6.5 Seasonal ARIMA models

A process \( \{X_t\} \) is called seasonal \( ARIMA(p,d,q) \times (P,D,Q) \), process with periods \( s \) if \( Y_t = (1 - B)^d(1 - B^s)^DX_t \) is a causal \( ARMA(p + sP, q + sQ) \) process as

\[
\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t
\]

with \( Z_t \sim WN(0, \sigma^2) \). Alternatively, one can write

\[
\phi^*(B)Y_t = \theta^*(B)Z_t
\]

in which the polynomial \( \phi^*(z) \) is of degree \( p + sP \) and the polynomial \( \theta^*(z) \) is of degree \( q + sQ \).

**Remark 1.** The process \( \{Y_t\} \) is causal if and only if \( \phi(z) \neq 0 \) and \( \Phi(z) \neq 0 \) for \( |z| \leq 1 \). In applications, \( D \) is rarely more than one, and \( P \) and \( Q \) are typically less than three.

**Remark 2.** Provided that \( p < s \) and \( q < s \), the constrains on the coefficients of \( \phi^*(.) \) and \( \theta^*(.) \) can all be expressed as multiplicative relations

\[
\phi^*_{is+j} = \phi^*_{is}\phi^*_{j}, \quad i = 1, 2, \ldots; \quad j = 1, \ldots, s - 1,
\]

and

\[
\theta^*_{is+j} = \theta^*_{is}\theta^*_{j}, \quad i = 1, 2, \ldots; \quad j = 1, \ldots, s - 1,
\]

**Remark 3.** The disadvantage of the classical decomposition model \( X_t = m_t + s_t + Y_t \) is that it assume that the seasonal component \( s_t \) repeats itself precisely the same way cycle after cycle. Seasonal ARIMA models allow for randomness in the seasonal pattern from one cycle to the next.

Example 6.5.1-6.5.3 The between year model with \( p = q = 0 \).
The identification and/or estimation of seasonal $ARIMA(p, d, q) \times (P, D, Q)_s$ model:

1. Find the right differencing to achieve stationarity (i.e., $s$, $D$ and $d$)
2. Examine the ACF $\hat{\rho}_{ks}$, $k = 1, 2, \ldots$ of $Y_t$ to choose $P$ and $Q$.
3. Examine the ACF $\hat{\rho}_k$, $k = 1, 2, \ldots, s$ to choose $p$ and $q$.
4. An alternative to the last two steps is to apply MLE and AICC criterion for ARMA
   model of the special form (i.e., the AR and MA polynomials are of the forms
   $\phi(B)\Phi(B^s)$ and $\theta(B)\Theta(B^s)$).

Example 6.5.4 Monthly accidental deaths
6.5.1 Forecasting SARIMA Processes

By a similar argument for ARIMA process, the h-step prediction is obtained by the following recursive formula:

\[ P_n X_{n+h} = P_n Y_{n+h} - \sum_{j=1}^{d+D} a_j P_n X_{n+h-j} \]

and the mean squared error of the h-step prediction is

\[ \sigma_n^2(h) = E(X_{n+h} - P_n X_{n+h})^2 = \sum_{j=0}^{h-1} \left( \sum_{r=0}^{j} \chi_r \theta_{n+h-r-1,j-r} \right)^2 v_{n+h-j-1}, \]

with

\[ \chi(z) = \sum_{r=0}^{\infty} \chi_r z^r = (\phi(z) \Phi(z^s)(1-z^d(1-z^s)^D))^{-1}, \]

and

\[ v_{n+h-j-i} = E \left( X_{n+h-j} - \hat{X}_{n+h-j} \right)^2 = E \left( Y_{n+h-j} - \hat{Y}_{n+h-j} \right)^2. \]

Note: When \( \theta(\cdot)\Theta(\cdot) \) is invertible, \( \sigma_n^2(h) \) can be approximated by

\[ \sigma_n^2(h) = \sum_{j=0}^{h-1} \psi_j^2 \sigma^2, \quad \psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\vartheta(z) \Theta(z^s)}{\phi(z) \Phi(z^s)(1-z^d(1-z^s)^D)}, \quad |z| < 1. \]

Example 6.5.5 Monthly accidental deaths