Dynamics of the Nash Map in the Game of Matching Pennies

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Abstract: An iterative procedure for the game of Matching Pennies is examined in which players use Nash’s map to respond to mixed strategies of the other players. It is shown that even though the game of Matching Pennies is a game with a unique Nash equilibrium point, the iterative procedure does not lead to convergence to the Nash equilibrium. In fact we find that if the game is played using this iterative procedure then the successive plays of the game will follow an orbit around the equilibrium point. This orbit has period eight and it is the only periodic orbit other than the equilibrium point.
1 Introduction

One way Nash [1950a] motivated his formulation of an equilibrium point for noncooperative games was based on a mass action interpretation of the game’s play. He went on to present three alternative proofs of an equilibrium point’s existence. Each mapping giving rise to an equilibrium at a fixed point can also be viewed as defining a discrete time evolutionary dynamical system. The object of our paper is to study the dynamics associated with the better response mapping formulated in the published form of the dissertation (Nash [1951]) for the game Matching Pennies and connect its dynamical properties to learning and innovative evolution in strategic form games.

Nash’s doctoral thesis [1950a] uses Brouwer’s Fixed Point Theorem to demonstrate that an equilibrium point of a noncooperative game exists. He constructs a response map that differs from the best response map appearing in his first published proof of an equilibrium point’s existence (Nash [1950b]).¹ Nash [1951] offers a refined version of the dissertation’s Brouwer Fixed Point Theorem argument. This better response map’s fixed points are Nash equilibria, and vice versa. We refer to this mapping from Nash [1951] as the Nash map. In fact, Becker and Chakrabarti [2005] show the Nash map belongs to a class of better response maps whose fixed points are Nash equilibrium points. In an interpretation of the mixed strategies that result when players use the Nash map, Nash [1950a] describes the mixed strategies as the distribution that results from the mass action of the players. Thus he states: “... we assume there is a population of participants of each position of the game. Let us also assume that the ‘average playing’ of the game involves $n$ participants selected at random from the $n$ populations, and that there is a stable average frequency with which each pure strategy is employed by the ‘average member’ of the appropriate population.” Thus a mixed strategy, according to this interpretation, is viewed as the distribution that results from each player of a given population playing a strategy that improves on the strategies played earlier.²

¹Debreu [1952] uses the best response map to generalize Nash’s existence theorem. This generality extends the basic noncooperative game paradigm to some forms of nonexpected utility theory that were only developed much later.

²We note that the players alternatively might be thought of as boundedly rational agents whose strategic revisions are guided by myopic better replies — a type of learning dynamics. There is a large literature on learning models in game theory; see the textbooks by Fudenberg and Levine [1998] and Samuelson [1997].
This interpretation of mixed strategies as the distribution of the pure strategies chosen by a population of players suggests a natural iterative procedure of playing the game. In this iterative procedure, the population of players in a position chooses a distribution of pure strategies that improves on the previous strategies, after having observed the empirical distribution of the pure strategies of the population of the players in the other position. If the population of players keep playing the game iteratively in this fashion then would the play converge to an equilibrium point? Or does the play lead to a path away from the equilibrium point?

Evolutionary game theory has studied the dynamical properties of response maps of different populations of players where the response map of a population of players' is connected to the players genetic code. For example, in the Hawk-Dove game, the Hawk population plays the game in a way that is distinct from the way the Dove population plays the game, and the responses of the two different populations are connected to their distinct genetic types. The object of our study here is to trace out how a particular game, Matching Pennies, is played when players from different populations meet players from the rival population of players in a random match when players play the *Nash map* in every iteration. Thus, in every round, players from one population play the game after being randomly matched with a player from the rival population. But unlike in evolutionary game theory, the players from a population are not programmed to play a specific strategy but respond by observing the distribution of strategies used by the rival players. Thus the response of the players against the observed distribution in every iteration is the Nash map, and differs from the *replicator* dynamics of evolutionary game theory.

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3This is a large literature. Hofbauer and Sigmund [2003] surveys it. They include a discussion of the best response dynamics as well as set up the Brown–von Neumann–Nash (BNN) dynamics motivated by Brown and von Neumann [1951]. The latter is an ordinary differential equation that describes the better response dynamics in continuous time. See Weibull’s presentation of innovative adaptation (in Kuhn et al [1996]). He argues this differential equation’s rest point is a Nash equilibrium. Hofbauer [2000] connects the BNN dynamics to the notion of an evolutionary stable equilibrium. Also see Sandholm (2001), (2005) and Swinkels (1993). Sandholm (2005) presents a microeconomic foundation for the BNN differential equation. Our paper focuses on the discrete time better response dynamics. Other recent surveys of evolutionary game dynamics include Cressman [2003], Friedman [1998], and Mailath [1998] as well as the texts by Hofbauer and Sigmund [1998], Vega-Redondo [1996], and Weibull [1995].

4Cressman [2003], Friedman [1998], and Hofbauer and Sigmund [2003] all discuss the
In this paper we investigate the convergence properties obtained by iterating the Nash map in the game of Matching Pennies. Our reasoning for examining the properties of this iterative procedure in this game is that Matching Pennies is a zero-sum game with a unique mixed strategy equilibrium. Since the equilibrium point is unique, it is a natural solution for the game. Moreover, as a mixed strategy equilibrium, it provides full scope for directly bringing into the analysis the “mass action” interpretation of mixed strategies.

In Matching Pennies there are two types of players, call them the Row and the Column players, respectively. A Row and Column player are drawn from a population of players and are matched randomly in every period. The payoff matrix is the same for every random match and in each period a player plays a pure strategy. Since the players do not know with whom they are going to be matched they play so as to maximize the expected value given the distribution of the pure strategies of the other type of player. Let $R_x$ denote the payoff matrix of the Row player or player X, and let $x$ denote the row vector of the distribution of strategies of player X. Similarly, let the row vector $y$ denote the distribution of the strategies used by the Column players or player Y. If a Row player decides to play the $i^{th}$ row (when column plays $y$), then the expected payoff of the Row player is $e_i R_x y^T$, where $e_i$ is the $i$th coordinate vector (denoted as a row vector). The Row player’s $x$ then gives the proportion of the Row players who play the different pure strategies. The average payoff of the Row players is then given by $x R_x y^T$. Similarly, if $R_y$ is the payoff matrix of the Column players, then the average expected payoff of the Column players is given by $x R_y y^T$.

