The Spectral Sequence of a Three Term Filtration:

We will consider a chain complex (C, ∂) with subcomplexes $A \subset B \subset C$.

The short exact sequences $0 \to A \to B \to B/A \to 0$ and $0 \to B \to C \to C/B \to 0$ (and the trivial s.e.s.'s $0 \to 0 \to A \to A \to 0$, $0 \to C \to C \to 0 \to 0$) give rise to LES's

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(B/A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_n(B/C) \xrightarrow{\partial} H_{n-1}(B/C)$$

which fit together to form the diagram on the next page:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & & & & & \\
H_{n+3}(C/B) \arrow[r] & H_{n+2}(B) \arrow[r, "j"] \arrow[d, "i"] & H_{n+2}(B/A) \arrow[r, "k"] & H_{n+1}(A) \arrow[r] \arrow[d, "i"] & H_{n+1}(A/0) \arrow[r] & 0 \arrow[d, "i"] & & & \\
0 \arrow[r, "k"] & H_{n+2}(C) \arrow[r, "j"] \arrow[d, "i"] & H_{n+2}(C/B) \arrow[r, "k"] & H_{n+1}(B) \arrow[r, "j"] \arrow[d, "i"] & H_{n+1}(B/A) \arrow[r, "k"] & H_n(A) \arrow[r, "j"] \arrow[d, "i"] & H_n(A/0) \arrow[r, "k"] & 0 \arrow[r, "j"] \arrow[d, "i"] & 0 \\
 & H_{n+2}(C) \arrow[r, "j"] & 0 \arrow[r, "k"] & H_{n+1}(C) \arrow[r, "j"] \arrow[d, "i"] & H_{n+1}(C/B) \arrow[r, "k"] & H_n(B) \arrow[r, "j"] \arrow[d, "i"] & H_n(B/A) \arrow[r, "k"] & H_{n-1}(A) \arrow[r, "j"] \arrow[d, "i"] & H_{n-1}(A/0) \\
 & & & H_{n+1}(C) \arrow[r, "j"] & 0 \arrow[r, "k"] & H_n(C) \arrow[r, "j"] \arrow[d, "i"] & H_n(C/B) \arrow[r, "k"] & H_{n-1}(B) \arrow[r, "j"] \arrow[d, "i"] & H_{n-1}(B/A) \\
 & & & & & H_n(C) \arrow[r, "j"] & 0 \arrow[r, "k"] & H_{n-1}(C) \arrow[r, "j"] & H_{n-1}(C/B)
\end{tikzcd}
$$

Diagram 1

We define $d = jk$, and note that $d^2 = j\underbrace{kj}_{0}k = 0$.

This gives us chain complexes

$$0 \xrightarrow{d} H_{n+2}(C/B) \xrightarrow{d} H_{n+1}(B/A) \xrightarrow{d} H_n(A/0) \to 0$$

$$0 \xrightarrow{d} H_{n+1}(C/B) \xrightarrow{d} H_n(B/A) \xrightarrow{d} H_{n-1}(A) \to 0$$

$$0 \xrightarrow{d} H_n(C/B) \xrightarrow{d} H_{n-1}(B/A) \xrightarrow{d} H_{n-2}A \to 0$$

and so on.

These form the first "page" of our spectral sequence. The homology groups of these complexes are the terms on the next page, but we need more info in order to determine the differentials.

Letting $T = \bigoplus_n \left(H_n(A) \oplus H_n(B) \oplus H_n(C)\right)$ and $E = \bigoplus_n \left(H_n(A) \oplus H_n(B/A) \oplus H_n(C/B)\right)$, we see that the maps i, j and k fit together to form an "exact couple"

$$\begin{tikzcd} T \arrow[rr, "i"] & & T \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd},$$

i.e. a (non-commuting) triangle exact at each vertex. Note that w/ $d_1 = jk$, exactness gives $d_1^2 = jkjk = 0$ for any exact couple.

Now, letting $E' = \ker d_1 / \mathrm{Im}\, d_1 = \bigoplus_n \frac{H_n(A)}{d(H_{n+1}B/A)} \oplus \frac{\ker(d|_{H_n B/A})}{d(H_{n+1}(C/B))} \oplus \ker d|_{H_n \frac{C}{B}}$

we can actually fit E' into a "derived" exact couple

$$\begin{tikzcd} T' \arrow[rr, "i'"] & & T' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

where $T' = i(T) = \bigoplus_n \Big(\underset{H_n(B)}{\underset{\cap}{i(H_nA)}} \oplus \underset{H_n(C)}{\underset{\cap}{i(H_nB)}} \oplus \underset{H_n(C)}{\underset{\|}{i(H_nC)}} \Big),$

and i', j', k' are given by:

$$i' = i|_{T'}$$

$$j'(ia) = [j(a)] \in E'$$

$$k'([e]) = k(e).$$

Note that i', j' and k' respect the direct sum structures of T' and E', i.e. i' sends classes in H_nB up to H_nC and fixes classes of H_nC, while j' sends classes in $i(H_nB) \subset H_nC$ to classes in the correct subquotient of $H_n(B/A)$ (etc.) and k' lowers the dimension of any given $[e]$.

