MICRODYNAMICS

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ABSTRACT. We describe a new type of the scaling phenomenon. It has some features similar to renormalization and some similar to intermittency. We call it microdynamics. In a one-parameter family of maps in dimension 2, when the parameter goes to 0, the maps converge to the identity. Nevertheless, after a linear rescaling of both space and time, we get maps with attracting invariant closed curves. As the parameter goes to 0, those curves converge in a strong sense to a certain circle.

1. INTRODUCTION

In the theory of dynamical systems, quite often various kinds of scaling are considered. The objects involved are phase space, parameter space and time. Each of those can be scaled.

The best known example of scaling is renormalization (see, e.g., [6],[7], [11], [12], [13], [17]). One looks at a small piece of the phase space and waits a long time until under the iterates of the map this piece returns to itself. Scaling both in space and in time is exponential. If we additionally look at a one-parameter family of maps (like quadratic maps of an interval), there is an additional possible exponential scaling of the parameter.

Another popular scaling phenomenon is intermittency (see, e.g., [5], [9], [14], [15], [16]). This one requires a parameter; just for one map there is nothing to say. Here we look at two pieces of the phase space, and wait until one of them goes under an iterate of the map to the other one. The only scaling is of time, relative to the parameter. This scaling is typically quadratic (time is of order of the square of the distance of the parameter from the limit parameter value).

Here we want to present a new type of scaling phenomenon. We call it *microdynamics*. We look at a small piece of the phase space, like in renormalization, but we do not wait until it returns. It is there all the time, so we wait until it rotates sufficiently under the iterates of of the map. As in intermittency, scaling is relative to the parameter. Thus, we scale both phase space and time, relative to the parameter. Both scalings are linear. An additional difference between microdynamics on one side and renormalization and intermittency on the other, is that while in the latter cases we get (at least locally) a limit map, in the former case it is only a limit type of dynamics.

The family of maps that we investigate comes from a model in economics and game theory. Each element of the family is a map of a square into itself, which is continuous but only piecewise smooth; it is a rational map in each quarter-square. As

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the parameter c goes to zero, the players become more and more cautious, so the map tends to the identity. However, by blowing up a neighborhood of the repelling fixed point by a factor of c and considering the $n(c)$-th iterate of the map, where $n(c)$ is of order $1/c$, we see an attracting simple closed curve that converges in a strong sense to a certain geometric circle, when $c \to 0$. Strictly speaking, taking the $n(c)$-th iterate of the map is not necessary, because the same curve is attracting for the map itself, but without it the attraction is practically invisible. Still, even with those rescalings, the attraction is weaker and weaker as $c \to 0$.

Note that the proof of the existence of an attracting curve that is a graph of a Lipschitz continuous map is more complicated here than in cases like Strange Nonchaotic Attractors (see, e.g., [1], [4], [8], [10]). The map is not a skew product and therefore subtle estimates of all entries of the matrix of the derivative are necessary.

2. Derivation of the family

We consider the Nash better response map for the game of Matching Pennies (see [2]). In this setting, X and Y are two players with strategies denoted by the row vectors $\bar{x} = (x, 1-x)$ and $\bar{y} = (y, 1-y)$, $0 \le x, y \le 1$. The payoff matrices for X and Y are $R_x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $R_y = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, respectively.

We take into account the *caution index* c. It varies from zero to infinity and indicates how cautious the players are, that is, how much they want to preserve the previous strategy versus how much they want to modify it. The value 0 for c means extreme caution, that is, no change in strategy is allowed. Very large values of c mean very large changes in strategy are possible. We will consider only strictly positive values of c and will concentrate on what happens when c is small.

Thus, we define

$$\begin{aligned}
t_x^{(c)} &= \quad x \quad + \; c \cdot \max\{0, (e_1 - \bar{x}) R_x \bar{y}^T\}, \\
t_{1-x}^{(c)} &= (1-x) + \; c \cdot \max\{0, (e_2 - \bar{x}) R_x \bar{y}^T\}, \\
t_y^{(c)} &= \quad y \quad + \; c \cdot \max\{0, \bar{x} R_y (e_1 - \bar{y})^T\}, \\
t_{1-y}^{(c)} &= (1-y) + \; c \cdot \max\{0, \bar{x} R_y (e_2 - \bar{y})^T\},
\end{aligned}$$

with e_1, e_2 the standard basis vectors.

