Fibrations, and Fibre Bundles

A map $p: E \to B$ is a fibration if, for any space X, homotopy $h: X \times I \to B$ and map $F: X \to E$ such that $p \circ F(x) = h(x,0)$, there is a homotopy $H: X \times I \to E$ such that

$H(x,0) = F$ and $pH = h$.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

__Example__ $p_2: \underset{\underset{E}{\shortparallel}}{F \times B} \to B$ is a fibration

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & F \times B \arrow[d, "p_2"] \\ X \times I \arrow[r, "h"'] & B \end{tikzcd}$$

we have $f(x) = (f_1(x), f_2(x))$ and

$p_2 f(x) = f_2(x)$.

Also $h(x,0) = p_2 f(x) = f_2(x)$

Define $H: X \times I \to F \times B$

$H(x,t) = (f_1(x), h(x,t))$

$\cancel{H(x,0)}$ $H i_0(x) = H(x,0) = (f_1(x), h(x,0)) = (f_1(x)\ f_2(x))$

[illegible]

The pullback construction:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} \cancel{E} \arrow[d, "p"] \\ \cancel{B} \end{tikzcd}$$

The pullback of f, g over Z is

$P \subset X \times Y$

$= \{ (x, y) \mid f(x) = g(y) \}$

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

$f' = p_2 | P \qquad g' = p_1 | P$

Give P the subspace topology.

~~Prelims~~ Then (P, f', g') has a univ property:

Let Q be a space equipped with maps $f_1 : Q \to \cancel{Z}\, Y$ ~~&~~ $g_1 : Q \to X$ such that $f g_1 = g f_1$

$$\begin{tikzcd} Q \arrow[dr, dashed] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "g_1"'] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

$\exists!\ \varphi : Q \to P$ such that $g' \circ \varphi = g_1$

$f' \circ \varphi = f_1$

Indeed, g_1, f_1 induce $(g_1, f_1) : Q \to X \times Y$
(univ prop of product)
and the condition

$$f g_1 = g f_1$$

implies that the image of $(g_1, f_1) : Q \to X \times Y$ actually lies in P. So we let φ be (g_1, f_1) viewed as map $Q \to P$

<u>Examples of pullbacks</u>

$$\begin{tikzcd}
P(f, id) \arrow[r] \arrow[d] & X \arrow[d, "id"] \\
X \arrow[r, "f"] & X
\end{tikzcd}$$

$$P(f, id) = \{ (x, y) \mid f(x) = y \}$$

is just the graph of f

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
A \subset & Z
\end{tikzcd}$$

inclusion

$$P(inc, g) = \{ (a, y) \mid g(y) = a \} \cong g^{-1}(A)$$

The diagram

$$\begin{tikzcd}
P(inc, g) \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
A \arrow[r, hook] & Z
\end{tikzcd}$$

is isomorphic to

$$\begin{tikzcd}
g^{-1}(A) \arrow[r, hook] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
A \arrow[r] & Z
\end{tikzcd}$$

<u>Proposition</u>: Let $p : E \to B$ be a fibration and let $f : B' \to B$ be a map. Form the pullback diagram

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Then $p' : E' \to B'$ is a fibration.

<u>Proof</u>: Consider the diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "i_0"] & E' \arrow[r, "f'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] & B' \arrow[r, "f"] & B \end{tikzcd}$$

in which the left half is the lifting problem we want to solve.

We have $p' F = h i_0$

so $(f \circ h) \circ i_0 = f \circ p' \circ F$

$= p \circ (f' \circ F)$

That is, there is an associated lifting problem for $p : E \to B$.

Since $p : E \to B$ is a fibration, there is a solution to the problem

$H : X \times I \to B$

such that $p \circ H = f \circ h$ and

$H i_0 = f' F$.

By the pullback property $pH = fh$ implies there is $H' : X \times I \to E'$ such that $p'H' = h$ and $f'H' = H$

The eqn $p'H' = h$ is part of what we need. We also need to show

$H' \circ i_0 = F$.

For this apply the uniqueness part of the universal property of pullback: we have

$$p' \circ H' \circ i_0 = h \circ i_0 = p' \circ F$$

Also

$$f' \circ H' \circ i_0 = H \circ i_0 = f' \circ F$$

In view of this uniqueness property

$$H' \circ i_0 = F$$

as needed.

