\times\mathbb{R}^n$, $W|_{U_2}\cong U_2\times\mathbb{R}^m$)

The Whitney sum is then the pullback
$$\begin{array}{ccc} \Delta^*(V\times W) & \to & V\times W \\ \downarrow & & \downarrow \\ B & \xrightarrow{\Delta} & B\times B \\ b & \longmapsto & (b,b) \end{array}.$$

- If $\{U_i\}_i$, $\{\varphi_{ji} : U_i\cap U_j \to GL_n\mathbb{R}\}_{i,j}$, and $\{\psi_{ji} : U_i\cap U_j \to GL_m\mathbb{R}\}$ give clutching data for V and W (respectively), then $\{\varphi_{ji}\oplus\psi_{ji} : U_i\cap U_j \to GL_{n+m}(\mathbb{R})\}$ gives clutching data for $V\oplus W$. Here $\varphi_{ji}\oplus\psi_{ji}$ is defined via the block-sum maps $GL_n\mathbb{R}\times GL_m\mathbb{R} \to GL_{n+m}\mathbb{R}$.
$$[A],\ [B] \longmapsto \begin{bmatrix} [A] & 0 \\ 0 & [B] \end{bmatrix}$$

To see this, we just need to look at the trivializations given above: the transitions for $V\oplus W$ have the form

$$(\varphi_j,\psi_j)^{-1}\circ(\varphi_i,\psi_i) : U_i\cap U_j\times\mathbb{R}^n\times\mathbb{R}^m \to U_i\cap U_j\times\mathbb{R}^n\times\mathbb{R}^m$$
$$u,x,y \longmapsto (u, \varphi_j^{-1}\varphi_i(u,x), \psi_j^{-1}\psi_i(u,y))$$

and the matrix for this transformation at $u\in U_i\cap U_j$ is exactly $\begin{bmatrix} \varphi_{ji}(u) & 0 \\ 0 & \psi_{ji}(u) \end{bmatrix}$.

In MS §3, a general construction is given, which works for other "continuous" functors such as $\otimes$, Hom, etc. It is easy to see that their description of $V \otimes W$ agrees with the first one given above. We'll return to the general construction later.

In order to prove the existence of Stiefel-Whitney and Chern classes, we'll study the cohomology of projective bundles.

Def'n: Let $\begin{matrix} E \\ \downarrow\pi \\ B \end{matrix}$ be a (real or cplx) vector bdle. The projective bundle associated to E is the space $P(E) = (E - E_0)/x \sim cx$ for all $x \in E_0$, $c \in \mathbb{R}$ (or $\mathbb{C}$). Here E_0, the zero section of E, consists of all the zero vectors.

Lemma: The natural projection $\begin{matrix} P(E) & [x] \\ \downarrow & \downarrow \\ B & \pi(x) \end{matrix}$ is a locally trivial fiber bdle, whose fiber is the projective space $\mathbb{R}P^{n-1}$ or $\mathbb{C}P^{n-1}$ (when V is a $\mathbb{R}^n$- or a $\mathbb{C}^n$-bdle, respectively).

Proof: It suffices to check that $(\mathbb{R}^n - \{0\}) \times U / x \sim cx$ is homeomorphic to $\mathbb{R}P^{n-1} \times U$, but this is immediate. $\square$

Our goal will be to understand the cohomology of projective space bldles (with $\mathbb{Z}$ coeff's in the cplx case, and with $\mathbb{Z}/2$ coeff's in the real case).

These cohomology groups will be described in terms of the Chern / Stiefel-Whitney classes of the tautological line bdle.

<u>Def'n</u>: If $\begin{smallmatrix}E\\ \downarrow\\ B\end{smallmatrix}$ is a vector bdle, the <u>tautological</u> <u>line</u> <u>bdle</u> $\begin{smallmatrix}\gamma_E\\ \downarrow\\ P(E)\end{smallmatrix}$ over $P(E)$ is defined by $\gamma_V = \{(L, \vec{v}) \in P(E) \times_B E \mid \vec{v} \in L\} \subseteq P(E) \times E$. (Here we think of points in $P(E)$ as lines through the origin in the fibers of E.)

Note that by def'n, the restriction of γ_E to any fiber of $\begin{smallmatrix}P(E)\\ \downarrow\\ B\end{smallmatrix}$ is precisely the tautological line bdle on that fiber.

<u>Lemma</u>: $\begin{smallmatrix}\gamma_E\\ \downarrow\\ P(E)\end{smallmatrix}$ is a locally trivial line bdle over $P(E)$.

