Math 643 - Topology III, Homework 1
Due date: February 23.

(1) Hatcher, Section 4.1 #1: prove that the various multiplications on π_n defined using composition in different coordinates of I^n are actually all the same.

(2) As explained in class, there are fibrations $S^{2n+1} \xrightarrow{p_n} \mathbb{C}\mathrm{P}^n$ with fiber S^1 for $n = 1, 2, \cdots$. Recall that if $X_1 \subset X_2 \subset X_3 \subset \cdots$ is an increasing sequence of topological spaces, then $X = \mathrm{colim}_n X_n$ is the topological space whose underlying set is $\bigcup_n X_n$ and whose topology is specified as follows: a set $U \subset X$ is open if and only if $X \cap X_n$ is open in X_n for each n.

(a) There are natural inclusions $S^{2n+1} \hookrightarrow S^{2(n+1)+1}$ and $\mathbb{C}\mathrm{P}^n \hookrightarrow \mathbb{C}\mathrm{P}^{n+1}$ compatible with the fibrations p_n. Prove that the natural map $\mathrm{colim}_n S^{2n+1} \xrightarrow{\mathrm{colim}_n p_n} \mathrm{colim}_n \mathbb{C}\mathrm{P}^n$ is a Serre fibration. (You may use the fact that every map from a compact set into the colimit of a sequence of inclusions of Hausdorff spaces has image lying inside one space in the sequence. This is a special case of a general fact about colimits of fibrations.)

(b) Prove that $\pi_* S^\infty = 0$ for $* = 0, 1, 2, \cdots$, where $S^\infty = \mathrm{colim}_n S^n = \mathrm{colim}_n S^{2n+1}$. (Hint: think about the map on homotopy groups induced by each inclusion $S^n \hookrightarrow S^{n+1}$.)

(c) Calculate $\pi_* \mathbb{C}\mathrm{P}^\infty$, where $\mathbb{C}\mathrm{P}^\infty = \mathrm{colim}_n \mathbb{C}\mathrm{P}^n$.

(3) Consider two surjective fibrations $X \xrightarrow{p} B$ and $X' \xrightarrow{p'} B'$, and say there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B'. \end{array}$$

Letting $F_b = p^{-1}(b)$ for $b \in B$ and $F'_{b'} = (p')^{-1}(b')$ for $b' \in B'$, note that f induces a map $F_b \to F'_{g(b)}$ for each $b \in B$.

(a) Choose a basepoint $x \in X$ and let $b = p(x)$. Prove that the maps on homotopy groups associated to the maps $F_b \to F'_{g(b)}$, $f\colon X \to X'$, and $g\colon B \to B'$ induce a *commutative* diagram of long exact sequences on homotopy groups. (Note: the only thing to check is that the squares involving the *boundary maps* actually commute.)

(b) Prove that if f induces bijections $\pi_0(X) \to \pi_0(X')$ and $\pi_0(F_x) \to \pi_0(F'_{f(x)})$ for all $x \in X$, then g induces a bijection $\pi_0(B) \to \pi_0(B')$ as well. (Hint: the proof is a bit like the proof of the 5-lemma.)

Remark: Recall that a map $f\colon X \to Y$ is called a *weak equivalence* if $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $n \geqslant 0$ (when $n = 0$, this just means that f induces a bijection between path components of X and Y).

This problem is one piece of a very useful 5-lemma type statement for fibrations, which says that, in the above situation, any two of the following properties imply the third:

- For each $b \in B$, the induced map $F_b \to F_{g(b)}$ is a weak equivalence;
- The map $g\colon X \to X'$ is a weak equivalence;
- The map $f\colon B \to B'$ is a weak equivalence.

In fact, if all the spaces involved are path connected, then this statement follows immediately by applying the 5-lemma to the diagram of long exact sequences induced by the diagram of fibrations from the first part of the problem.

(4) Let $C_* = F_nC_* \supset F_{n-1}C_* \supset \cdots \supset F_0C_* = 0$ be a filtered chain complex.

As we will see in class, there is an associated spectral sequence converging to the homology of C_*. More specifically, the E^∞ term of the spectral sequence is isomorphic to the associated graded group of the filtration $i_*H(F_0C_*) \subset i_*H(F_1C_*) \subset \cdots \subset i_*H(C_*)$, where i denotes the various inclusions $F_iC_* \hookrightarrow C_*$.

(a) Prove that if $\phi\colon C_* \to D_*$ is a map of filtered complexes that preserves the filtrations, then ϕ induces a map $\phi^1, \phi^2, \cdots$ of spectral sequences. (Here a map of spectral sequences $\mathbb{E} = (E^1, E^2, \ldots) \to \mathbb{G} = (G^1, G^2, \ldots)$ is simply a sequence of chain maps $E^i \to G^i$ that respect any additional gradings.)

(b) Prove that if the induced map ϕ^k between the kth pages of the spectral sequences associated to C_* and D_* is an isomorphism for some k, then in fact ϕ induces an isomorphism in homology $H_*(C_*) \xrightarrow{\cong} H_*(D_*)$.