Bartlett's Formula:

If \( \{X_t\} \) is a bivariate Gaussian time series with covariances satisfying

\[
\sum_{h=-\infty}^{\infty} |\gamma_{ij}(h)| < \infty, \quad i, j = 1, 2,
\]

then

\[
\lim_{n \to \infty} n \text{Cov}(\hat{\rho}_{12}(h), \hat{\rho}_{12}(k)) = \sum_{j=-\infty}^{\infty} \left[ \rho_{11}(j)\rho_{22}(j + k - h) + \rho_{12}(j + k)\rho_{21}(j - h) \\
- \rho_{12}(h)\{\rho_{11}(j)\rho_{12}(j + k) + \rho_{22}(j)\rho_{21}(j - k)\} \\
- \rho_{12}(k)\{\rho_{11}(j)\rho_{12}(j + h) + \rho_{22}(j)\rho_{21}(j - h)\} \\
+ \rho_{12}(h)\rho_{12}(k) \left\{ \frac{1}{2} \rho_{11}^2(j) + \rho_{12}^2(j) + \frac{1}{2} \rho_{22}^2(j) \right\} \right]
\]

Corollary 7.3.1 If \( \{X_t\} \) satisfies the conditions for Bartlett’s formula, if either \( \{X_{t1}\} \) or \( \{X_{t2}\} \) is white noise, and if

\[
\rho_{12}(h) = 0, \quad h \not\in [a, b],
\]

then

\[
\lim_{n \to \infty} n \text{Var}(\hat{\rho}_{12}(h)) = 1, \quad h \not\in [a, b].
\]

**Example 7.3.2** Sales with leading indicator.
7.4 Multivariate ARMA Processes

Definition 7.4.1 \( \{X_t\} \) is an ARMA(p,q) process if \( \{X_t\} \) is stationary and if for every \( t \),

\[
X_t - \Phi_1 X_{t-1} - \cdots - \Phi_p X_{t-p} = Z_t + \Theta_1 Z_{t-1} - \cdots - \Theta_q Z_{t-q},
\]

where \( \{Z_t\} \sim WN(0, \Sigma) \). \( \{X_t\} \) is an ARMA(p,q) process with mean \( \mu \) if \( \{X_t - \mu\} \) is an ARMA}(p,q) process.\)

Causality: An ARMA(p,q) process \( \{X_t\} \) is causal, or causal function of \( \{Z_t\} \), if there exist matrices \( \{\Psi_j\} \) with absolutely summable components such that

\[
X_t = \sum_{j=0}^{\infty} \Psi_j Z_{t-j}, \quad \text{for all } t.
\]

Causality is equivalent to the condition

\[
\det \Phi(z) \neq 0, \quad \text{for all } z \in \mathbb{C} \text{ such that } |z| \leq 1.
\]

The matrices \( \Psi_j \) are found recursively from the equations

\[
\Psi_j = \Theta_j + \sum_{k=1}^{\infty} \Phi_k \Psi_{j-k}, \quad j = 0, 1, \ldots,
\]

where we define \( \Theta_0 = I, \Theta_j = 0 \text{ for } j > q, \Phi_j = 0 \text{ for } j > p, \text{ and } \Psi_j = 0 \text{ for } j < 0.\)

Example 7.4.1-2 The multivariate AR(1) process

Remark 3
Invertibility: An ARMA(p,q) process \( \{X_t\} \) is invertible, if there exist matrices \( \{\Pi_j\} \) with absolutely summable components such that
\[
Z_t = \sum_{j=0}^{\infty} \Pi_j t_{t-j}, \text{ for all } t.
\]

Causality is equivalent to the condition
\[
\det \Theta(z) \neq 0, \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1.
\]
The matrices \( \Pi_j \) are found recursively from the equations
\[
\Pi_j = -\Phi_j - \sum_{k=1}^{\infty} \Theta_k \Pi_{j-k}, \quad j = 0, 1, \ldots,
\]
where we define \( \Phi_0 = -I, \Phi_j = 0 \) for \( j > p \), \( \Theta_j = 0 \) for \( j > q \), and \( \Pi_j = 0 \) for \( j < 0 \).

7.4.1 The covariance matrix function of a causal ARMA process

The covariate matrix \( \Gamma(h) = E(X_{t+h}X_t') \) of the causal process is
\[
\Gamma(h) = \sum_{j=0}^{\infty} \Psi_{h+j} \Sigma \Psi_j', \quad h = 0, \pm 1, \ldots.
\]
The covariance matrices \( \Gamma(h) \) can also be found by solving the Tule-Walker equations
\[
\Gamma(j) - \sum_{r=1}^{p} \Phi_r \Gamma(j-r) = \sum_{j \leq r \leq q} \Theta_r \Sigma \Psi_{r-j}, \quad j = 0, 1, 2, \ldots,
\]
obtained by postmultiplying our model equation by \( X_{t-j}' \) and taking expectations.