Lecture 9

Another nice application of the Splitting Principle is the uniqueness of Chern / Stiefel-Whitney classes.

<u>Theorem</u>: The classes w_i, c_i we have defined are the <u>only</u> sequences of real/cplx char. classes satisfying the 3 axioms.

<u>Pf</u>: Let $\begin{array}{c}E\\ \downarrow\\ X\end{array}$ be a (cplx, say) bdle, and let $X' \xrightarrow{f} X$ be a map s.t. f^* is injective on cohomology, and $f^*E = L_1 \oplus - \oplus L_k$ for line bdles $L_1, \ , L_k$. Then if $\beta = 1 + \beta_1 + \cdots + \beta_k$ are char. classes satisfying the axioms, we have

$$f^*(\beta(E)) = \beta(f^*E) = \beta(L_1 \oplus \cdots \oplus L_k) \overset{WSF}{=} \prod_{i=1}^{k} \beta(L_i)$$

$$= \prod_{i=1}^{k} (1 + \beta_1(L_i)) = \prod_{i=1}^{k} (1 + c_1(L_i)) \overset{WSF}{=} c(\oplus L_i)$$

axioms $\Rightarrow$ $\beta_2, \beta_3, \cdot$ vanish on line bdles

the axioms determine values on line bdles

$$= c(f^*E) = f^* c(E).$$

Since f^* is injective, we have $\beta(E) = c(E)$. $\square$

Lecture 12

Bdles over Paracpt Spaces

We have used the following result (to define $c_1(L_E) \in$ ~~...~~ for example):

Theorem: If $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ is a complex, n-dim'l, vector bdle over a paracompact Hausdorff space X, then there exists a diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \gamma_n \arrow[d] \\ X \arrow[r] & Gr_n(\mathbb{C}^\infty) \end{tikzcd}.$$

The corresponding result holds in the real case as well.
[Rmk: MS proves this without assuming X is Hausdorff!]

Proof: First, we claim that it suffices to construct a continuous, linear injection $E \xhookrightarrow{j} \mathbb{C}^\infty \cong \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots$.

Given such a map j, we define

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\ulcorner", very near start] & \gamma_n \arrow[d] \\ X \arrow[r, "f"] & Gr_n \mathbb{C}^\infty \end{tikzcd}$$

by $\tilde{f}(e) = \left(j\left(\begin{smallmatrix}\text{fiber} \\ \text{through } e\end{smallmatrix}\right), j(e) \right)$ and $f(x) = j(\pi^{-1}x)$.

Note that locally, f has the form

$$\begin{tikzcd} & E|_U \times \cdots \times E|_U \arrow[r, "j \times \cdots \times j"] \arrow[d] & V_n(\mathbb{C}^\infty) \subseteq \underbrace{\mathbb{C}^\infty \times \cdots \times \mathbb{C}^\infty}_{n} \arrow[d] \\ x \arrow[r, mapsto, "\text{local basis}"] & U \arrow[u, bend left] \arrow[r, dashed, "f"] & Gr_n \mathbb{C}^\infty \end{tikzcd}$$

So f is continuous.

So we must construct a map

$$E \longrightarrow \mathbb{C}^\infty \cong \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots$$

<u>Lemma</u>: If X is paracompact and Hausdorff, and $\{U_i\}_{i \in I}$ is an open cover of X, then there exists a <u>countable</u> open cover $\{V_k\}_{k=1}^{\infty}$ of X such that

1) Each V_k can be written as a disjoint union

$$V_k = \coprod_{j \in J} V_k^j$$

with $V_k^j \subseteq U_i$ for some $i = i(j)$

2) There is a $\boxed{\text{locally finite}}$ partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ with

$$\operatorname{supp}(\varphi_k) \subseteq V_k.$$

Assuming the Lemma, we can easily construct the desired map $j: E \hookrightarrow \mathbb{C}^\infty$:

Let $\{U_i\}_{i \in I}$ be a cover of X over which E is trivial, and let $\{V_k\}_{k=1}^{\infty}$ be the cover in the Lemma.