Suppose some of the Column players change their strategy in the hope of increasing their payoff given the distribution $x$ of the strategies of the Row players observed in the last period and some of the Row players do the same given the distribution $y$ of the strategies of the Column players. If the Row players and the Column players continue to change their strategies given the observed distribution of the strategies of the rival players, then we have an iterative process in which some of the players change their strategy in order to increase their expected payoff. Such an iterative procedure could have been what Nash had in mind when he talked about mixed strategies as being the “mass action” of the players in the population in which each player

replicator dynamics for bimatrix games. Nachbar (1990) studies discrete time selection dynamics that include the replicator dynamics, but excludes the Nash dynamics.
plays a pure strategy, see e.g. Nash [1950a]. Certainly according to this procedure the “mass” of the Row players respond with a better response to the “mass actions” of the Column players and vice versa. This interpretation also explains why the “mass” of Row players respond with a better response rather than the best response. It could be that some of the players respond with a best response to the last observed distribution of the strategies of the opponents while the other players, being uncertain about which opponent they will be matched with, concludes that they may not want to play the best response and may hedge by playing some other response, not necessarily the best response. The diversity of the responses of the players in the population then leads to an “average” response that is a better response rather than a best response.

This “mass action” interpretation of an equilibrium point could explain the specific formulation of the Nash map that Nash [1951] used to demonstrate an equilibrium point exists. This map sends a mixed strategy of each player to another mixed strategy of the player which is better for each player, given the strategies of the other players.\(^5\) Nash’s better response map can be taken as the basis for an iterative scheme. The resulting Nash dynamics’ rest point(s) correspond to the map’s fixed point(s) and its equilibrium point(s).

Skyrms [1990] explored the properties of the iterative procedure outlined above in various two person games with two pure strategies available to each player (called \(2 \times 2\) games). He motivated his analysis by exploring the more general notion of deliberational dynamics. He thinks of an arbitrary mixed strategy for each player in a \(2 \times 2\) person game as representing a state of indecision. Players are assumed to share common knowledge about the game’s structure. They are presumed able to calculate alternative actions according to a simple, qualitatively Bayesian, dynamical rule.\(^6\) It is important to note he does not assume players compute their best responses to the given strategy profiles. Rather, he assumes each player seeks the good. By this, he means a player uses two criteria to modify the current mixed strategy vector, called the status quo, given the other player’s mixed strategy. The first criterion states the player’s adjustment must raise the probability of a particular pure strategy only if its expected utility is greater than the status

\(^5\)This contrasts to Nash’s [1950b] first published existence proof for noncooperative games which mapped mixed strategies to best replies and relied on Kakutani’s Fixed Point Theorem. Also see Nash’s interview with Ron Howard from the DVD extras of A Beautiful Mind as well as Leonard [1994]

\(^6\)See Skyrms ([1990], p. 30).
quo payoff. The second criterion is the adjustment rule raises the sum of the probabilities of all pure strategies with payoffs exceeding the status quo’s expected utility. The Nash better response map satisfies these two conditions.\textsuperscript{7}

Skyrms goes on to study the evolution of each player’s mixed strategy vector for different specifications of the adjustment dynamics. He examines the Nash dynamics as well as the replicator, or \textit{Darwinian dynamics}, and some explicitly Bayesian adjustment processes.

The idea of using an iterative method to find a solution of a game goes back at least to the idea of fictitious play in discrete zero-sum games that originates with Brown [1951]. According to that procedure, a player after observing the sequence of strategies used by the rival player, computes the empirical mixed strategy and \textit{plays the best response} against the empirical mixed strategy. Robinson [1951] shows that the payoff of the players from such fictitious play will converge in the limit to the value of the two-person, zero-sum games. We note here that the result is given only for two-person zero-sum games and shows the convergence of the payoffs and \textit{not} of the strategies.\textsuperscript{8}

Evolutionary game theorists focused on the continuous time Brown–von Neumann–Nash (BNN) dynamics have proposed it as alternative to the replicator and best reply dynamics. Berger and Hofbauer (2005) note the BNN dynamics “picks out the attractive feature of both the replicator and best reply dynamics.” In particular, the BNN dynamics are \textit{innovative}.\textsuperscript{9} This means that it is possible for a pure strategy, played with zero probability in the dynamics’ initial condition, to achieve a positive probability of play at some later date. A strategy initially played with positive probability can also go extinct at a future date. The discrete time Nash map shares these properties with the continuous time BNN system.

Another interest in the Nash map’s dynamics is it links to the “as if” rationality hypothesis associated with Weibull (1994). This hypothesis is

\textsuperscript{7}We stress that deliberational dynamics rests on assuming players’ preferences are represented by an expected utility function. The generalized Nash map constructed in Becker and Chakrabarti [2005] can be reinterpreted as setting up a deliberational dynamic when players evaluate payoffs by nonexpected utility criteria.


\textsuperscript{9}See Weibull’s comments in Kuhn, et al (1996).
supported if the BNN dynamics asymptotically lead to rational play at a game’s equilibrium point(s). Our results for Matching Pennies are negative as applied to the discrete time Nash map. This may not be surprising.\textsuperscript{10} We note that the non-trivial periodic orbit found here has the property that at each point in that orbit, both pure strategies are played with positive probability (in contrast to the best response map’s fixed point of period four consisting of each pair of the two players’ pure strategies).

