7.1 Bivariate time series

A two-dimentional process \( \{X_t\} \) with process \( \{X_t = (X_{t1}, X_{t2})'\} \) is called a bivariate time series. Weak stationarity means

\[
\mu_t = E(X_t), \quad \Gamma(t+h, t) = Cov(X_{t+h}, X_t) = \begin{bmatrix}
Cov(X_{t+h,1}, X_{t,1}) & Cov(X_{t+h,1}, X_{t,2}) \\
Cov(X_{t+h,2}, X_{t,1}) & Cov(X_{t+h,2}, X_{t,2})
\end{bmatrix}
\]

are independent of \( t \), in which we use the notation

\[
\mu = E(X_t) = \begin{bmatrix}
E(X_{t1}) \\
E(X_{t2})
\end{bmatrix}
\]

\[
\Gamma(h) = Cov(X_{t+h}, X_t) = \begin{bmatrix}
\gamma_{11}(h) & \gamma_{12}(h) \\
\gamma_{21}(h) & \gamma_{22}(h)
\end{bmatrix}
\]

Note: \( \Gamma(h) = \Gamma'(-h) \) and \( \gamma_{ii}(h) = \gamma_{ii}(-h) \).

The natural estimator of \( \mu \) and \( \Gamma(h) \) are:

\[
\hat{\mu} = \bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t
\]

\[
\hat{\Gamma}(h) = n^{-1} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X}_n)(X_{t+h} - \bar{X}_n)', \quad 0 \leq h \leq n - 1.
\]

\[
\hat{\Gamma}(h) = \hat{\Gamma}'(-h), \quad -n + 1 \leq h < 0.
\]

Example 7.1.1 (DJAO2.TSM and DJAOPC2.TSM) shows the motivation to study the joint structure of two series.

Example 7.1.2 Sales with a leading indicator; LS2.TSM
7.2 Second-Order Properties of Multivariate Time series

We define
\[ X_t := \begin{bmatrix} X_{t1} \\ \vdots \\ X_{tm} \end{bmatrix}, \quad t = 0, \pm 1, \cdots \]
as the vector form of \( m \)-variate multivariate time series \( \{X_t\} \). It’s mean vectors and covariance matrices then are
\[ \mu_t = E X_t := \begin{bmatrix} \mu_{t1} \\ \vdots \\ \mu_{tm} \end{bmatrix} \]
\[ \Gamma(h, t) = Cov(X_{t+h}, X_t) = \begin{bmatrix} \gamma_{11}(t+h, t) & \cdots & \gamma_{12}(t+h, t) \\ \vdots & \ddots & \vdots \\ \gamma_{m1}(t+h, t) & \cdots & \gamma_{mm}(t+h, t) \end{bmatrix} \]
where \( \gamma_{ij}(t+h, t) = Cov(X_{t+h,i}, X_{t,j}) \).

**Definition 7.2.1** The \( m \)-variate series \( \{X_t\} \) is (weakly) stationary if
1. \( \mu_X(t) = \mu \) is independent of \( t \); and
2. \( \Gamma_X(t+h, t) = \Gamma(h) \) is independent of \( t \) for each \( h \).

**Remark:** The correlation matrix is related to the covariance matrix as follows:
\[ R(h) = \begin{pmatrix} \gamma_{11}^{-1/2}(0) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \gamma_{mm}^{-1/2}(0) \end{pmatrix} \Gamma(h) \begin{pmatrix} \gamma_{11}^{-1/2}(0) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \gamma_{mm}^{-1/2}(0) \end{pmatrix}. \]

**Example 7.2.1**
Basis Properties of $\Gamma(.)$:

1. $\Gamma(h) = \Gamma'(-h)$,

2. $|\gamma_{ij}(h) \leq [\gamma_{ii}(0)\gamma_{jj}(0)]^{1/2}$, $i, j = 1, \ldots, m$,

3. $\gamma_{ij}(.)$ is an autocovariance function for all $i = 1, \ldots, m$, and

4. $\sum_{j,k=1}^{n} a_j' \Gamma(j-k)a_k \geq 0$ for all $n \in \{1, 2, \ldots\}$ and $a_1, \ldots, a_n \in \mathbb{R}^m$.

Definition 7.2.2, 7.2.3, 7.2.4
7.3 Estimation of the Mean and Covariance Function

7.3.1 Estimation of \( \mu \)

A natural estimator of \( \mu \) is:

\[
\hat{\mu} = \bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t.
\]

Using the same argument in getting Proposition 2.4.1, we have

**Proposition 7.3.1** If \( \{X_t\} \) is a stationary multivariate time series with mean \( \mu \), then as \( n \to \infty \),

\[
E(\bar{X}_n - \mu)'(\bar{X}_n - \mu) \to 0, \quad \text{if } \gamma_{ii}(n) \to 0, \quad 1 \leq i \leq m,
\]

\[
nE(\bar{X}_n - \mu)'(\bar{X}_n - \mu) = \sum_{i=1}^{m} \sum_{h=-n}^{n} \left( 1 - \frac{|h|}{n} \right) \gamma_{ii}(h)
\]

\[
\to \sum_{i=1}^{m} \sum_{h=-\infty}^{\infty} \gamma_{ii}(h), \quad \text{if } \sum_{h=-\infty}^{\infty} \gamma_{ii}(h) < \infty, \quad 1 \leq i \leq m.
\]

A confidence region for \( \mu \) is given by

7.3.2 Estimation of \( \Gamma(h) \)

The natural estimator of \( \Gamma(h) \) is:

\[
\hat{\Gamma}(h) = n^{-1} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X}_n)(X_{t+h} - \bar{X}_n)', \quad 0 \leq h \leq n-1.
\]

\[
\hat{\Gamma}(h) = \hat{\Gamma}'(-h), \quad -n + 1 \leq h < 0.
\]

**Theorem 7.3.1** If the two time series \( X_{t1} \) and \( X_{t2} \) are linear processes with independent IID noises \( \{Z_{t1}\} \) and \( \{Z_{t2}\} \) respectively, i.e., \( \{Z_{t1}\} \sim \text{IID}(0, \sigma_1^2) \) and \( \{Z_{t2}\} \sim \text{IID}(0, \sigma_2^2) \). Then for all integers \( h \) and \( k \) with \( h \neq k \),

\[
\sqrt{n} \begin{pmatrix} \hat{\rho}_{12}(h) \\ \hat{\rho}_{12}(k) \end{pmatrix} \to_d \mathcal{N} \left( 0, \begin{pmatrix} \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j) & \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j + k - h) \\ \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j + k - h) & \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j) \end{pmatrix} \right)
\]

**Remark:**

1. If one of the two processes in the theorem is white noise, then \( \sqrt{n} \hat{\rho}_{12}(h) \to_d \mathcal{N}(0, 1) \). It is straightforward to test for correlation between two time series.
2. Any test for independence of the two component series cannot be based solely on estimated values of $\rho_{12}(h)$, $h = 0, \pm 1, \cdots$, without taking into account the nature of the two component series.

Example 7.3.1