Note that condition 1) implies that $E|_{V_k}$ is trivial for each k.
Choose trivializations $\psi_k : E|_{V_k} \longrightarrow V_k \times \mathbb{C}^n$, and let Ψ_k denote $E|_{V_k} \to V_k \times \mathbb{C}^n \xrightarrow{\pi_2} \mathbb{C}^n$.
Letting φ_k denote the partition of unity in 2), we define

$$j(e) = \bigoplus_{k=1}^{\infty} \varphi_k(\pi e) \cdot \Psi_k \in \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots \cong \mathbb{C}^\infty.$$

Since only finitely many φ_k are non-zero at $\pi(e)$, this point lies in $\bigoplus_1^\infty \mathbb{C}^n$.

Also, j is injective b/c at each $x \in X$, some φ_k must be non-zero (b/c $\sum_{k=1}^{\infty} \varphi_k(x) = 1$ at each $x \in X$).
This completes the proof of the theorem.

<u>Pf of Lemma</u>: Since X is paracpt Hausdorff, $\exists$ a (locally finite) part. of 1 subordinate to $\{U_i\}_{i \in I}$.

This means a collection of fcns $\{\varphi_j\}_{j \in J}$ s.t.

- $\varphi_j : X \to \mathbb{R}_{\geq 0}$
- $\mathrm{supp}(\varphi_j) = \overline{\varphi_j^{-1}(\mathbb{R}_{>0})}$ is contained in some U_i
- For each $x \in X$, $\exists$ an open nbhd $W \ni x$ s.t. only finitely ~~many~~ φ_j are non-zero on W.

Define, for each <u>finite</u> set $S \subseteq I$,

$$V_S = \{ x \in X \mid \forall s \in S, \forall i \notin S, \varphi_s(x) > \varphi_i(x) \}.$$

Note that if $x \in X$, $\exists$ $W \ni x$ st. only $\varphi_{i_1}, \ldots, \varphi_{i_n}$ are non-zero on W, so

$$V_S \cap W = \bigcap_{\substack{s \in S \\ j = 1, \ldots, n}} \{ x \in W \mid \varphi_s(x) > \varphi_{i_j}(x) \}$$

$$= \left[\bigcap_{\substack{s \in S \\ j = 1, \ldots, n}} (\varphi_s - \varphi_{i_j})^{-1}(\mathbb{R}_{>0}) \right] \cap W$$

which is a finite intersection of open sets in X. Hence V_S is open in X for each (finite) set $S \subset I$.

Note that $V_S \subseteq U_i$ if $\mathrm{supp}(\varphi_s) \subseteq U_i$ for some $s \in S$, b/c each φ_S $(s \in S)$ is positive on V_S.

Let $V_k = \bigcup_{|S|=k} V_S$. We claim that

S contains k elts

$\{V_k\}_{k=1}^{\infty}$ is the desired cover of X. It's certainly a cover, since for any $x \in X$, $x \in V_{\{s \in I \mid \varphi_s(x) > 0\}}$.

Next, we claim that $V_S \cap V_{S'} = \emptyset$ if $|S| = |S'|$. Since $S \not\subseteq S'$, $S' \not\subseteq S$, we can choose $s \in S \setminus S'$, $s' \in S' \setminus S$. Then any point $x \in V_S \cap V_{S'}$ would have to satisfy both

$$\varphi_s(x) > \varphi_{s'}(x)$$

$$\underline{\text{and}} \quad \varphi_{s'}(x) > \varphi_s(x),$$

which is impossible.

So $V_k = \coprod_{|S|=k} V_S$, and each V_S lies in some U_i.

Finally, ~~let~~ consider a part. of 1 sub. to $\{V_k\}_{k=1}^{\infty}$, say $\{\varphi_\alpha\}_{\alpha \in A}$ and let $\varphi_k = \sum \{\varphi_\alpha : \operatorname{supp}(\varphi_\alpha) \subseteq V_k$ but not in $V_1, \ldots, V_{k-1}\}$.

