

## Fibrations, and Fibre Bundles

A map  $p: E \rightarrow B$  is a fibration if, for any

space  $X$ , homotopy  $h: X \times I \rightarrow B$

and map  $F: X \rightarrow E$  such that

$p \circ F(x) = h(x, 0)$ , there is a

homotopy  $H: X \times I \rightarrow E$  such that

$H(x, 0) = F$  and  $p \circ H = h$ .

$$\begin{array}{ccc} X & \xrightarrow{F} & E \\ \downarrow \cong & \searrow H & \downarrow p \\ X \times I & \xrightarrow{h} & B \end{array}$$

Example  $p_2: F \times B \rightarrow B$  is a fibration

$$\begin{array}{ccc} X & \xrightarrow{f} & F \times B \\ \downarrow \cong & & \downarrow p_2 \\ X \times I & \xrightarrow{h} & B \end{array}$$

we have  $f(x) = (f_1(x), f_2(x))$  and

$$p_2 f(x) = f_2(x).$$

Also  $h(x, 0) = p_2 f(x) = f_2(x)$

Define  $H: X \times I \rightarrow F \times B$

$$H(x, t) = (f_1(x), h(x, t))$$

$$H(x, 0) = H(x, 0) = (f_1(x), h(x, 0)) = (f_1(x), f_2(x))$$

Exercise

The pullback construction:

$$\begin{array}{ccc}
 & Y & \xrightarrow{g} \\
 & \downarrow & \downarrow \\
 X & \xrightarrow{f} & Z
 \end{array}$$

The pullback of  $f, g$  over  $Z$  is

$$P \subset X \times Y$$

$$= \{ (x, y) \mid f(x) = g(y) \}$$

$$P \xrightarrow{f'} Y$$

$$g' \downarrow \quad \downarrow g$$

$$X \xrightarrow{f} Z$$

$$f' = p_2|_P \quad g' = p_1|_P$$

Give  $P$  the subspace topology.

Prove the  $(P, f', g')$  has a univ

property:

Let  $Q$  be a space equipped with

$$\text{maps } f_1: Q \rightarrow Y \quad g_1: Q \rightarrow X$$

$$\text{such that } f g_1 = g f_1$$

$$\begin{array}{ccc}
 Q & \xrightarrow{f_1} & Y \\
 \downarrow g_1 & \searrow f' & \downarrow g \\
 X & \xrightarrow{f} & Z
 \end{array}$$

$$\exists! \varphi: Q \rightarrow P$$

$$\text{such that } g' \circ \varphi = g_1$$

$$f' \circ \varphi = f_1$$

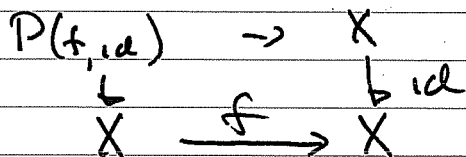
Indeed,  $g_1, f_1$  induce  $(g_1, f_1) : Q \rightarrow X \times Y$   
 (univ prop of product)  
 and the condition

$$f \circ g_1 = g_2 \circ f_1$$

implies that the image of  $(g_1, f_1) : Q \rightarrow X \times Y$   
 actually lies in  $D$ . So we let  $\varphi$

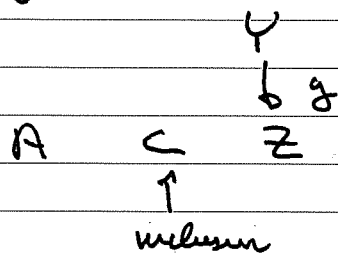
be  $(g_1, f_1)$  viewed as map  $Q \rightarrow D \subset X \times Y$