A natural problem concerns the stability properties of each deliberational dynamical system in different games. We have already noted there is a substantial literature on replicator dynamics. There is substantially less literature on the properties of games evolving according to the Nash dynamics.\textsuperscript{11} Skyrms [1990] presents examples several examples of Nash dynamics in discrete time using phase diagrams and computer simulations.\textsuperscript{12} In particular, he explores Nash dynamics for the familiar zero-sum game Matching Pennies.\textsuperscript{13} Each player has the pure strategies Heads and Tails. Player 1 (the Row player in a payoff matrix) wins a penny when the players match their strategic choices, and loses a penny otherwise. Player 2, the Column player, loses when Row wins and vice versa. This game is well-known to have a unique equilibrium point in mixed strategies where each player uses the pure strategy Heads with probability one-half. The Nash dynamics evolve on the unit square, $[0, 1] \times [0, 1] := [0, 1]^2$ where each coordinate represents the probability a player chooses the pure strategy Heads. Skyrms infers from his simulation of the Nash dynamics for Matching Pennies that the equilibrium

\textsuperscript{10}Hofbauer ([2000], p. 83) asserts that the Nash map can yield chaotic dynamics with two pure strategies and argues that the discrete time dynamics might be more reasonably replaced by better behaved continuous time dynamics. However, Hart and Mas-Collel (2003) observe that the lack of general results on convergence of dynamics (such as BNN) has more to do with the uncoupled nature of the system than whether it is modeled in discrete or continuous time.

\textsuperscript{11}See Weibull’s discussion in Kuhn, et al [1996]. He refers to the Nash dynamics as an example of an innovative dynamical system in continuous time. However, he limits his discussion to noting that this system’s stationary equilibrium point(s) correspond to Nash equilibrium point(s) in a given game. Also, see Hofbauer [2000]. There has been some work on dynamics with better response sets in continuous time setups: for example, see Demichelis and Ritberger [2003] and Ritzberger and Weibull [1995].

\textsuperscript{12}Skyrms presents continuous time and discrete time Nash dynamic adjustment processes. However, his computer code is developed for the discrete time system only, and not as a numerical approximation for a continuous time dynamical system.

\textsuperscript{13}Skyrms specifies this game in a strategically equivalent non-zero sum format. It is clearly best response equivalent to the standard zero-sum setup we use below.
point is strongly stable, meaning the unit square is the basin of attraction. He plots a simulated trajectory, initiated at a non-equilibrium pair of mixed strategies for each player.\textsuperscript{14} The graph shows a spiral trajectory. He interprets it to mean that the equilibrium point is being approached asymptotically.

Given that $R_x$ denotes the payoff matrix of player X or the Row player and $R_y$ denotes the payoff matrix of player Y or the column player, if $x$ is the distribution of the strategies of the population of Row players and $y$ is the distribution of strategies of the Column players then the “average” payoff of player X is $xR_xy^T$ and the “average” payoff of payer Y is $xR_yy^T$. Given $xR_xy^T$, player X plays a strategy $x^1R_xy^T$ such that

$$x_1R_xy^T > xR_xy^T$$

and given $xR_xy^T$, player X plays a strategy $y_1$ such that

$$xR_yy_1^T > xR_yy^T.$$  

Such a process of revision of the strategies can continue until the players reach a distribution of strategies such that it is no longer possible to “improve” ones payoff given the distribution of the strategies of the other type of players. Such a distribution of strategies is clearly an equilibrium distribution of strategies. It is of some interest to ask whether the iterative dynamics of the better response maps as described here will lead to convergence to such an equilibrium distribution of strategies.

The object of our paper is to prove the discrete time Nash dynamics for Matching Pennies is unstable. The equilibrium point is a repeller. Moreover, we show there is an orbit of period eight which attracts every point in the unit square except the equilibrium point. Starting the dynamics from an arbitrary initial condition, no matter how close to the equilibrium point, leads rational deliberating players away from the equilibrium point. Nash’s map, and the deliberational dynamics constructed from it, need not be (asymptotically) stable.\textsuperscript{15} The Nash map is formally defined and its properties for Matching Pennies are:

\textsuperscript{14}See Skyrms ([1990], Figure 3.2, p. 65.

\textsuperscript{15}There are games for which the discrete time Nash (or, deliberational) dynamics yield the game’s equilibrium point as the limit obtained by iterating the map from any initial condition in the unit square. Consider Prisoner’s Dilemma where the two pure strategies are called Defect and Cooperate. It is easy to define the Nash map, show each player’s component of the map is independent of the other player’s mixed strategy, and the probability of playing the strategy Defect increases over time, and converges to one. Of course, this follows because Defect is a dominant strategy.
Pennies are stated in Section 2. Proofs are given in Section 3 and concluding comments appear in Section 4.

2 Nashing Pennies

Here we find that even in games with a unique mixed strategy equilibrium, the equilibrium point may be not stable in the sense that for any initial condition (other than that equilibrium point), the system never converges to the equilibrium. This is best seen in the context of the game of Matching Pennies, which we analyze in detail. Let \( X \) and \( Y \) be two players with strategies denoted by the row vectors \( \bar{x} = (x, 1 - x) \) and \( \bar{y} = (y, 1 - y) \), \( 0 \leq x, y \leq 1 \), respectively. Matching Pennies has payoff matrices \( R_x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \) and \( R_y = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \) for \( X \) and \( Y \), where \( X \) wins when the pennies match and \( Y \) wins when they are opposite. Define

\[
\begin{align*}
t_x &= x + \max \{0, (e_1 - \bar{x}) R_x \bar{y}^T\}, \\
t_{1-x} &= (1 - x) + \max \{0, (e_2 - \bar{x}) R_x \bar{y}^T\}, \\
t_y &= y + \max \{0, \bar{x} R_y (e_1 - \bar{y})^T\}, \\
t_{1-y} &= (1 - y) + \max \{0, \bar{x} R_y (e_2 - \bar{y})^T\},
\end{align*}
\]

with \( e_1, e_2 \) the standard basis vectors.