~~Then $\{\varphi_k\}_{k=1}^{\infty}$ is still locally~~

Then $\operatorname{supp}(\varphi_k) \subseteq \bigcup \operatorname{supp}(\varphi_\alpha) \subseteq V_k$; $\sum_{k=1}^{\infty} \varphi_k(x) = \sum_{\alpha \in A} \varphi_\alpha(x) = 1$; and $\{\varphi_k\}_1^{\infty}$ is locally finite b/c $\{\varphi_\alpha\}$ was locally finite, and each φ_α appears as a summand in just one φ_k. $\square$

Some Important Facts about Char. Classes

Theorem: If X is a

then line bdles over X are completely determined by their first Chern class (cplx case) or their first Stiefel-Whitney class (real case).

Proof: We need to show that if $c_1(L) = c_1(M)$ then $L \cong M$. Let $f: X \to \mathbb{C}P^\infty$, $g: X \to \mathbb{C}P^\infty$ be classifying maps for L and M (resp.). Then the induced maps

$$f^*, g^* : H^*\mathbb{C}P^\infty = \mathbb{Z}[\alpha] \longrightarrow H^*X$$
$$\alpha \longmapsto c_1 L = c_1 M$$

are completely determined by the image of α, so $f^* = g^*$. Hence we need to show that maps from CW cplxs into $\mathbb{C}P^\infty$ are completely determined (up to htpy) by their effect in cohomology (w/ $\mathbb{Z}$-coeff's).

Theorem: If Z is a space with just one non-zero htpy group $\pi_n(Z) = \pi$ (with π abelian if $n=1$) then $[X, Z] \cong H^n(X; \pi)$ for any CW cplx X. unbased htpy classes of maps

[<u>Rmk</u>: Spaces like Z, with one non-zero htpy gp, are called Eilenberg-MacLane spaces, and are usually denoted $Z = K(\pi, n)$. Up to htpy, there is a <u>unique</u> CW model for $K(\pi, n)$.]

This result applies to both $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ and $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, b/c

- $\mathbb{C}P^\infty = Gr(1, \mathbb{C}^\infty) = BU(1) \Rightarrow \pi_* \mathbb{C}P^\infty = \pi_{*-1} U(1)$
$$= \begin{cases} \mathbb{Z}, & * = 2 \\ 0, & \text{else} \end{cases}$$
Note: $U(1) \cong S^1$

- $\mathbb{R}P^\infty = Gr(1, \mathbb{R}^\infty) = BO(1) \Rightarrow \pi_* \mathbb{R}P^\infty = \pi_{*-1} O(1)$
$$= \begin{cases} \mathbb{Z}/2, & * = 1 \\ 0, & \text{else} \end{cases}$$
$O(1) = \{\pm 1\}$

[<u>Rmk</u>: The isomorphism $\pi_1 BG \cong \pi_0 G$ <u>is</u> an isom. of <u>groups</u>.]

~~The isomorphism $[X, Z] \cong H^n Z$~~

~~Obs~~ The isomorphism

$$[X, K(\pi,n)] \xrightarrow{\cong} H^n(X)$$

is given by sending $f: X \to K(\pi,n)$ to $f^*(c)$ for a particular "universal class" $c \in H^n(K(\pi,n))$. ← cohom. w/ π-coeff's

Hence if two maps $f, g: X \to K(\pi,n)$ have the same effect in cohomology, they are homotopic. This completes the proof. $\square$

<u>The universal class in $H^n(K(\pi,n);\pi)$:</u>

This class can be described in terms of the Hurewicz map ~~Obs~~

$$\pi_n Z \longrightarrow H_n(Z;\mathbb{Z}).$$

$$\alpha: S^n \to Z \longmapsto \alpha_* \underbrace{[S^n]}_{\text{fundamental class in } H_n(S^n;\mathbb{Z})}$$

~~If applied to $Z = K(\pi,n)$,~~

<u>Theorem</u> [Hurewicz Thm]: If $\pi_k Z = 0$ for $k < n$, then the Hurewicz map $\pi_n Z \to H_n(Z;\mathbb{Z})$ is an isom. (when $n=1$, we must also assume $\pi_1 Z$ is abelian).