The *Cautious Nash map* on the pair of probability vectors is defined to be

$$(\bar{x}, \bar{y}) \mapsto \left(\left(\frac{t_x^{(c)}}{t_x^{(c)} + t_{1-x}^{(c)}}, \frac{t_{1-x}^{(c)}}{t_x^{(c)} + t_{1-x}^{(c)}} \right), \left(\frac{t_y^{(c)}}{t_y^{(c)} + t_{1-y}^{(c)}}, \frac{t_{1-y}^{(c)}}{t_y^{(c)} + t_{1-y}^{(c)}} \right) \right).$$

All information about it is contained in the continuous map of the unit square to itself

$$n^{(c)} = (n_1^{(c)}, n_2^{(c)}) : [0,1]^2 \to [0,1]^2,$$

defined by

$$\begin{aligned} n_1^{(c)}(x,y) &= \frac{t_x^{(c)}}{t_x^{(c)}+t_{1-x}^{(c)}} \\ &= \frac{x+c\cdot\max\{0,(e_1-\bar{x})R_x\bar{y}^T\}}{1+c\cdot\max\{0,(e_1-\bar{x})R_x\bar{y}^T\}+c\cdot\max\{0,(e_2-\bar{x})R_x\bar{y}^T\}}, \end{aligned}$$

$$\begin{aligned} n_2^{(c)}(x,y) &= \frac{t_y^{(c)}}{t_y^{(c)}+t_{1-y}^{(c)}} \\ &= \frac{y+c\cdot\max\{0,\bar{x}R_y(e_1-\bar{y})^T\}}{1+c\cdot\max\{0,\bar{x}R_x(e_1-\bar{y})^T\}+c\cdot\max\{0,\bar{x}R_y(e_2-\bar{y})^T\}}, \end{aligned}$$

with $\bar{x}$ and $\bar{y}$ as before.

Now we translate the unit square to a unit square centered at the origin and get the map

$$\eta^{(c)} = (\eta_1^{(c)}, \eta_2^{(c)}) : [-1/2, 1/2]^2 \to [-1/2, 1/2]^2,$$

given by

$$\eta^{(c)}(x,y) = n^{(c)}\left(x+\frac{1}{2}, y+\frac{1}{2}\right) - \left(\frac{1}{2}, \frac{1}{2}\right).$$

The formulas for $\eta^{(c)}$ are

$$\eta_1^{(c)}(x,y) = \frac{x+c(1-2x)\max\{0,y\}-c(1+2x)\max\{0,-y\}}{1+2c(1-2x)\max\{0,y\}+2c(1+2x)\max\{0,-y\}},$$

$$\eta_2^{(c)}(x,y) = \frac{y+c(1-2y)\max\{0,-x\}-c(1+2y)\max\{0,x\}}{1+2c(1-2y)\max\{0,-x\}+2c(1+2y)\max\{0,x\}}.$$

Note that this map commutes with the counterclockwise rotation by $\pi/2$, which we will denote ζ.

We get the formula for $\eta^{(c)}$ in the North-East quadrant:

$$NE\eta^{(c)}(x,y) = \left(\frac{x+cy-2cxy}{1+2cy-4cxy}, \frac{y-cx-2cxy}{1+2cx+4cxy}\right).$$

Lemma 2.1. *The map $\eta^{(c)}$ maps the square $[-1/2, 1/2]$ homeomorphically into itself.*

Proof. The proof is practically the same as in [2] for the case $c = 1$ and we will not repeat the details here. The idea is to compute the images of the boundaries of the quadrants, show that the map is a homeomorphism on the boundary of the square, compute the Jacobian and show that it is positive everywhere, and use the result of Derrick saying that a planar map that is a homeomorphism on a Jordan curve and a local homeomorphism inside, is a homeomorphism of the set bounded by this curve onto the set bounded by the image of the curve. □

Moreover, it is easy to check that the origin is a repelling fixed point.

