Math 643 - Topology III, Homework 3
Due date: May 1.

(1) **Simple systems of generators.** Prove Theorem 5.13 from McCleary's User's Guide to Spectral Sequences (see page 154). This is a purely algebraic statement about the relationship between a filtered graded ring and the ring formed by taking the direct sum of all the quotient groups of the filtration (the associated bi-graded ring, one might call it). For us, the key example of such filtered rings are the cohomology of the total space of a fibration (over a CW complex), and the associated bi-graded ring is then $\bigoplus E_\infty^{p,q}$.

(2) **The cohomology of $U(n)$.** In this problem, all homology groups have coefficient group $\mathbb{Z}$.

In class, we proved by induction that the Serre Spectral Sequence (in cohomology) for the fibration

$$U(n-1) \xrightarrow{i} U(n) \xrightarrow{p} S^{2n-1}$$

collapses at the E_2 page (in other words, we showed that all differentials d_r with $r \geqslant 2$ are zero). We also proved that, assuming $H^*(U(n-1))$ is an exterior algebra on generators $x_1, x_3, \ldots, x_{2n-3}$ (with $x_i \in H^i(U(n-1))$), the ring $\bigoplus E_2^{p,q}$ is an exterior algebra on generators $\bar{x}_1, \bar{x}_3, \ldots, \bar{x}_{2n-3}, \bar{x}_{2n-1}$ (with $\bar{x}_i \in E_2^{0,i}$ for $i < 2n-1$ and $\bar{x}_{2n-1} \in E_2^{2n-1,0}$). In this problem, we'll show that this does in fact imply that $H^*(U(n)) \cong \Lambda(x_1, x_3, \ldots, x_{2n-3}, x_{2n-1})$, where $x_i \in H^i(U(n))$ is any element mapping to $\bar{x}_i$ under the natural projection map $H^i(U(n)) \longrightarrow \bigoplus_{p+q=i} E_\infty^{p,q}$ (this projection is the map from the filtered group $H^i(U(n))$ to the associated graded group). Note here that $E_\infty^{p,q} = E_2^{p,q}$ is zero except when $p = 0$ and when $p = 2n-1$ (why?), and free abelian otherwise, so these direct sums contain at most two terms and the projection map is an isomorphism.

a) Prove that the groups $E_\infty^{0,q} \cong E_2^{0,q}$ are precisely

$$\frac{H^q(U(n))}{\ker(H^q(U(n)) \xrightarrow{i^*} H^q(U(n-1))},$$

where $i\colon U(n-1) \to U(n)$ is the inclusion, and the groups $E_\infty^{2n-1,q} \cong E_2^{2n-1,q}$ are precisely

$$\ker(H^q(U(n)) \xrightarrow{i^*} H^q(U(n-1).$$

(Hint: consider the cell structure on S^{2n-1} with one 0-cell and one $(2n-1)$–cell.)

b) Prove that if $x_i \in H^i(U(n))$ are classes mapping the to exterior generators $\bar{x}_i \in \bigoplus E_2^{p,q}$, then $x_i^2 = 0$. Use this fact to construct a ring map $\Lambda(x_1, x_3, \ldots, x_{2n-3}, x_{2n-1}) \to H^*(U(n))$ (sending x_i to x_i).

c) Prove that the map $\Lambda(x_1, x_3, \ldots, x_{2n-3}, x_{2n-1}) \to H^*(U(n))$ from part b) is an isomorphism. (Hint: use Problem 1.)

(3) **Reduced and unreduced K-theory.** Throughout this problem, the word *bundle* is short for complex vector bundle.

For a compact Hausdorff space X, the group $\widetilde{K}^0(X)$ is defined to be the quotient of the set $\mathrm{Vect}(X)$ of isomorphism classes of (complex vector) bundles over X, modulo the equivalence relation of stable isomorphism: $[E] \sim [F]$ if there exist trivial bundles ϵ^n, ϵ^m over X such that $E \oplus \epsilon^n \cong F \oplus \epsilon^m$. We will write $[E]_s$ for the stable isomorphism class of a bundle $E \to X$, to distinguish from the isomorphism class $[E]$. We proved in class that this set is a group under the operation of direct sum. We also defined $K^0(X)$ to be the Grothendieck group of the monoid $\mathrm{Vect}X$; this group consists of formal differences between elements of $\mathrm{Vect}(X)$. To avoid confusion, we will write elements in $K^0(X)$ as pairs $([E],[F])$, where $[E],[F] \in \mathrm{Vect}(X)$.

a) Note that for each $x_0 \in X$, the restriction map $i_{x_0}^*\colon K^0(X) \longrightarrow K^0(\{x_0\})$ (induced by the inclusion $\{x_0\} \overset{i_{x_0}}{\hookrightarrow} X$) restricts to an isomorphism from the subgroup

$$\{([\epsilon^n],[\epsilon^m]) \,:\, n,m \in \mathbb{Z}\} \subset K^0(X)$$

to $K^0(x_0)$. This allows us to view $K^0(\{x_0\})$ as a subgroup of $K^0(X)$. Note that this subgroup is naturally isomorphic to the integers.

Prove that for each point $x_0 \in X$, there is a short exact sequence

$$0 \longrightarrow K^0(\{x_0\}) \xrightarrow{j} K^0(X) \xrightarrow{\pi} \widetilde{K}^0(X) \longrightarrow 0,$$

where j is defined via the isomorphism described in the previous paragraph and $\pi([E],[F]) = [E]_s - [F]_s$ (the difference, in the group $\widetilde{K}^0(X)$, between the stable isomorphism classes of E and F).

b) Show that there is a well-defined homomorphism

$$\widetilde{K}^0(X) \xrightarrow{s} K^0(X)$$

defined by $s([E]_s) = ([E],[\epsilon^{\dim_{x_0}(E)}])$.

c) Prove that the short exact sequence from a) has (natural) splittings given by the restriction map

$$i_{x_0}^*\colon K^0(X) \longrightarrow K^0(\{x_0\})$$

and the map

$$\widetilde{K}^0(X) \xrightarrow{s} K^0(X)$$

from part b), and show that the sequence

$$0 \longrightarrow \widetilde{K}^0(X) \xrightarrow{s} K^0(X) \xrightarrow{i_{x_0}^*} K^0(\{x_0\}) \longrightarrow 0$$

is also exact.

(4) **Applications of the Atiyah–Hirzebruch Spectral Sequence.** The Atiyah–Hirzebruch Spectral Sequence for the complex K–theory of a CW complex X has $E_2^{p,q} = H^p(X; K^q(*))$, where $*$ denotes the one-point space, and converges to $K^*(X)$.

This spectral sequence behaves essentially like the Serre spectral sequence for cohomology: the differential on the rth page has the form

$$d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

and we have $E_\infty^{p,n-p} = F_pK^n(X)/F_{p+1}K^n(X)$, where $F_pK^n(X) = \ker(K^n(X) \to K^n(X^{(p-1)}))$.

a) Use this spectral sequence to calculate the complex K–theory of the orientable surface M^g of genus g.

b) Use this spectral sequence to calculate the complex K–theory of the Klein bottle and of RP^2.