A diagram chase shows that this derived couple is exact, and so the previous paragraph amounts to the existence of long exact sequences

$$\cdots \to 0 \xrightarrow{i'} i(H_nA) \xrightarrow{j'} \overset{=E^2_{n,A}}{\frac{\ker d_1|_{H_nA/0}}{\operatorname{Im} d_1|_{H_{n+1}B/A}}} \xrightarrow{k'} 0 \to \cdots$$

$$\to i(H_nA) \xrightarrow{i'} i(H_nB) \xrightarrow{j'} \overset{=E^2_{n,B}}{\frac{\ker d_1|_{H_nB/A}}{\operatorname{Im} d_1|_{H_{n+1}C/B}}} \xrightarrow{k'} 0 \xrightarrow{i'} i(H_{n-1}A) \xrightarrow{j'} \overset{=E^2_{n-1,A}}{\frac{\ker d_1|_{H_{n-1}A/0}}{\operatorname{Im} d_1|_{H_nB/A}}} \to$$

$$\to i(H_nB) \xrightarrow{i'} i(H_nC) \xrightarrow{j'} \overset{=E^2_{n,C}}{\frac{\ker d_1|_{H_nC/B}}{0}} \xrightarrow{k'} i(H_{n-1}A) \xrightarrow{i'} i(H_{n-1}B) \xrightarrow{j'} E^2_{n-1,B} \to \cdots$$

$$\to 0 \to i(H_nC) \xrightarrow{i'} i(H_nC) \xrightarrow{j'} 0 \xrightarrow{k'} i(H_{n-1}B) \to i(H_{n-1}C) \to E^2_{n-1,C} \to \cdots$$

(and some more trivial ones above and below) which fit together to form the diagram on the next page, where I'll write each term in the same spot as the term in Diagram 1 from which it was derived (either as a subgroup or subquotient):

$$
\begin{tikzcd}
E^2_{n+3,C} \arrow[r, "k'"] & i(H_{n+2}A) \arrow[d, "i'"] & E^2_{n+2,B} \arrow[r, "k'"] & 0 \arrow[d, "i'"] & E^2_{n+1,A} \arrow[r, "k'"] & 0 \arrow[d, "i'"] & 0 \arrow[r] & 0 \arrow[d] & 0 \\
0 \arrow[r, "k'"] & i(H_{n+2}B) \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n+2,C} \arrow[r, "k'"] & i(H_{n+1}A) \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n+1,B} \arrow[r, "k'"] & 0 \arrow[d, "i'"] \arrow[ur] & E^2_{n,A} \arrow[r, "k'"] & 0 \arrow[d, "i'"] \arrow[ur] & 0 \\
 & i(H_{n+2}C) \arrow[ur, "j'"] & 0 \arrow[r] & i(H_{n+1}B) \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n+1,C} \arrow[r, "k'"] & i(H_nA) \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n,B} \arrow[r, "k'"] & 0 \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n-1,A} \\
 & & & i(H_{n+1}C) \arrow[ur, "j'"] & 0 \arrow[r, "k'"] & i(H_nB) \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n,C} \arrow[r, "k'"] & i(H_{n-1}A) \arrow[d, "i'"] \arrow[ur, "j'"] & E^2_{n-1,B} \\
 & & & & 0 \arrow[r] & i(H_nC) \arrow[ur, "j'"] & 0 \arrow[r, "k'"] & i(H_{n-1}B) \arrow[ur, "j'"] & E^2_{n-1,C}
\end{tikzcd}
$$

Diagram 2

The next page:

$$
\begin{tikzcd}
E^3_{n+1,A} \arrow[r] & 0 & 0 & \\
E^\infty_{n+1,B} \arrow[r] & 0 & E^3_{n,A} \arrow[r, "k_3"] & 0 \arrow[d, "i_3"] \\
E^3_{n+1,C} \arrow[r] & 0 \arrow[d, "i"] & E^\infty_{n,B} \arrow[r, "k_3"] & 0 \arrow[d, "i_3"] \\
0 \arrow[r] & i^2(H_nA) \arrow[d, "i"] \arrow[uur, "j_3"] & E^3_{n,C} \arrow[r, "k_3"] & 0 \arrow[d, "i_3"] \\
0 \arrow[r] & i^2(H_nB) \arrow[d, "i"] \arrow[uur, "j_3"] & 0 \arrow[r, "k_3"] & i(H_nA) \\
0 \arrow[r] & i^2(H_nC) \arrow[uur, "j_3"] & &
\end{tikzcd}
$$

Exact Sequences

$0 \to i^2 H_n A \xrightarrow{\cong} E^3_{n,A} \to 0$, $\quad E^3_{n,A} = E^\infty_{n,A}$

$0 \to i^2 H_n A \to i^2 H_n B \to E^\infty_{n,B} \to 0$

$0 \to i^2 H_n B \to i^2 H_n C \to E^3_{n,C} \to 0$, $\quad i^2 H_n C = H_n C$, $\quad E^3_{n,C} = E^\infty_{n,C}$

All differentials are zero

Diagram 3

What does the 3^{rd} page say?