The \textit{Nash map} on the pair of probability vectors is defined to be

\[
(\bar{x}, \bar{y}) \mapsto \left( \frac{t_x}{t_x + t_{1-x}}, \frac{t_{1-x}}{t_x + t_{1-x}} \right), \left( \frac{t_y}{t_y + t_{1-y}}, \frac{t_{1-y}}{t_y + t_{1-y}} \right).
\]

All information is contained in the \textit{Essential Nash map} on the unit square

\[
n = (n_1, n_2) : [0, 1]^2 \to [0, 1]^2,
\]

defined by

\[
n_1(x, y) = \frac{t_x}{t_x + t_{1-x}} = \frac{x + \max \{0, (e_1 - \bar{x}) R_x \bar{y}^T\}}{1 + \max \{0, (e_1 - \bar{x}) R_x \bar{y}^T\} + \max \{0, (e_2 - \bar{x}) R_x \bar{y}^T\}},
\]

\[
n_2(x, y) = \frac{t_y}{t_y + t_{1-y}} = \frac{y + \max \{0, \bar{x} R_y (e_1 - \bar{y})^T\}}{1 + \max \{0, \bar{x} R_y (e_1 - \bar{y})^T\} + \max \{0, \bar{x} R_y (e_2 - \bar{y})^T\}}.
\]
with \( \bar{x} \) and \( \bar{y} \) as before. It is clear from the definition that in this setting the Essential Nash map is a continuous map of the unit square into itself.

It is well-known that there is one Nash equilibrium point for the Matching Pennies game. It is the mixed strategy where both players play heads with probability \( 1/2 \).

First we set some terminology (see Figure 1). There is the center, eight regions and eight borders between adjacent regions. We refer to the regions by points on the compass. They are the north-northwest NNW, east-northeast ENE and so on. We also speak of clockwise neighbors. It means the ENE region is the clockwise neighbor of the NNE region, etc. We will also distinguish four quadrants (NE, SE, SW and NW) and four hemisquares (northern, southern, eastern and western).

In particular, the boundary (in the mathematical sense) of each region is the union of 3 or 4 smooth curves (some of them are straight line segments); two of them are borders. We will refer to those curves as northern, southern, eastern and western boundaries of the region. For instance, the ENE region has all 4 boundaries, while the NNE region has only northern, eastern and western boundaries.

The borders, starting with the border between NNW and NNE and proceeding in the clockwise direction, are given by the equations:

\[
\begin{align*}
    x &= 1/2, \\
    1/2 &\leq y \leq 1; \quad &x &= 1 - 1/(4y), \\
    1/2 &\leq y \leq 1; \quad &y &= 1/(4x), \\
    1/2 &\leq x \leq 1; \quad &x &= 1/2, \\
    0 &\leq y \leq 1/2; \quad &x &= 1/(4 - 4y), \\
    1/2 &\leq y \leq 1; \quad &y &= 1/2, \\
    0 &\leq x \leq 1/2; \quad &y &= 1 - 1/(4 - 4x), \\
    1/2 &\leq x \leq 1. 
\end{align*}
\]

Define the set \( \Omega = \cap_{k \geq 0} n^k([0, 1]^2) \) to be the infinite intersection of the images of the unit square under the iterates of the Essential Nash map. It is pictured in Figure 2. Iterates of the map are defined in the usual way:

\[
    n^k(x, y) = n(n^{k-1}(x, y)); \quad n^0(x, y) = (x, y).
\]

Now we list the properties of the Essential Nash map and the set \( \Omega \) as a theorem.

**Theorem 1**  
1. The Essential Nash map \( n \) is a continuous map of the unit square into itself. It is a homeomorphism from the unit square onto its image.

2. Each region is mapped into its clockwise neighbor by \( n \), homeomorphically onto its image. For instance, NNW goes to NNE. Each region is mapped into itself by \( n^8 \).
Figure 1: Eight regions and periodic points

Figure 2: Intersection of the images
3. Each border is mapped into its clockwise neighbor by \( n \), homeomorphically onto its image. For instance, the border between ENE and ESE goes into the border between ESE and SSE. Each border is mapped into itself by \( n^8 \).

4. The center is \((1/2, 1/2)\). It is the unique fixed point for the Nash map and it is repelling.

5. There is an orbit \( P \) of period 8 with one point on each of the borders. The orbit is \((1/2, 3/4), (2/3, 3/4), (3/4, 1/2), (3/4, 1/3), (1/2, 1/4), (1/3, 1/4), (1/4, 1/2), (1/4, 2/3)\).

6. All points, except \((1/2, 1/2)\), on a border between two regions are attracted under iterations of \( n^8 \) to the point of \( P \) contained in this border.

7. Under the iterations of \( n^8 \), every point in a region, except those on its counterclockwise boundary, is attracted to the point of \( P \) contained in its clockwise boundary. For instance, all points in the WNW region except those on its southern boundary are attracted to the point \((1/4, 2/3)\) under iterations of \( n^8 \).

8. Every point in the square except for \((1/2, 1/2)\) is attracted to the orbit \( P \) under the iterations of \( n^8 \).

9. The fixed point and the points of \( P \) are the only periodic points.

10. The set \( \Omega \) is a compact, connected set and contains \((1/2, 1/2)\) in its interior. The points of \( P \) are on its boundary. The map \( n \) restricted to \( \Omega \) is a homeomorphism.

11. The set \( \Omega \) is the union of the unstable manifolds of the points of \( P \) and of the point \((1/2, 1/2)\). Its interior is the unstable manifold of \((1/2, 1/2)\) and its boundary is the union of the unstable manifolds of the points of \( P \).

3 Proofs

To study the dynamics of the map \( n \) we translate the unit square to a unit square centered at the origin. This results in the centered Essential Nash
map. It is \( \eta = (\eta_1, \eta_2) : [-1/2, 1/2]^2 \to [-1/2, 1/2]^2 \), given by

\[
\eta(x, y) = n \left( x + \frac{1}{2}, y + \frac{1}{2} \right) - \left( \frac{1}{2}, \frac{1}{2} \right).
\]

The formulas for \( \eta \) are

\[
\eta_1(x, y) = \frac{x + (1 - 2x) \max\{0, y\} - (1 + 2x) \max\{0, -y\}}{1 + 2(1 - 2x) \max\{0, y\} + 2(1 + 2x) \max\{0, -y\}},
\]

\[
\eta_2(x, y) = \frac{y + (1 - 2y) \max\{0, -x\} - (1 + 2y) \max\{0, x\}}{1 + 2(1 - 2y) \max\{0, -x\} + 2(1 + 2y) \max\{0, x\}}.
\]

We will prove properties 1-11 in this new setup.