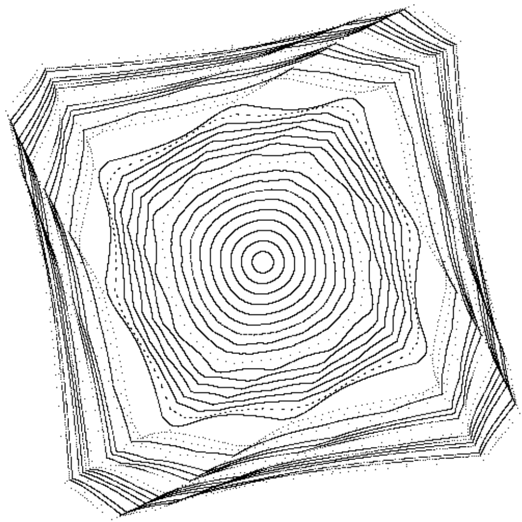

Figure 1. Attractors for various values of c; the phase space

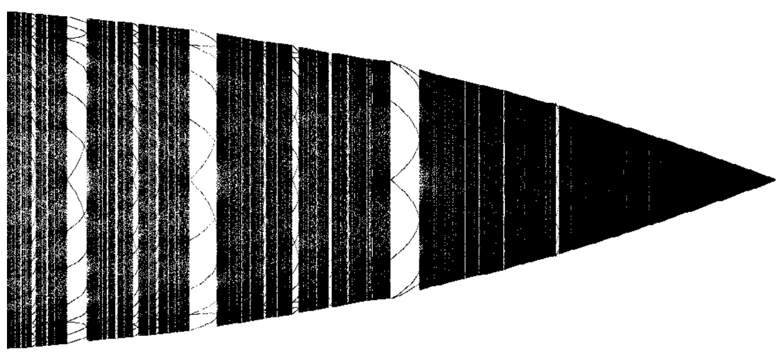

Figure 2. Dependence of the attractor on c

3. Computer experiments

Computer experiments suggest that for all values of c there exists an invariant Jordan curve that goes around the origin and the curve attracts the trajectories of all points except the origin. This curve appears to be mapped onto itself by an orientation preserving homeomorphism. The rotation number of this homeomorphism changes as c varies. If it is rational, there is an attracting periodic orbit on this curve and this periodic orbit attracts the trajectories of all points except the origin. Note that if $c = 1$ this has proved to be the case [2]; the rotation number is $-1/8$ and there is a periodic orbit of period 8 attracting the trajectories of all points except the origin.

Figure 1 shows the (minimal) attractors for various values of c. As c decreases, the size of the invariant curve also decreases and the shape of curve becomes more and more similar to a circle. Figure 2 shows the same picture, but in the c, x-plane (and more values of c are used). The c-axis is horizontal and c decreases as we move to the right. At the "tip" $c = 0$. Finally, Figure 3 shows the same picture as Figure 2,

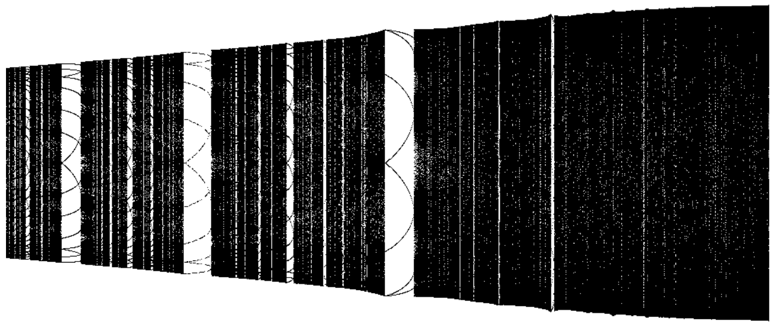

FIGURE 3. Dependence of the attractor on c; the vertical axis is rescaled

except that the x-axis is rescaled, so the vertical axis shows x/c rather than x.

Computations show that the limit circle, after the rescaling of the phase space (that is, using x/c and y/c instead of x and y) is centered at the origin and has radius slightly smaller than 0.3. And indeed, we will show later that this radius is $3\pi/32 \approx 0.294524311274043$.

