Lecture 14

K-theory and The Chern Character:

Def'n: For any topological space X, we define

$$K^0(X) = Gr(\underline{Vect}(X))$$

where $\underline{Vect}(X)$ denotes the commutative monoid of isomorphism classes of cplx vector bdles over X, and $Gr(-)$ denotes group completion, i.e. the Grothendieck construction. The monoid structure is given by Whitney sum.

By def'n, $K^0(X) = \underline{Vect}(X) \times \underline{Vect}(X) / \sim$, where the equivalence rel'n $\sim$ is:

$$([V],[W]) \sim ([V'],[W']) \text{ if } V \oplus W' \cong V' \oplus W.$$

We think of $([V],[W]) \in K^0(X)$ as the formal difference b/w $[V]$ and $[W]$, and write this el't as

$$[V]-[W].$$

Then $[V]-[W] = [V']-[W'] \iff V \oplus W' \cong V' \oplus W$,
$\iff [V]+[W'] = [V']+[W]$ in $\underline{Vect}(X)$.

<u>Universal Property:</u>

If $\underline{\mathrm{Vect}}(X) \xrightarrow{f} G$ is any homomorphism from $\underline{\mathrm{Vect}}(X)$ to an abelian group A, then there exists a unique extension of f to $K^0(X)$:

$$\begin{tikzcd} \underline{\mathrm{Vect}}(X) \arrow[r, "f"] \arrow[dr] & A \\ & K^0(X) \arrow[u, dashed, "\tilde{f}"'] \end{tikzcd}$$

$$[V] \longmapsto [V]-[X]$$

← the unique 0-dim'l bdle $\begin{smallmatrix} X \\ \downarrow \\ X \end{smallmatrix}$

<u>Pf:</u> Since $\tilde{f}$ must be a homomorphism, we must have

$$\tilde{f}\big([V]-[W]\big) = \tilde{f}[V] - \tilde{f}[W] = f[V] - f[W].$$

↑ inverses exist in A

So $\tilde{f}$ is unique, and this formula does give a well-defined homomorphism. $\square$

We can also define $K^1(X)$:

<u>Def'n:</u> $K^1(X) = \tilde{K}^0(SX)$, where $S(X) = X\times I / \begin{smallmatrix} X\times 0, \\ X\times 1 \end{smallmatrix}$ is the (unreduced) suspension of X, and

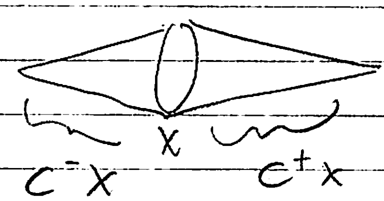
$$\tilde{K}^0(Z) := \ker\big(K^0 X \to K^0(x_0)\big)$$

<u>Note:</u> $K^0(pt) = \mathbb{Z}$, b/c $\underline{\text{Vect}}(pt) = \mathbb{N}$.

The map $K^0(X) \longrightarrow K^0(x_0) \cong \mathbb{Z}$ just sends $[V]-[W]$ to $\dim V - \dim W$, so is defined independently of $x_0 \in X$.

<u>Relation to Banach Algebras:</u>

Vector bdles $\begin{array}{c} E \\ \downarrow \\ SX \end{array}$ are automatically trivial over the two cones $C^+X, C^-X \subseteq SX$:



Hence bdles over SX are determined by a clutching function $X \longrightarrow GL_n \mathbb{C}$, i.e. an element in

$$GL_n(\underbrace{C^\circ}_{\text{cts complex-valued fcns on } X} X)$$

<u>Theorem:</u>
For any finite CW cplx X, there are isomorphisms of $\mathbb{Q}$-vector spaces

$$\bigoplus_{i=0}^{\infty} H^{2i}(X;\mathbb{Q}) \cong K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\bigoplus_{i=0}^{\infty} H^{2i+1}(X;\mathbb{Q}) \cong K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, $\operatorname{Rank}(K^0X) = \sum_{i=0}^{\infty} \operatorname{Rank} H^{2i}(X;\mathbb{Z})$
$\operatorname{Rank}(K^1X) = \sum_{0}^{\infty} \operatorname{Rk}(H^{2i+1}(X;\mathbb{Z}))$.

In fact, there is a ring structure on $K^*X = K^0X \oplus K^1X$ such that the above maps give a ring isomorphism

$$K^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^*(X; \mathbb{Q}).$$

This isomorphism is known as the Chern Character, and is defined in terms of the Chern classes of $[V] \in K^*(X)$.

Ring Structure on K^*X:

The ring structure is graded, so

$$K^0X \otimes K^0X \longrightarrow K^0X, \quad K^1X \otimes K^0X \rightarrow K^1X, \text{ etc.}$$

On K^0X, it is defined simply by tensor product:

$$([V]-[W]) \cdot ([V']-[W'])$$
$$= [V \otimes V'] - [W \otimes V'] - [V \otimes W'] + [W \otimes W'].$$

We'll more or less ignore the rest of the ring structure, which is harder to describe.

To begin, we'll construct the Chern Character

$$Ch: K^0(X) \longrightarrow \bigoplus_{i=0}^{\infty} H^{2i}(X; \mathbb{Q}).$$

We want
$$ch(E \oplus F) = ch(E) + ch(F)$$
$$ch(E \otimes F) = ch(E)\, ch(F).$$

By the Splitting Principle, we expect that it will suffice to define $ch(L_1 \oplus \cdots \oplus L_n)$ for L_i line bdles.

<u>Idea</u>: For line bdles,
$$c_1(L_1 \otimes L_2) = c_1 L_1 + c_1 L_2$$
and we need to switch the operations $+$ and $\cdot$ on the right.

This is achieved by exponentiating:

<u>Def'n</u>: For a line bdle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ (with X a <u>finite</u> CW cplx) we set
$$ch(L) = e^{c_1 L} = 1 + c_1 L + (c_1 L)^2/2! + (c_1 L)^3/3! + \cdots$$
$$\in H^*(X; \mathbb{Q}).$$

(Here $c_1 L$ denotes the image of $c_1 L \in H^2(X;\mathbb{Z})$ under the map $H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Q})$ induced by $\mathbb{Z} \hookrightarrow \mathbb{Q}$.)

<u>Note</u>: The sum defining $e^{c_1 L}$ is finite b/c X is finite dim'l.

If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bdles, then we <u>must</u> set $ch(E) = \sum_{i=1}^{n} ch(L_i) = \left(1 + c_1 L_1 + (c_1 L_1)^2/2 + \cdots\right) + \cdots + \left(1 + c_1 L_n + (c_1 L_n)^2/2 + \cdots\right)$