We have a filtration on $H_n C$ given by

$$0 \subset i^2 H_n A \subset \underset{i^2 H_n B}{\overset{\shortparallel}{i H_n B}} \subset H_n C$$

and the 3^{rd} page tells us what each quotient is:

$$i^2 H_n A \cong E^\infty_{n,A}$$

$$iH_nB \big/ i^2 H_n A \cong E^\infty_{n,B}$$

$$H_n C \big/ i H_n B \cong E^\infty_{n,C}.$$

Thus we have shown:

<u>Theorem</u> $H_n C \cong E^\infty_{n,A} \oplus E^\infty_{n,B} \oplus E^\infty_{n,C}$. [I'm assuming these are vector spaces.]

What <u>is</u> $E^\infty_{n,-}$?

First, $E^\infty_{n,A} = E^3_{n,A}$ is a quotient of $E^2_{n,A}$, which in turn is a quotient of $H_n(A)$. So in effect, $E^\infty_{n,A}$ is just a subquotient of $H_n A$. To be precise,

$$E^2_{n,A} \cong H_n A \big/ \partial\left(H_{n+1}(B/A)\right),$$

and $E^3_{n,A} \cong E^2_{n,A} / d_2(E^2_{n+1,C})$. To go further, we'll need to understand $E^2_{n+1,C}$. We have

$$E^2_{n+1,C} \cong \ker\left(d_1\left(H_{n+1} C/B\right)\right),$$

so $E^2_{n+1,C}$ is a subgroup of $H_{n+1} C/B$; namely those elts whose bdry lies in A.

What is $E^\infty_{n,B}$? We know

$$E^\infty_{n,B} \cong E^2_{n,B}$$

and $E^2_{n,B}$ is a subquotient of $H_n(B/A)$, namely

$$\frac{\ker(\partial : H_n(B/A) \to H_n A)}{\mathrm{Im}\ H_{n+1} C/B}.$$

These are the elts of $H_n B/A$ which come from $H_n B$ modulo the boundaries from $H_{n+1} C$.

What is $E^\infty_{n,C}$? We have

$$E^\infty_{n,C} \subseteq E^3_{n,C} = \ker(d_2 : E^2_{n,C} \to E^2_{n-1,A}),$$

or $E^\infty_{n,C}$ is a subgroup of $E^2_{n,C} = \ker(d_1 : H_n C/B \to H_{n-1} B/A)$, a subgroup of $H_n C/B$. So $E^\infty_{n,C}$ is just a subgroup of $H_n C/B$.

CW-complex X, filtered by skeleta:

Page 1:

$$
\begin{tikzcd}
\overset{0}{\parallel} & \overset{0}{\parallel} & \overset{0}{\parallel} & \overset{0}{\parallel} \\[-1em]
H_{k+1}(X^k) \arrow[r, "j"] \arrow[d, "i"] & H_{k+1}(X^k/X^{k-1}) \arrow[r, "\partial"] & H_k(X^{k-1}) \arrow[r, "j"] \arrow[d, "i"] & H_k(X^{k-1}/X^{k-2}) \\
H_{k+1}(X^{k+1}) \arrow[r, "j"] & H_{k+1}(X^{k+1}/X^k) \arrow[r, "\partial"] \arrow[rr, bend right, "\text{bdry in cellular complex}"'] & H_k(X^k) \arrow[r, "j"] & H_k(X^k/X^{k-1})
\end{tikzcd}
$$

Page 2:

$$
\begin{tikzcd}
0 \arrow[d] & 0 & 0 \arrow[d] & 0 & 0 \arrow[d] \\
0 \arrow[d] \arrow[ur] & E^2_{k+1,X^{k+1}} \arrow[r] & 0 \arrow[d] \arrow[ur] & E^2_{k,X^k} \arrow[r] & 0 \\
i(H_{k+1}X^{k+1}) \arrow[ur] & 0 \arrow[r] & i(H_kX^k) \arrow[d] \arrow[ur] & E^2_{k,X^{k+1}} \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & i(H_kX^{k+1}) \arrow[ur] & 0 &
\end{tikzcd}
$$

So $E^2_{k,X^k} \cong i(H_k(X^k))$, and moreover,

$$0 = E^2_{k,X^{k+1}} = \frac{iH_kX^{k+1}}{i^2H_kX^k} \cong \frac{H_kX^{k+1}}{iH_kX^k}, \text{ so}$$

$$iH_k(X^k) = H_kX^{k+1} = H_k(X).$$