4. Main estimates

We want to investigate the behavior of $\eta^{(c)}$ for small values of c (that is, when the players are very cautious). As we stated in the preceding section, numerical experiments suggest that $\eta^{(c)}$ has an attracting invariant simple closed curve around the origin and that the size of that curve is of order c. Therefore it makes sense to rescale the square $[-1/2, 1/2]^2$ by a factor of $1/c$ to make this curve of a similar size for all small c. This means that we have to conjugate $\eta^{(c)}$ by the homothety $(x, y) \mapsto (x/c, y/c)$. In such a way we get a map that we will denote F_c. In the first (Northeast) quadrant it is given by the formula

$$NEF_c(x,y) = \left(\frac{x+cy-2c^2xy}{1+2c^2y-4c^3xy}, \frac{y-cx-2c^2xy}{1+2c^2x+4c^3xy}\right). \tag{4.1}$$

Since F_c commutes with ζ, the formulas for other quadrants can be easily obtained from this one. By Lemma 2.1, F_c is a homeomorphism of its domain onto its image.

Our aim is to investigate the limit dynamics of the iterates of F_c as $c \to 0$. We cannot do this in a simple way, by looking at one iterate, since F_c converges to the identity, although the dynamics for all $c > 0$ are non-trivial. Therefore, we will have to make another rescaling, in this case not of space, but of time. For a fixed c we will examine the $n(c)^{th}$ iterate of F_c where $n(c) = \lfloor \pi/2c \rfloor$, the greatest integer such that $n(c)c \le \pi/2$. It will be easier to work with a representation of F_c in polar coordinates, which we will call G_c. We will consider only small values of c and use the notation

$$G_c^k(r_0, \vartheta_0) = (r_k, \vartheta_k) \tag{4.2}$$

without referring explicitly to the dependence of r_k and ϑ_k on c (but of course keeping this dependence in mind).

We start by expanding NEF_c in powers of c (up to degree 2). This gives

$$\frac{x+cy-2c^2xy}{1+2c^2y-4c^3xy} = x + cy - c^2 4xy + c^3\left(\frac{4x^2y - 2y^2 + 12cxy^2 - 16c^2x^2y^2}{1+2c^2y-4c^3xy}\right) \tag{4.3}$$

and

$$\frac{y-cx-2c^2xy}{1+2c^2x+4c^3xy} = y-cx-c^24xy+c^3\left(\frac{-4xy^2+2x^2+12cx^2y+16c^2x^2y^2}{1+2c^2x+4c^3xy}\right). \tag{4.4}$$

Thus,

$$NEF_c(x,y) = (x,y)+c(y,-x)-c^2(4xy,4xy)+c^3(q_1(c,x,y),q_2(c,x,y)), \tag{4.5}$$

where q_1, q_2 are rational functions. In particular they are analytic for $c \in [0, 1/4]$ and $\|(x,y)\| \le 2$ with (x,y) in the first quadrant. Moreover, each term in the numerators of q_1 and q_2 is of degree at least two in x and y.

It is easy to check that if $c > 0$ then the origin is a repelling fixed point for $\eta^{(c)}$. Since F_c is smoothly conjugate with $\eta^{(c)}$, and the origin goes to the origin under the conjugacy, the origin is also a repelling fixed point of F_c. We will show that the domain of repulsion is uniformly large in c, provided c is small enough.

Lemma 4.1. *There is a constant $C \in (0, 1/4]$ such that if $0 < c \le C$, (x,y) is in the first quadrant and $\|(x,y)\| \le 1/16$, then $\|NEF_c(x,y)\| \ge (1+c^2/4)\|(x,y)\|$.*

Proof. By (4.3) and (4.4) we get

$$\|NEF_c(x,y)\|^2 = (1+c^2)(x^2+y^2)-8c^2xy(x+y)+c^3q_3(c,x,y), \tag{4.6}$$

where q_3 is a rational function analytic in the first quadrant for $c \in [0, 1/4]$ and $\|(x,y)\| \le 2$. Each term in the numerator of q_3 is of degree at least two in x and y, while the denominator at $(c,0,0)$ is 1. This means the function $q_3(c,x,y)/\|(x,y)\|^2$ is bounded, so there is a constant $C \in (0,1/4]$ such that $c|q_3(c,x,y)|/\|(x,y)\|^2 \le 1/4$ if $0 < c \le C$.

If $\|(x,y)\| \le 1/16$ then $x, y \in [0, 1/16]$, so

$$(x^2+y^2)-8xy(x+y) \ge x^2+y^2-xy \ge \frac{x^2+y^2}{2}.$$

Thus, if $0 < c \le C$ then

$$\frac{\|NEF_c(x,y)\|^2}{\|(x,y)\|^2} \ge 1+\frac{c^2}{2}+c^3\frac{q_3(c,x,y)}{x^2+y^2} \ge 1+\frac{c^2}{4}.$$

□

Now we compute r_1 and ϑ_1.

