Section 12.4. Application of second order differential equation

We already discussed in class the main ideas of the physical problem represented in Figure 12.1. The main goal of this lecture is to investigate the problem in more detail and to find its solutions by applying the techniques on differential equations that we learnt in class. This section also include the electrical analog, but we will focus only on the problem of a weight oscillating on a spring. These notes are just a guideline to help you identifying the main concepts in the text. You still have to read the book though!

The main equation modeling the motion of a weight oscillating on a spring is given by equation (12.15) page 446:

\[ m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = f(t). \]

The above equation is a mathematical translation of the well-known Newton’s law

\[ F = ma. \]

You can rewrite the equation for the weight on the spring as

\[ f(t) - b \frac{dx}{dt} - kx = m \frac{d^2x}{dt^2} \]

and then you see that you obtain the Newton’s law by setting

\[ F = f(t) - b \frac{dx}{dt} - kx \quad \text{and} \quad a = \frac{d^2x}{dt^2}. \]

So this tells you that:

1. the force \( F \) acting on the system is given by three terms:
   
   (a) an external force \( f(t) \) (e.g. you are intentionally pushing and/or pulling the spring). This force may depend on time (e.g. you are not constantly pushing and/or pulling, but you may do so every 2 seconds);

   (b) a damping term proportial to the velocity \( -b \frac{dx}{dt} \). This maybe given by the resistance of the surrounding air. In general, \( b \) is greater than or equal to zero. If in the system there is no damping, then just set \( b = 0 \). This will be useful to solve the homework.

   (c) a term proportional to the spring displacement \( -kx \). This term is due to the Hooke’s law and \( k \) is the elastic constant (\( k > 0 \)).

2. The acceleration \( a \) is given by the second derivative of the displacement with respect to time, namely:

\[ a = \frac{d^2x}{dt^2}. \]

Remember that \( x \) is the displacement, \( dx/dt \) is the velocity, and \( d^2x/dt^2 \) is the acceleration.
Now that you know the meaning of all the terms in the equation, let us go through the examples in the book to see how to use it.

**Example 1.** Read the statement of the problem.

1. The last sentence says *assuming that the damping force is negligible*. This tells you that you will not have in the equation the term $-b \, dx/dt$, so just set $b = 0$ and forget about it. So now you are left with:

   $$m \frac{d^2x}{dt^2} + kx = f(t).$$

2. The text does not mention any external force applied on the system. It only says that the weight is then pulled 4in. below the equilibrium position and then released. This means that $f(t) = 0$ and that at the initial time the displacement at the initial time is equal to 4 in. and that the initial velocity is zero. This is explained at the beginning of page 443. So now your equation is simply:

   $$m \frac{d^2x}{dt^2} + kx = 0.$$

3. First thing you do: use the condition in the test to find the elastic constant $k$.

4. Solve the differential equation as you are used to.

5. Plug in the initial conditions.

**Example 2.** Read the statement of the problem. This is very similar to Example 1, with the only difference that you now have a damping force. To solve this problem you basically follow all the steps as in the previous example.

**Example 3.** Read the statement of the problem. Now you have no damping, but you have an external force. So now the equation is non-homogeneous, but you know how to solve it.

Let me know if you have any questions! Enjoy!