<u>Pf</u>: Say $\begin{smallmatrix}P(E)|_U \cong U \times \mathbb{R}P^{n-1}\\ \downarrow\\ U\end{smallmatrix}$ for some U. If $\begin{smallmatrix}\gamma_1\\ \downarrow\\ \mathbb{R}P^{n-1}\end{smallmatrix}$ is trivial over $W \subseteq \mathbb{R}P^{n-1}$, then $\begin{smallmatrix}(\gamma_E)|_{U\times W}\\ \downarrow\\ U \times W \subseteq U \times \mathbb{R}P^{n-1}\end{smallmatrix}$ is trivial as well. $\square$

<u>The Projective Bdle Theorem:</u>

Let $\begin{smallmatrix}P(E)\\ \downarrow \pi\\ B\end{smallmatrix}$ be the projective bdle associated to a cplx n-plane bdle $\begin{smallmatrix}E\\ \downarrow\\ B\end{smallmatrix}$. Then the map $\pi^*: H^n(B;\mathbb{Z}) \to H^n(P(E);\mathbb{Z})$ is injective, and there is an isomorphism of graded $H^*(B;\mathbb{Z})$-<u>modules</u>

$$H^*(B;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[X]/(X^{n+1}) \xrightarrow[\cong]{\alpha} H^*(P(E);\mathbb{Z}),$$

where $\deg(X) = 2$, and for each i, $\alpha(1 \otimes X^i) = c_1(L_E)^i$, the i^{th} cup-power of $c_1(L_E)$. In particular, $H^*(P(E);\mathbb{Z})$ is free as an $H^*(B;\mathbb{Z})$-module.

If $\begin{smallmatrix}E\\ \downarrow\\ B\end{smallmatrix}$ is a real n-plane bdle, we have

$$H^*(B;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[X]/(X^{n+1}) \xrightarrow[\alpha]{\cong} H^*(PE;\mathbb{Z}/2) \quad \text{for } i = 0, \dots n!!$$
$$1 \otimes X^i \longmapsto w_1(L_E)^i$$

<u>Note</u>: - This theorem <u>does</u> <u>not</u> describe the full ring structure of $H^*(PE)$.

- To make sense of the first Chern / Stiefel-Whitney classes of $\begin{matrix} L_E \\ \downarrow \\ PE \end{matrix}$, we need to know that bdles over PE are pulled back from $\begin{matrix} V_1(\mathbb{C}^\infty) \\ \downarrow \\ Gr_1(\mathbb{C}^\infty) \end{matrix}$. This is somewhat subtle. The argument in MS 5.6, which shows that all bundles over paracompact spaces are pulled back from $\begin{matrix} V_n\mathbb{R}^\infty \\ \downarrow \\ Gr_n\mathbb{R}^\infty \end{matrix}$, can be modified to work in the complex case, and there's also a "projective" version which shows that $\begin{matrix} LE \\ \downarrow \\ PE \end{matrix}$ has a classifying map. This is explained in Hatcher's Vector Bundles notes

In the real case, we'll see that there's a simple general definition of w_1 that can be used.

<u>Grothendieck's Definition of Chern / Stiefel-Whitney Classes</u>:

Given a cplx n-plane bdle $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, the class $c_1(L_E)^n \in H^*(PE;\mathbb{Z})$ must be expressible in terms of the basis $1, c_1(L_E), \cdots, c_1(L_E)^{n-1}$ for $H^*(PE;\mathbb{Z})$ as an $H^*(B;\mathbb{Z})$-module. In other words, there are unique elements $c_1(E), \cdots, c_n(E) \in H^*(B;\mathbb{Z})$ s.t.

$$(\star) \quad c_1(L_E)^n = (-1)^{n+1} c_n(E)\cdot 1_{H^*(PE;\mathbb{Z})} + (-1)^n c_{n-1}(E)\cdot c_1(L_E) + \cdots + c_1(E)\cdot c_1(L_E)^{n-1}.$$

<u>Rmk</u>: For Stiefel-Whitney classes, the signs have no effect b/c we work with $\mathbb{Z}/2$ coeffs.

<u>Def'n</u>: The class $c_i(E) \in H^*(B;\mathbb{Z})$ appearing in $(\star)$ is the i^{th} Chern class of the bdle E, and the Stiefel-Whitney classes of real n-plane bdles are defined analogously.

Here is a straightforward, general definition of the class $w_1\left(\begin{smallmatrix} L \\ \downarrow \\ E\end{smallmatrix}\right)$, where L is a real line bundle. (One can define $w_1\left(\begin{smallmatrix} E \\ \downarrow \\ X\end{smallmatrix}\right) \in H^1(X;\mathbb{Z}/2)$ similarly for any real bundle E.)