Lemma 4.2. *There exist analytic functions ξ_r and ξ_t such that when $c \in [0, C]$, $r_0 \in [1/32, 2]$ and $\vartheta_0 \in [0, \pi/2]$, then*

$$r_1 = r_0 + c^2r_0\left(\tfrac{1}{2}-4r_0\cos\vartheta_0\sin\vartheta_0(\cos\vartheta_0+\sin\vartheta_0)\right)+c^3\xi_r(c,r_0,\vartheta_0) \tag{4.7}$$

and

$$\vartheta_1 = \vartheta_0 - c + c^24r_0\cos\vartheta_0\sin\vartheta_0(\sin\vartheta_0-\cos\vartheta_0)+c^3\xi_t(c,r_0,\vartheta_0). \tag{4.8}$$

Proof. To get the r-component of $NEF_c(x,y)$, use (4.6) to compute

$$\|NEF_c(x,y)\|^2 = r^2\big(1+c^2(1-8r\cos\vartheta\sin\vartheta(\cos\vartheta+\sin\vartheta))\big)+c^3\tilde{q}_3(c,r,\vartheta),$$

where $\tilde{q}_3$ is an analytic function. Now by taking the square root we get formula (4.7).

To get (4.8), we use the formula $\tan\alpha = |x\widetilde{y}-y\widetilde{x}|/|x\widetilde{x}+y\widetilde{y}|$ for the tangent of the angle α between the vectors (x,y) and $(\widetilde{x},\widetilde{y}) = NEF_c(x,y)$ (it is basically the length of the vector product divided by the length of the scalar product). We will write $\approx$ if the difference of both sides is of the form c^3 times an analytic function of c, x, y. We have

$$\tan\alpha \approx \frac{|x(y-cx-4c^2xy)-y(x+cy-4c^2xy)|}{|x(x+cy-4c^2xy)+y(y-cx-4c^2xy)|} = \frac{|-c(x^2+y^2)+4c^2xy(y-x)|}{|(x^2+y^2)-4c^2xy(x+y)|}$$
$$= \frac{\left|-c+4c^2\frac{xy(y-x)}{(x^2+y^2)}\right|}{\left|1-4c^2\frac{xy(y-x)}{(x^2+y^2)}\right|} \approx \left|-c+4c^2\frac{xy(y-x)}{(x^2+y^2)}\right|.$$

The power series expansion of $\arctan z$ begins with $z - z^3/3$, so we get

$$\alpha \approx \left|-c+4c^2\frac{xy(y-x)}{(x^2+y^2)}\right|.$$

For small c the main contribution to $(\widetilde{x},\widetilde{y})-(x,y)$ is $c(y,-x)$, so ϑ decreases under the application of NEF_c. Therefore

$$\vartheta_1 - \vartheta_2 \approx -c + 4c^2\frac{xy(y-x)}{(x^2+y^2)},$$

and (4.8) follows. □

Next we compute the derivatives of the maps.

Lemma 4.3. *When $c \in [0, C]$, $r_0 \in [1/32, 2]$ and $\vartheta_0 \in [0, \pi/2]$, we have*

$$\frac{\partial r_1}{\partial r_0} = 1 + c^2\left(\tfrac{1}{2} - 8r_0\cos\vartheta_0\sin\vartheta_0(\cos\vartheta_0+\sin\vartheta_0)\right) + c^3\frac{\partial \xi_r}{\partial r_0}(c, r_0, \vartheta_0), \tag{4.9}$$
$$\frac{\partial r_1}{\partial \vartheta_0} = 4c^2r_0^2(\sin\vartheta_0-\cos\vartheta_0)(1+3\cos\vartheta_0\sin\vartheta_0) + c^3\frac{\partial \xi_r}{\partial \vartheta_0}(c, r_0, \vartheta_0), \tag{4.10}$$
$$\frac{\partial \vartheta_1}{\partial r_0} = 4c^2\cos\vartheta_0\sin\vartheta_0(\sin\vartheta_0-\cos\vartheta_0) + c^3\frac{\partial \xi_t}{\partial r_0}(c, r_0, \vartheta_0), \tag{4.11}$$
$$\frac{\partial \vartheta_1}{\partial \vartheta_0} = 1 + 4c^2r_0(\cos\vartheta_0+\sin\vartheta_0)(3\cos\vartheta_0\sin\vartheta_0-1) + c^3\frac{\partial \xi_t}{\partial \vartheta_0}(c, r_0, \vartheta_0). \tag{4.12}$$