We will keep for \([-1/2, 1/2]^2 \) and \( \eta \) the terminology we introduced for \([0, 1]^2 \) and \( n \).

Observe that the terms which appear in \( \eta_1 \) depend only on the sign of \( y \) and not on the sign of \( x \). Thus, the formulas for \( \eta_1 \) on the northern and southern hemisquares are

\[
N\eta_1(x, y) = \frac{x + y - 2xy}{1 + 2y - 4xy},
\]

\[
S\eta_1(x, y) = \frac{x + y + 2xy}{1 - 2y - 4xy},
\]

respectively. Those formulas agree on the equator \( y = 0 \). Similarly, the formulas for \( \eta_2 \) on the eastern and western hemisquares are

\[
E\eta_2(x, y) = \frac{y - x - 2xy}{1 + 2x + 4xy},
\]

\[
W\eta_2(x, y) = \frac{y - x + 2xy}{1 - 2x + 4xy},
\]

respectively, and they agree on the meridian \( x = 0 \). We will also use the notation \( NE\eta = (N\eta_1, E\eta_2) \), etc.

Let \( \theta : [-1/2, 1/2]^2 \to [-1/2, 1/2]^2 \) denote the (counterclockwise) rotation by \( \pi/2 \), that is, \( (x, y) \mapsto (-y, x) \). Observe that it commutes with the map \( \eta \), that is, \( \theta \circ \eta = \eta \circ \theta \). This follows from computations showing that

\[
S\eta_1(x, y) = -N\eta_1(-x, -y),
\]

\[
W\eta_2(x, y) = -E\eta_2(-x, -y),
\]

\[
E\eta_2(x, y) = -N\eta_1(-y, x).
\]
This observation allows us to prove some properties of \( \eta \) only for one quadrant, and the same properties for other quadrants will follow automatically.

**Proof of property 3.** In the new setup, the western boundary of NNE is given by \( x = 0, 0 \leq y \leq 1/2 \). We have \( NE\eta(0,y) = (y/(1+2y), y) \). Therefore the image under \( \eta \) of the western boundary of NNE is the eastern boundary of NNE, which is given by \( x = y/(1+2y), 0 \leq y \leq 1/2 \), and \( \eta \) on the western boundary of NNE is a homeomorphism.

We have \( NE\eta(y/(1+2y), y) = (2y/(1+4y), 0) \). Therefore the image under \( \eta \) of the eastern boundary of NNE is a part of the southern boundary of ENE, which is given by \( y = 0, 0 \leq y \leq 1/2 \), and \( \eta \) on the eastern boundary of NNE is a homeomorphism.

The rest of property 3 follows directly from what we proved and from the fact that \( \eta \) commutes with \( \theta \).

Before proceeding further, we state a useful lemma.

**Lemma 1** Suppose that \( C \) is a Jordan curve in the plane and \( D \) is the closure of the region bounded by \( C \). Let \( f \) be a map from \( D \) into \( \mathbb{R}^2 \), which is a local homeomorphism on the interior of \( D \) and a homeomorphism from \( C \) onto its image. Then:

1. \( f(C) \) is a Jordan curve and \( f \) of the interior of \( D \) is the region bounded by \( f(C) \);

2. \( f \) is a homeomorphism of \( D \) onto its image.

**Proof.** The first statement is an elementary exercise in point set topology. The second statement was proved by Derrick [1973].

**Proof of property 2.** We already know where the western and southern boundaries of the NE quadrant are mapped. Let us look at the northern and eastern boundaries. The northern one is \( y = 1/2, 0 \leq x \leq 1/2 \). We have \( NE\eta(x, 1/2) = (1/(4-4x), (1-4x)/(2+8x)) \). Therefore the image under \( \eta \) of the northern boundary of the NE quadrant is the curve \( y = (1-3x)/(10x-2), 1/4 \leq x \leq 1/2 \), and \( \eta \) on the northern boundary of the NE quadrant is a homeomorphism. Moreover, this image is contained in our square.

The eastern boundary of the NE quadrant is \( x = 1/2, 0 \leq y \leq 1/2 \). We have \( NE\eta(1/2, y) = (1/2, -1/(4+4y)) \). Therefore the image under \( \eta \) of the
eastern boundary of the NE quadrant is the straight line segment \( x = 1/2, \)
\(-1/4 \leq y \leq -1/6, \) and \( \eta \) on the eastern boundary of the NE quadrant is a homeomorphism. Moreover, this image is contained in our square.

This proves that \( \eta \) is a homeomorphism from the boundary of the NE quadrant onto its image and that this image is contained in our square.

Next we compute the derivative of \( NE\eta \), and we get the following formulas.

\[
\frac{\partial}{\partial x} N\eta_1(x, y) = \frac{1}{(1 + 2y - 4xy)^2}, \\
\frac{\partial}{\partial y} N\eta_1(x, y) = \frac{(1 - 2x)^2}{(1 + 2y - 4xy)^2}, \\
\frac{\partial}{\partial x} E\eta_2(x, y) = \frac{- (1 + 2y)^2}{(1 + 2x + 4xy)^2}, \\
\frac{\partial}{\partial y} E\eta_2(x, y) = \frac{1}{(1 + 2x + 4xy)^2}.
\]

From these equations it follows that the Jacobian of \( \eta \) is strictly positive on the closed NE quadrant. Therefore \( \eta \) is a local homeomorphism there. Now we apply Lemma 1 to conclude that \( \eta \) is a homeomorphism from the NE quadrant onto its image and that the image is contained in our square. The image of the NE quadrant under \( \eta \) is shown on Figure 3.