Examples of pullbacks



$$P(f, id) = \{ (x, y) \mid f(x) = y \}$$

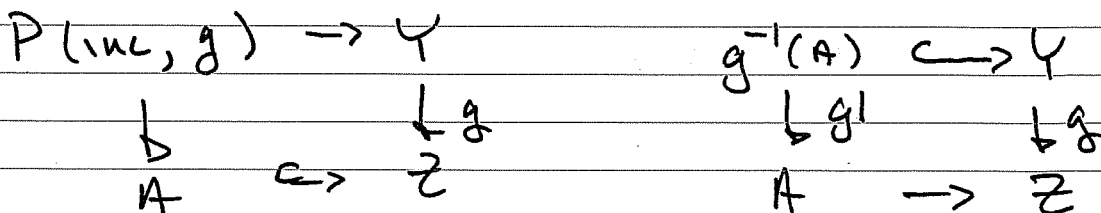
is just the graph of  $f$



$$P(inc, g) = \{ (a, y) \mid g(y) = a \}$$

$$\cong g^{-1}(A)$$

The diagram is isomorphic to



Proposition: Let  $p: E \rightarrow B$  be a fibration and let  $f: B' \rightarrow B$  be a map. Form the pullback diagram

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

Then  $p': E' \rightarrow B'$  is a fibration.

Proof: Consider the diagram

$$\begin{array}{ccccc} X \times \mathbb{I} & \xrightarrow{F} & E' & \xrightarrow{f'} & E \\ \downarrow \alpha & & \downarrow p' & & \downarrow p \\ X \times \mathbb{I} & \xrightarrow{h} & B' & \xrightarrow{f} & B \end{array}$$

in which the left half is the lifting problem we want to solve.

We have  $p' \circ F = h \circ \alpha$   
 so  $(f \circ h) \circ \alpha = f \circ p' \circ F$   
 $= p \circ (f' \circ F)$

That is, there is an associated lifting problem for  $p: E \rightarrow B$ .

Since  $p: E \rightarrow B$  is a fibration, there is a solution to the problem

$$H: X \times \mathbb{I} \rightarrow B$$

such that  $p \circ H = f \circ h$  and  
 $H \circ i_0 = f' \circ F$ .

By the pull back property  $pH = fh$   
implies there is  $H': X \times I \rightarrow E'$

such that  $p'H' = h$  and  $f'H' = H$

The eqn  $p'H' = h$  is part of what  
we need. We also need to show

$$H' \circ i_0 = F.$$

For this apply the uniqueness part  
of the universal property of pull back:  
we have

$$p' \circ H' \circ i_0 = h \circ i_0 = p' \circ F$$

Also

$$f' \circ H' \circ i_0 = H \circ i_0 = f' \circ F$$

In view of the uniqueness property

$$H' \circ i_0 = F$$

as needed.

Observation: Let  $p: E \rightarrow B$  be a fibra-  
tion and let  $\{U_\alpha\}$  be an open cover  
of  $B$ . In view of the proposition,

the diagrams

$$\begin{array}{ccc} E|U_\alpha = E_\alpha = p^{-1}(U_\alpha) & \longrightarrow & E \\ \downarrow p|p^{-1}(U_\alpha) & & \downarrow p \\ U_\alpha & \hookrightarrow & B \end{array}$$

as  $\alpha$  varies ~~and~~ all ~~part~~ show

that  $p|p^{-1}(U_\alpha) : p^{-1}(U_\alpha) \rightarrow U_\alpha$

is a fibration. Better:

if  $p : E \rightarrow B$  is a fibration, so is

each  $p_\alpha = p|p^{-1}(U_\alpha) : E|U_\alpha = p^{-1}(U_\alpha) \rightarrow U_\alpha$

There is a partial converse

Theorem (Hurewicz, Huebsch)

let  $p : E \rightarrow B$  be a map and

suppose  $B$  is Hausdorff and

paracompact. If there is a

covering of  $B$  by open sets

$U_\alpha$  such that  $p|p^{-1}(U_\alpha) : p^{-1}(U_\alpha) \rightarrow U_\alpha$

is a fibration for each  $U_\alpha$ , then

$p : E \rightarrow B$  is also a fibration.