First, we reinterpret the group $H^1(X;\mathbb{Z}/2)$.

By the Univ. Coeff. Thm, we have

$$H^1(X;\mathbb{Z}/2) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}/2) \oplus \mathrm{Ext}(H_0(X;\mathbb{Z}),\mathbb{Z}/2)$$
$$\cong \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}/2)$$

b/c $H_0(X;\mathbb{Z}) = \mathbb{Z}$ is free. Since $H_1(X;\mathbb{Z}) = \pi_1(X)^{ab}$, we find that

$$H^1(X;\mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1 X, \mathbb{Z}/2).$$

<u>Claim</u>: The first Stiefel-Whitney class $w_1\left(\begin{smallmatrix} L \\ \downarrow \\ X\end{smallmatrix}\right)$ corresponds to the function $\pi_1 X \xrightarrow{w_1} \mathbb{Z}/2$ defined by

$$w_1([\alpha]) = \begin{cases} 1, & \alpha^* L \text{ is non-trivial} \\ 0, & \text{else} \end{cases}$$

<u>Pf</u>: The function is well-defined by the Bundle Htpy Thm.

First, let's check the formula on the universal line bdle $\begin{smallmatrix} \gamma_1 \\ \downarrow \\ \mathbb{R}P^\infty\end{smallmatrix}$. We know that $\pi_1(\mathbb{R}P^\infty) \cong \pi_1 \mathbb{R}P^2 \cong \mathbb{Z}/2$, so both $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$ and $\mathrm{Hom}(\pi_1 \mathbb{R}P^\infty, \mathbb{Z}/2)$ have a single non-zero element, which by <u>def'n</u> is $w_1(\gamma_1)$. On the other hand, the pullback of γ_1 along the generator $\alpha: S^1 = \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$

is precisely the tautological bdle over $\mathbb{RP}^1$, which we have shown is non-trivial. So our new def'n also gives us the unique non-zero map $\pi_1 \mathbb{RP}^\infty \longrightarrow \mathbb{Z}/2$.

To complete the proof, we just note that the new def'n is natural under pullbacks, and the diagram

$$
\begin{tikzcd}
H^1(X;\mathbb{Z}/2) \arrow[d, "\cong"] & H^1(\mathbb{RP}^\infty;\mathbb{Z}/2) \arrow[l, "f^*"'] \arrow[d, "\cong"] \\
\mathrm{Hom}(X,\mathbb{Z}/2) & \mathrm{Hom}(\pi_1\mathbb{RP}^\infty,\mathbb{Z}/2) \arrow[l] \\
\varphi\circ f_* & \varphi \arrow[l, mapsto]
\end{tikzcd}
$$

commutes for any $f: X \to \mathbb{RP}^\infty$.

~~<u>Proof of Lemma 2 (Real Case)</u>: Say $L_1 \searrow X \swarrow L_2$ are line bdles. We must show that $w_1(L_1 \otimes L_2) = w_1 L_1 + w_1 L_2$. In terms of maps $\pi_1 X \to \mathbb{Z}/2$, this means we must check that if $N_1 \searrow S^1 \swarrow N_2$ are line bdles, then $N_1 \otimes N_2$ is non-trivial if and only if exactly one of the N_i is non-trivial. The only case to check is that $\gamma_1^1 \otimes \gamma_1^1 \downarrow S^1 = \mathbb{RP}^1$ is trivial. But a choice of metric on γ_1^1 gives an isomorphism $\gamma_1^1 \cong (\gamma_1^1)^*$ (the dual bdle) so $\gamma_1^1 \otimes \gamma_1^1 \cong \gamma_1^1 \otimes (\gamma_1^1)^* \cong S^1 \times \mathbb{R}$. (Note: the last step can also be done by considering the clutching fcns of $\gamma_1^1 \otimes \gamma_1^1$.) $\square$~~

<u>Rmk</u>: We could take this as <u>our</u> def'n of the first Stiefel-Whitney class of line bdles. In fact, it makes sense for <u>any</u> line bdle $\begin{smallmatrix} L \\ \downarrow \\ X \end{smallmatrix}$, even if X is not para cpt (and then L need not be pulled back from $\begin{smallmatrix} \gamma_1 \\ \downarrow \\ \mathbb{R}P^\infty \end{smallmatrix}$).

So in the real case, we don't need to use Milnor's result (MS §5) that bdles over para cpt spaces are pulled back from the universal bdle. (Although we have just shown that this new def'n agrees with the old ~~when~~ whenever Milnor's result applies.)