We know the images of the boundaries of the NE quadrant. In the proof of property 3 we computed the image of the border between the NNE and ENE sectors. Using these computations and the fact that \( \eta \) is a homeomorphism from the NE quadrant onto its image we conclude that property 2 holds for the NNE and ENE sectors. The rest of property 2 follows directly from what we proved and from the fact that \( \eta \) commutes with \( \theta \).

**Proof of property 1.** The map \( \eta \) is continuous, because the formulas that define it give continuous functions that agree on common boundaries. The second claim of property 1 follows directly from property 2.

**Proof of property 4.** We know that \((0, 0)\) is a fixed point for \( \eta \). In the proof of property 2 we computed the four partial derivatives of \( \eta \) in the NE quadrant. From this we see that the derivative of \( NE\eta \) at \((0, 0)\) is
\[
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
\]
which is rotation by \(-\pi/4\) composed with multiplication by \(\sqrt{2}\).
The derivative at \((0,0)\) of each of the other three quadrant maps is identical, because \(\eta\) commutes with \(\theta\). It follows that \((0,0)\) is a repelling fixed point. It is unique by property 2. 

\[\square\]

**Proof of property 5.** This follows by computing the orbit of \((0,1/4)\) for \(\eta\). 

\[\square\]

At this point we introduce a new map. Let \(\Lambda\) be the closed region NNE. We use property 2 and the fact that \(\eta\) commutes with rotation by \(\pi/2\) to define a map \(\zeta = (\zeta_1, \zeta_2) : \Lambda \rightarrow \Lambda\) by \(\zeta = \theta \circ \eta^2\). By property 1, \(\zeta\) is a homeomorphism onto its image. The important point is that on \(\Lambda\), \(\zeta^4 = \eta^8\). The formulas for \(\zeta\) are

\[
\begin{align*}
\zeta_1(x, y) &= \frac{2x(1 + 2y + 2y^2 - 2xy - 4xy^2)}{1 + 4x + 4y + 4y^2 + 4xy - 8x^2y - 16x^2y^2}, \\
\zeta_2(x, y) &= \frac{2(y + 2x^2 - 2xy + 2xy^2 + 2x^2y - 4x^2y^2)}{1 + 4y + 4x^2 - 4xy + 8xy^2 - 16x^2y^2}.
\end{align*}
\]

**Proof of property 6.** By property 3, \(\zeta\) maps the western boundary of \(\Lambda\) (that is, the segment \(x = 0, 0 \leq y \leq 1/2\)) into itself. On this boundary we
have
\[ \zeta_2(0, y) = \frac{2y}{1 + 4y}. \]
Since \( 2y/(1+4y) - y = y(1-4y)/(1+4y) \), we have \( \zeta_2(0, y) > y \) if \( 0 < y < 1/4 \) and \( \zeta_2(0, y) < y \) if \( 1/4 < y \leq 1 \). Therefore, all points on the western boundary of \( \Lambda \) are attracted to the point \((0, 1/4)\) under the iterates of \( \zeta \).

Since \( \zeta^4 = \eta^8 \), the same is true for the iterates of \( \eta^8 \).

From this, property 2, and the fact that \( \eta^8 \) commutes with \( \eta \) it follows that the analogous statement is true also for all other 7 borders between adjacent regions. \( \Box \)

**Proof of property 7.** First we examine the region \( \Lambda_1 \subset \Lambda \) where \( 0 < x < 1/6 \) and \( 0 < y \leq 1/4 \). We claim that in this region \( \zeta_1(x, y) - x > 0 \). We know that \( \zeta_1(x, y) - x = 0 \) in \( \Lambda \) when \( x = 0 \) and that it is also zero at \((1/6, 1/4)\). In \( \Lambda \) the denominator of the formula for \( \zeta_1 \) is positive, so when \( x > 0 \), the inequality \( \zeta_1(x, y) - x > 0 \) is equivalent to \( F_1(x, y) > 0 \), where
\[ F_1(x, y) = 1 - 4x - 8xy - 8xy^2 + 8x^2y + 16x^2y^2. \]
We have
\[ \frac{\partial F_1}{\partial y}(x, y) = 16xy(2x - 1) + 8x(x - 1) < 0. \]
Therefore in order to show that \( F_1(x, y) > 0 \) in \( \Lambda_1 \) it is enough to show it on the northern boundary of \( \Lambda_1 \), that is, the straight line segment joining \((0, 1/4)\) with \((1/6, 1/4)\). On this segment we have \( F_1(x, 1/4) = 3(x-2)(x-1/6) > 0 \). This proves our claim.

Next we examine the region \( \Lambda_2 \subset \Lambda \) where \( 0 \leq x \leq 1/4 \) and \( 1/4 \leq y \leq 1/2 \). We claim that in this region \( \zeta_2(x, y) - y < 0 \) except at the points \((0, 1/4)\) and \((1/6, 1/4)\). This is equivalent to \( F_2(x, y) < 0 \), where
\[ F_2(x, y) = y - 4y^2 + 4x^2 - 4xy + 8xy^2 - 8x^2y^2 - 8xy^3 + 16x^2y^3. \]
We have
\[ \frac{\partial F_2}{\partial y}(x, y) = 1 - 8y - 4x + 16xy - 16x^2y - 24xy^2 - 48x^2y^2 \leq (1 - 8y) + (16xy - 24xy^2). \]
In \( \Lambda_2 \) we have \( 1 - 8y \leq -1 \) and
\[ 16xy - 24xy^2 = 8x \left( \frac{1}{3} - 3 \left( y - \frac{1}{3} \right)^2 \right) \leq \frac{2}{3}. \]
Therefore $\partial F_2/\partial y \leq -1/3$, so in order to show that $F_2(x, y) < 0$ in $\Lambda_2$, it is enough to show it on the southern boundary of $\Lambda_2$. This boundary consists of the segment $y = 1/4$ from $(0, 1/4)$ to $(1/6, 1/4)$ and the part of the eastern boundary of $\Lambda$ between $(1/6, 1/4)$ and $(1/4, 1/2)$. On the segment where $y = 1/4$ we get $F_2(x, 1/4) = (15/4)x(x - 1/6)$, so $F_2(x, 1/4) < 0$ except at the points $(0, 1/4)$ and $(1/6, 1/4)$. By properties 3 and 6, the second curve is mapped into itself by $\zeta$ and all its points are attracted to $(1/6, 1/4)$. Therefore $\zeta_2(x, y) - y < 0$ (so $F_2(x, y) < 0$) on this curve. This proves our claim.