(a <sup>sort</sup> ~~of~~ local to global principle)

Defn (Paracompact) Let  $X$  be a

Hausdorff space.  $X$  is paracompact

if every open cover  $\mathcal{U}$  has a locally finite refinement  $\mathcal{V}$ .

This means that for each  $V \in \mathcal{V}$  there is  $U \in \mathcal{U}$  such that  $V \subset U$  and, for each  $x \in X$ , there is an open nbhd  $W$  of  $x$  such that  $W \cap V \neq \emptyset$  for at most only finitely many  $V \in \mathcal{V}$ .

Examples:  $X$  compact  $\Rightarrow X$  paracompact

Manifolds are paracompact

CW-complexes are paracompact

Sample application

$\mathbb{C}P^n =$  space of lines in  $\mathbb{C}^{n+1}$

is usually defined as a quotient space of  $\mathbb{C}^{n+1} - \{0\}$  under a certain equiv. relation.

Since any line in  $\mathbb{C}^{n+1}$  meets the unit sphere  $S^{2n+1}$ , we can also

view  $\mathbb{C}P^n$  as a quotient of  $S^{2n+1}$ . To-

day it is convenient to view  $\mathbb{C}P^n$

as the orbit space of  $S^{2n+1}$  under the group action  $S^1 \times S^{2n+1} \rightarrow S^{2n+1}$

$$(z, (z_0, \dots, z_n)) \mapsto (zz_0, \dots, zz_n)$$

$$\text{let } p: S^{2n+1} \rightarrow \mathbb{C}P^n$$

$$p(z_0, \dots, z_n) = [z_0, \dots, z_n]$$

be the quotient map.

$p$  can be seen to be a fibration as follows.

$$\text{Let } U_i \subset \mathbb{C}P^n$$

$$= \{ [z_0, \dots, z_n] \mid z_i \neq 0 \}$$

$$\text{so } p^{-1}(U_i) \subset S^{2n+1}$$

$$= \left\{ (z_0, \dots, z_n) \mid \begin{array}{l} z_i \neq 0 \\ \sum |z_j|^2 = 1 \end{array} \right\}$$

We define maps  $f, g$  which are mutually inverse and make the  $\Delta$  commute

$$\begin{array}{ccc} p^{-1}(U_i) & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} & S^1 \times U_i \\ & \searrow p & \swarrow p_2 \\ & & U_i \end{array}$$

$$f(z_0, \dots, z_n) = \left( \frac{z_i}{|z_i|}, [z_0, \dots, z_n] \right)$$

$$g(z, [w_0, \dots, w_n])$$

$$= \left( \frac{zw_0/|w_i|}{w_i}, \dots, z|w_i|, \dots, \frac{zw_n/|w_i|}{w_i} \right)$$



Since  $[\omega_0, \dots, \omega_n] = [\omega'_0, \dots, \omega'_n]$  means there is  $\lambda$ ,  $|\lambda|=1$  such that

$$\omega_j = \lambda \omega'_j \quad \text{for } j=0, \dots, n, \quad g(z, [\omega_0, \dots, \omega_n])$$

is well defined. It is also easy to

$$\text{check } |g(z, [\omega_0, \dots, \omega_n])|^2 = 1$$

$$\text{if } \sum_i |\omega_i|^2 = 1.$$

With a little patience

$$f \circ g = \text{id}_{S^1 \times U_i}$$

$$\text{and } g \circ f = \text{id}_{p^{-1}(u_i)}$$

This shows that  $p: p^{-1}(u_i) \rightarrow u_i$

is isomorphic to  $p_2: S^1 \times U_i \rightarrow U_i$ ,

which is a fibration.

Consequently  $p: p^{-1}(u_i) \rightarrow u_i$  is

a fibration.

By the "local to global" theorem,

$p: S^{2n+1} \rightarrow \mathbb{C}P^n$  is a fibration

(actually we are showing  $p: S^{2n+1} \rightarrow \mathbb{C}P^n$

is a fibre bundle, with fibre  $S^1$ )