<u>Observation</u>: Let $p : E \to B$ be a fibration and let $\{U_\alpha\}$ be an open cover of B. In view of the proposition,

the diagrams

$$\begin{array}{ccc} E|U_\alpha = E_\alpha = p^{-1}(U_\alpha) & \longrightarrow & E \\ \downarrow p|p^{-1}(U_\alpha) & & \downarrow p \\ U_\alpha & \hookrightarrow & B \end{array}$$

as α varies ~~as~~ all ~~prod~~ show that $p|p^{-1}(U_\alpha) : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration. Better:

If $p : E \to B$ is a fibration, so is each $p_\alpha = p|p^{-1}(U_\alpha) : E|U_\alpha = p^{-1}(U_\alpha) \to U_\alpha$

There is a partial converse

<u>Theorem</u> (Hurewicz, Huebsch)

Let $p : E \to B$ be a map and suppose B is Hausdorff and paracompact. If there is a covering of B by open sets U_α such that $p|p^{-1}(U_\alpha) : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each U_α, then $p : E \to B$ is also a fibration.

(a ~~test~~ sort of local to global principle)

<u>Defn</u> (Paracompact) Let X be a Hausdorff space. X is paracompact

if every open cover ~~has~~ $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$.

This means that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$ and, for each $x \in X$, there is an open nbhd W of x such that $W \cap V \neq \emptyset$ for ~~at most~~ only finitely many $V \in \mathcal{V}$

<u>Examples</u>: X compact $\Rightarrow$ X paracompact

Manifolds are paracompact

CW-complexes are paracompact

<u>Sample application</u>

$\mathbb{C}P^n$ = space of lines in $\mathbb{C}^{n+1}$ is usually defined as a quotient space of $\mathbb{C}^{n+1} - \{0\}$ under a certain equiv. relation. Since any line in $\mathbb{C}^{n+1}$ meets the unit sphere S^{2n+1}, we can also view $\mathbb{C}P^n$ as a quotient of S^{2n+1}. Today it is convenient to view $\mathbb{C}P^n$ as the orbit space of S^{2n+1} under the group action $S^1 \times S^{2n+1} \to S^{2n+1}$

$$(z, (z_0, \dots, z_n)) \mapsto (zz_0, \dots, zz_n)$$

let $p : S^{2n+1} \to \mathbb{CP}^n$

$$p(z_0, \dots, z_n) = [z_0, \dots, z_n]$$

be the quotient map.

p can be seen to be a fibration as follows.

Let $\mathcal{U}_i \subset \mathbb{CP}^n$

$$= \{ [z_0, \dots, z_n] \mid z_i \neq 0 \}.$$

so $p^{-1}(\mathcal{U}_i) \subseteq S^{2n+1}$

$$= \left\{ (z_0, \dots, z_n) \;\middle|\; \begin{array}{l} z_i \neq 0 \\ \sum |z_j|^2 = 1 \end{array} \right\}$$

We define maps f, g which are mutually inverse and make the Δ commute

$$\begin{tikzcd}
p^{-1}(\mathcal{U}_i) \arrow[r, shift left, "f"] \arrow[dr, "p"'] & S^1 \times \mathcal{U}_i \arrow[l, shift left, "g"] \arrow[d, "p_2"] \\
& \mathcal{U}_i
\end{tikzcd}$$

$$f(z_0, \dots, z_n) = \left(\frac{z_i}{|z_i|}, [z_0, \dots, z_n] \right)$$

$$g(z, [w_0, \dots, w_n])$$

$$= \left(\frac{z w_0 |w_i|}{w_i}, \dots, z|w_i|, \dots, \frac{z w_n |w_i|}{w_i} \right)$$

Since $[\omega_0, \dots, \omega_n] = [\omega_0', \dots, \omega_n']$ means there is λ, $|\lambda| = 1$ such that $\omega_j = \lambda \omega_j'$ for $j = 0, \dots n$, $g(z, [\omega_0 \cdots \omega_n])$ is well defined. It is also easy to check $|g(z, [\omega_0, \dots, \omega_n])|^2 = 1$ if $\Sigma_i |\omega_j|^2 = 1$.

With a little patience

$$f \circ g = \mathrm{Id}_{S^1 \times U_i}$$

and

$$g \circ f = \mathrm{Id}\big|_{p^{-1}(U_i)}$$

This shows that $p : p^{-1}(U_i) \to U_i$ is isomorphic to $p_2 : S^1 \times U_i \to U_i$, which is a fibration.

Consequently $p : p^{-1}(U_i) \to U_i$ is a fibration.

By the "local to global" theorem, $p : S^{2n+1} \to \mathbb{C}P^n$ is a fibration

(Actually we are showing $p : S^{2n+1} \to \mathbb{C}P^n$ is a fibre bundle, with fibre S^1)