Consider the closure of $\Lambda_1$. The map $\zeta$ restricted to either the eastern or western boundaries of this set is a homeomorphism of the boundary to itself. The image of the northern boundary of $\Lambda_1$ under $\zeta$ is a curve connecting the fixed endpoints $(0, 1/4)$ and $(1/6, 1/4)$. By the observation concerning $\zeta_2(x, y) - y$, we see that for all $(x, 1/4)$ with $0 < x < 1/6$ we have $\zeta_2(x, 1/4) < 1/4$. Since $\zeta$ is a homeomorphism onto its image it means that $\Lambda_1$ is mapped into itself by $\zeta$.

If $(x, y) \in \Lambda_1$, with $x > 0$ then it stays there under all iterations of $\zeta$. Since each iterate of $\zeta$ increases the $x$ coordinate, its trajectory converges to $(1/6, 1/4)$.

If $(x, y) \in \Lambda_2$ and $x > 0$, there are two possibilities for its trajectory. The first is that its second coordinate is always greater than $1/4$ and it converges to $(1/6, 1/4)$ and the second is that it goes into $\Lambda_1$ and then converges to $(1/6, 1/4)$. Suppose this is not the case. Then the trajectory of $(x, y)$ must have an accumulation point on the $y = 1/4$ boundary of $\Lambda_1$. If the accumulation point has its first coordinate greater than zero it must be $(1/6, 1/4)$ because $\zeta$ maps all other points on the boundary into the interior of $\Lambda_1$. The remaining possibility is that the trajectory converges to $(0, 1/4)$. However, $F_1(x, y)$ is close to 1 for small positive $x$, so for all points of $\Lambda$ close to $(0, 1/4)$ with the $x$ coordinate positive, that coordinate increases under iterates of $\zeta$. This excludes $(0, 1/4)$ as a limit point of such trajectory.

Thus, property 7 holds for the NNE region. From this, property 2, and the fact that $\eta^8$ commutes with $\eta$ it follows that the analogous statement is true also for all other 7 regions. \hfill $\blacksquare$

**Proof of properties 8 and 9.** These are immediate consequences of property 7. \hfill $\blacksquare$

**Proof of property 10.** Let $\Omega_c = \bigcap_{k \geq 0} \eta^k([-1/2, 1/2]^2)$. It is compact
and connected by definition. Since \( \eta \) is a homeomorphism onto its image and \( \eta(\Omega_c) = \Omega_c \) we see that \( \eta \) restricted to \( \Omega_c \) is a homeomorphism. By property 4 the point \((1/2, 1/2)\) is in the interior of \( \Omega_c \) and by property 7 the points of period eight are on its boundary. \(\square\)

**Proof of property 11.** For this we consider \( \zeta : \Lambda \to \Lambda \). Let \( \Omega'_c = \bigcap_{k \geq 0} \zeta^k(\Lambda) \), which is a compact, connected set and \( \zeta \) restricted to it is a homeomorphism.

Let \( \Lambda_1 \) and \( \Lambda_2 \) be defined as in the proof of property 7. Note that \( \overline{\Lambda_1} \) and \( \Lambda_2 \) are compact sets, \( \Lambda_1 \cup \Lambda_2 = \Lambda \) and \( \Lambda_1 \cap \Lambda_2 \) is the line segment \( \{(x, 1/4) : 0 \leq x \leq 1/6\} \).

We examine the image of \( \overline{\Lambda_1} \) under \( \zeta \). Since \( \zeta \) is a homeomorphism onto its image, we only need to understand the image of the boundary of \( \overline{\Lambda_1} \).

The set \( \overline{\Lambda_1} \) has three boundary pieces, eastern, western and northern. In the proof of property 6 we saw that \( \zeta \) maps the eastern and western boundaries onto themselves, while fixing the three points \((0,0)\), \((0,1/4)\) and \((1/6,1/4)\). In the proof of property 7 we saw that the image of the northern boundary, with the exception of the endpoints, is mapped into \( \Lambda_1 \). Consequently, the boundary of \( \overline{\Lambda_1} \) is mapped into \( \overline{\Lambda_1} \), so by Lemma 1, \( \zeta(\overline{\Lambda_1}) \subset \overline{\Lambda_1} \).

We claim that \( \Omega'_c \subset \overline{\Lambda_1} \). To prove this, observe that a point \( p \) belongs to \( \Omega'_c \) if and only if \( \zeta^{-n}(p) \) is defined for all \( n \geq 0 \). If \( p \in \Lambda \setminus \overline{\Lambda_1} \) then, since \( \zeta(\overline{\Lambda_1}) \subset \overline{\Lambda_1} \), either \( \zeta^{-1}(p) \in \Lambda \setminus \overline{\Lambda_1} \subset \Lambda_2 \), or \( \zeta^{-1}(p) \) does not exist. In the former case, in view of what we showed in the proof of property 7, the \( y \)-coordinate of \( \zeta^{-1}(p) \) is larger than the \( y \)-coordinate of \( p \). Therefore, if the points \( \zeta^{-n}(p) \) exist for all \( n \geq 0 \), their \( y \)-coordinates increase as \( n \) increases. The sequence \( (\zeta^{-n}(p))_{n=0}^{\infty} \) has an accumulation point \( q \) in \( \Lambda \setminus \overline{\Lambda_1} \), but this contradicts the fact that the \( y \)-coordinate of \( \zeta^{-1}(q) \) is larger than the \( y \)-coordinate of \( q \). This proves our claim.