Now if Z is a $K(\pi,n)$, we define $\iota \in H^n(K(\pi,n),\pi)$ to be the image of $Id: \pi \to \pi$ under the maps

$$Hom(\pi,\pi) \cong Hom(\pi_n(K(\pi,n)),\pi) \xrightarrow[\text{Hurewicz}]{\cong} Hom(H_n(K(\pi,n);\mathbb{Z}),\pi)$$
$$\xrightarrow{UCT} H^n(K(\pi,n);\pi).$$

This class ι the depends only on our identification $\pi \cong \pi_n(K(\pi,n))$, and if we replace π by the isomorphic group $\pi_n(K(\pi,n))$ everywhere, ι becomes canonical.

For a proof that
$$\begin{array}{ccc} H^n(X;\pi) & \cong & [X, K(\pi,n)] \\ f^*(\iota) & \longleftarrow & f \end{array}$$
is an isom., see Hatcher, Chap. 4. (Possibly there will be a HW exercise containing another proof.)

We have defined and studied Chern classes and Stiefel-Whitney classes, and we observed that w_1, c_1 are not always zero (b/c there exist non-trivial line bdles).

<u>Theorem</u>: The Chern classes of $\begin{array}{c}\gamma_n\\ \downarrow\\ Gr_n\mathbb{C}^\infty\end{array}$ and the Stiefel-Wh. classes of $\begin{array}{c}\gamma_n\\ \downarrow\\ Gr_n\mathbb{R}^\infty\end{array}$ are all non-zero.

Pf: It suffices to show that there exist bdles w/ w_k, c_k non-zero. We'll work in the cplx case; the real case is identical.

Let $\pi_i : \underbrace{\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty}_{k} \to \mathbb{C}P^\infty$ denote the i^{th} projection, and consider the bdle $\gamma_1 \times \cdots \times \gamma_1 \cong \pi_1^* \gamma_1 \oplus \cdots \oplus \pi_k^* \gamma_1$.

We have $c(\gamma_1 \times \cdots \times \gamma_1) = \prod c(\gamma_1) = \prod (1 + c_1 \gamma_1)$, and we claim that each term in this sum is non-zero (i.e. each Chern class $c_1, \ldots, c_k$ of $\gamma_1 \times \cdots \times \gamma_1$ is non-zero). This follows from the Kunneth Thm, which says that

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \cong \bigotimes^k H^* \mathbb{C}P^\infty,$$

$$\pi_1^* x_1 \cup \cdots \cup \pi_k^* x_k \longleftarrow\!\shortmid\; x_1 \otimes \cdots \otimes x_k$$

meaning that there can be no relations among the classes $\pi_i^*(c_1\gamma)$. Since the degree ℓ term in $\prod(1 + c_1(\gamma_1))$ is a poly. in $c_1(\gamma_1)$, it must be non-zero. $\square$

In fact, more is true:

Thm: ~~[illegible]~~ $\mathbb{Z}$-coeff's

$$H^*(Gr_n(\mathbb{C}^\infty)) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$$

where $c_i = c_i(\gamma_n)$, the Chern classes of the universal bdl, and

$$H^*(Gr_n \mathbb{R}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$$

w/ $w_i = w_i(\gamma_n)$.

This theorem says that up to multiplicative combinations, the Stiefel-Whitney / Chern classes account for all characteristic classes of vector bdles.

Sketch of Proof (MS §7):

We have shown that $H^*(Gr_n \mathbb{C}^\infty; \mathbb{Z})$ and ~~[illegible]~~ $H^*(Gr_n \mathbb{R}^\infty; \mathbb{Z}/2)$ contain poly. algebras on $c_1, \ldots, c_n$ / $w_1, \ldots, w_n$, b/c relations among these classes would imply relations among the Chern classes of $\gamma_1 \times \cdots \times \gamma_1$. MS §5 gives a cell structure on the Grassmannians, which provides the ~~[illegible]~~ corresponding upper bd on H^*. □