3.2.3 The Partial Autocorrelation Function

The PACF is \( \alpha(0) = 1, \alpha(h) = \phi_{hh}, h \geq 1 \), while the sample PACF is \( \hat{\alpha}(0) = 1 \), and \( \hat{\alpha}(h) = \hat{\phi}_{hh}, h > 1 \). The PACF is actually the correlation of \( X_h - P(X_h | X_1, \ldots, X_{h-1}) \) and \( X_0 - P(X_0 | X_1, \ldots, X_{h-1}) \), hence the name partial autocorrelation. Recall that \( \rho(h) = 0, h > q \) for MA(q). Similarly, \( \alpha(h) = 0, h > q \) for AR(p).

Example 3.2.6-Example 3.2.9
3.3 Forecasting ARMA Processes

Suppose \( \{X_t\}_{t=-\infty}^{\infty} \) is a mean zero causal ARMA\((p,q)\) process
\[
\phi(B)X_t = \theta(B)Z_t, \quad \text{where} \quad Z_t \sim WN(0, \sigma^2)
\]

Define a new process \( \{W_t\}_{t=1}^{\infty} \) as
\[
W_t = \left\{ \begin{array}{ll}
\sigma^{-1}X_t & 1 \leq t \leq m = \max\{p,q\} \\
\sigma^{-1}\phi(B)X_t & t > m
\end{array} \right.
\]

Then for the new process \( \{W_t\}_{t=1}^{\infty} \), one can calculate
\[
\text{COV}(W_i, W_j) = R_{ij}
\]
where
\[
R_{ij} = \left\{ \begin{array}{ll}
\sigma^2/2 & 1 \leq i < j < m \\
\sigma^2[\phi_{ij} - \phi_{i+1,j}] & m \leq i < j < 2m \\
\frac{q}{j} \theta_j & i = j \leq m \\
\theta_j & j < i \\
0 & \text{otherwise}
\end{array} \right.
\]

Note: 1. \( \theta_0 = 1 \) and \( \theta_j = 0, j > q \)

2. Through calculation, \( R_{r,s} = 0 \) if \( r > m \) and \( |r-s| > q \)

Applying the innovations algorithm to the process \( \{W_t\} \), we have
\[
\begin{align*}
\hat{W}_{nt} &= \sum_{j=1}^{n} \theta_j (\hat{W}_{n+1,t-j} - \hat{W}_{nt-j}) \quad 1 \leq n < m \\
\hat{W}_{n+1,t} &= \sum_{j=1}^{q} \theta_j (\hat{W}_{n,t-j} - \hat{W}_{n+1,t-j}) \quad n \geq m
\end{align*}
\]

Meanwhile, using the linear properties of \( P_n \), we have
\[
\begin{align*}
\hat{W}_{nt} &= \sigma^{-1} \hat{X}_{nt} \quad 0 \leq n < m \\
\hat{X}_{n+1,t} &= \sigma^{-1} \left[ \hat{X}_{n+1,t} \theta_j - \phi_i \hat{X}_{n+t} \right] \quad n \geq m
\end{align*}
\]

Hence,
\[
\hat{X}_{n+1,t} - \hat{X}_{nt} = \sigma (\hat{W}_{n+1,t} - \hat{W}_{nt}) \quad n = 0, 1, 2, \ldots
\]

and
\[
\begin{align*}
\hat{X}_{nt} &= \sum_{j=1}^{n} \theta_j (\hat{X}_{n+1,t-j} - \hat{X}_{nt-j}) \quad 1 \leq n < m \\
\hat{X}_{n+1,t} &= \sum_{j=1}^{q} \phi_j \hat{X}_{n+t-j} + \frac{q}{j} \theta_j (\hat{X}_{n+1,t-j} - \hat{X}_{n+t-j}) \quad n \geq m
\end{align*}
\]
And also, \( E(X_{n+1} - \hat{X}_{n+1})^2 = \sigma^2 E(W_{n+1} - \hat{W}_{n+1})^2 = \sigma^2 r_n, n = 0, 1, 2, \ldots \)

Remark: It can shown that if \( \{X_t\} \) is invertible, then

1. \( E(X_n - \hat{X}_n - Z_n)^2 \rightarrow 0 \) and \( \theta_{nj} \rightarrow \theta_j, j = 1, \ldots, q \).
2. \( r_n \rightarrow 1 \).

Example 3.3.1 Prediction of an AR(\(p\)) process
Example 3.3.2 Prediction of an MA(\(q\)) process
Example 3.3.3 Prediction of an ARMA(1,1) process
Example 3.3.4 Numerical prediction of an ARMA(2,3) process

Exam 1 on March 9, 2010.