The unstable manifolds (for \( \zeta \)) \( W^u(0,0) \) and \( W^u(0,1/4) \) of the fixed points \((0,0)\) and \((0,1/4)\) are the sets of points \( p \in \Lambda \) for which the sequence \( (\zeta^{-n}(p))_{n=0}^{\infty} \) converges to \((0,0)\) and \((0,1/4)\) respectively. For such \( p \), \( \zeta^{-n}(p) \) exists for every \( n \geq 0 \), so both unstable manifolds are contained in \( \Omega'_c \). The fixed point \((1/4,1/6)\) also belongs to \( \Omega'_c \). We will prove this is all, that is, that \( \Omega'_c = K \), where \( K = W^u(0,0) \cup W^u(0,1/4) \cup \{(1/4,1/6)\} \).

Take a point \( p \in \Omega'_c \). Then either \( p = (1/4,1/6) \), or the \( x \)-coordinate of \( p \) is 0, or \( p \in \Lambda_1 \). In the first case \( p \in K \). In the second case either \( p = (0,1/4) \) and then \( p \in W^u(0,1/4) \subset K \), or \( p = (0,t) \) for some \( t \in [0,1/4) \) and then by what we showed in the proof of property 6, the \( y \)-coordinate
of $\zeta^{-n}(p)$ decreases as $n$ increases, so $p \in W^u(0,0) \subset K$. If $p \in \Lambda_1$, then in view of what we showed in the proof of property 7, the $x$-coordinate of $\zeta^{-1}(p)$ is smaller than the $x$-coordinate of $p$. The same applies to $\zeta^{-n}(p)$ replacing $p$. Thus, the sequence $(\zeta^{-n}(p))_{n=0}^{\infty}$ has an accumulation point $q$ on the western boundary of $\Lambda_1$. The unstable manifold $W^u(0,0)$ for $\zeta$ is an open set and it contains the whole western boundary of $\Lambda_1$ except $(0,1/4)$. Therefore, if $q \neq (0,1/4)$ then one of the points $\zeta^{-n}(p)$ belongs to $W^u(0,0)$ and hence also $p \in W^u(0,0) \subset K$. The remaining possibility is that the sequence $(\zeta^{-n}(p))_{n=0}^{\infty}$ converges to $(0,1/4)$, and then $p \in W^u(0,1/4) \subset K$. This proves that $\Omega'_c = K$.

It remains to prove that $W^u(0,1/4)$ is the northern boundary of $W^u(0,0)$. Since those sets are invariant for $\zeta$, it is enough to show it close to the fixed point $(0,1/4)$. The derivative of $\zeta$ at $(0,1/4)$ is 
\[
\begin{bmatrix}
13/9 & 0 \\
-5/16 & 1/2
\end{bmatrix}.
\]
Thus, $(0,1/4)$ is a saddle with eigenvalues $1/2$ and $13/9$ and corresponding eigenvectors $(0,1)$ and $(1,-45/136)$. There is a neighborhood $U$ of $(0,1/4)$ where the map can be linearized, so there the notions “above $W^u(0,1/4)$” and “below $W^u(0,1/4)$” make sense. Fix a sufficiently small $\varepsilon > 0$. If a point $p$ is in $U$, above $W^u(0,1/4)$ and sufficiently close to $W^u(0,1/4)$, then there is $n \geq 0$ such that if $\zeta^{-n}(p) = (x,y)$ then $y > 1/4 + \varepsilon$ and $x$ is as small as we want. Then $\zeta^{-n}(p) \in \Lambda_2$, so $p \notin \Omega'_c$. If a point $p$ is in $U$, below $W^u(0,1/4)$ and sufficiently close to $W^u(0,1/4)$, then there is $n \geq 0$ such that if $\zeta^{-n}(p) = (x,y)$ then $y < 1/4 - \varepsilon$ and $x$ is as small as we want. Then, since the whole western boundary of $\Lambda_1$ except $(0,1/4)$ is contained in $W^u(0,0)$, and $W^u(0,0)$ is open, we get $\zeta^{-n}(p) \in \Omega'_c$, so $p \in \Omega'_c$. Therefore locally (and thus globally) $W^u(0,1/4)$ is the northern boundary of $W^u(0,0)$.

Property 11 follows from these observations, property 2, and the fact that $\eta^8$ commutes with $\eta$. $\square$

4 Conclusion

Here we have studied the convergence properties of the process generated by iterating the Nash map. Our reasons for doing so was to test the “mass action” interpretation of the Nash equilibrium as being the rest point of a dynamic process in which the players respond to the distribution of strategies of the players from the other populations. The players play a strategy that
is a good response to the entire distribution of the strategies of the rival players as they do not know with whom they are going to be matched in the next period. In judging the relevant distribution to which they should respond, the players use the most recently observed distribution of strategies of the players from the rival population. In a game-theoretic setting where players have complete discretion about what strategies to adopt this seems to be a reasonable choice, especially if the players have some bias towards the status-quo. We find here that in the game of Matching Pennies this process need not converge to the equilibrium point in the limit. Since the game of Matching Pennies has a unique equilibrium point our result shows that the process need not converge even if the equilibrium point is the unique equilibrium point of the game. Since some earlier discussion in the literature had suggested that such innovative dynamics converge to an equilibrium point, see for instanceSkyrms [1990] and Hofbauer and Sigmund [2003], our result here is significant because it shows that these observations have to be accepted with more caution.

The result here raises many other issues. In particular, it raises questions about the nature of these kind of innovative dynamics in other games. For instance, if the games have pure strategy equilibrium points then will that lead to different results and is it then true that the iterations of the Nash map asymptotically yields an equilibrium point. It would also be of interest to know how the convergence properties would change if one replaced the Nash map with a selection from the best responses. It is perhaps significant that in the game of Matching Pennies this would cause the play to swing from one pure strategy to the other unless the opponent is observed to be playing exactly the equilibrium mixed strategy.
References