Proof. Use the formulas from Lemma 4.2 and compute the derivatives. □

Now we will get similar formulas that are valid on the whole plane, not just the first quadrant. We use the fact that F_c commutes with the rotation ζ (rotation by $\pi/2$). Let $q(\vartheta) = \vartheta \mod \pi/2$, $E_1 = \{(r,\vartheta) : 1/16 \le r \le 1\}$ and $E_2 = \{(r,\vartheta) : 1/32 \le r \le 2\}$. Set

$$\varphi_r(r,\vartheta) = \tfrac{1}{2} - 4r\cos q(\vartheta)\sin q(\vartheta)(\cos q(\vartheta)+\sin q(\vartheta)), \tag{4.13}$$
$$\varphi_t(r,\vartheta) = 4r\cos q(\vartheta)\sin q(\vartheta)(\sin q(\vartheta)-\cos q(\vartheta)), \tag{4.14}$$
$$\psi_{rr}(r,\vartheta) = \tfrac{1}{2} - 8r\cos q(\vartheta)\sin q(\vartheta)(\cos q(\vartheta)+\sin q(\vartheta)), \tag{4.15}$$
$$\psi_{rt}(r,\vartheta) = 4r^2(\sin q(\vartheta)-\cos q(\vartheta))(1+3\cos q(\vartheta)\sin q(\vartheta)), \tag{4.16}$$
$$\psi_{tr}(r,\vartheta) = 4\cos q(\vartheta)\sin q(\vartheta)(\sin q(\vartheta)-\cos q(\vartheta)), \tag{4.17}$$
$$\psi_{tt}(r,\vartheta) = 4r(\cos q(\vartheta)+\sin q(\vartheta)(3\cos q(\vartheta)\sin q(\vartheta)-1), \tag{4.18}$$

and let $\tau_{rr}, \tau_{rt}, \tau_{tr}, \tau_{tt}$ be the corresponding functions for $\partial\xi_r/\partial r_0$, $\partial\xi_r/\partial\vartheta_0$, $\partial\xi_t/\partial r_0$, $\partial\xi_t/\partial\vartheta_0$. All ten functions are analytic in each quadrant for $c \in [0, C]$, $(r, \vartheta) \in E_2$ and consequently each is bounded.

With this notation Lemmas 4.2 and 4.3 immediately generalize to the plane. On the lines where the quadrants meet the derivative of G_c may not exist, but then we mean the derivatives from each side of those lines.

Lemma 4.4. *When $c \in [0, C]$ and $(r_0, \vartheta_0) \in E_2$, then*

$$r_1 = r_0 + c^2 r_0 \varphi_r(r_0, \vartheta_0) + c^3 \xi_r(c, r_0, q(\vartheta_0)), \tag{4.19}$$

$$\vartheta_1 = \vartheta_0 - c + c^2 \varphi_t(r_0, \vartheta_0) + c^3 \xi_t(c, r_0, q(\vartheta_0)). \tag{4.20}$$

The derivative DG_c of G_c at (r_0, ϑ_0) is

$$\begin{pmatrix} 1 + c^2\psi_{rr}(r_0, \vartheta_0) + c^3\tau_{rr}(c, r_0, \vartheta_0) & c^2\psi_{rt}(r_0, \vartheta_0) + c^3\tau_{rt}(c, r_0, \vartheta_0) \\ c^2\psi_{tr}(r_0, \vartheta_0) + c^3\tau_{tr}(c, r_0, \vartheta_0) & 1 + c^2\psi_{tt}(r_0, \vartheta_0) + c^3\tau_{tt}(c, r_0, \vartheta_0) \end{pmatrix}. \tag{4.21}$$

The next lemma is the key lemma for the whole paper.

Lemma 4.5. *There exist positive constants $K_c, K_s, K_\vartheta, K_r$ such that $K_c \le C$ and for all $c \in (0, K_c]$ and $(r_0, \vartheta_0) \in E_1$ we have*

$$|r_k - r_0| \le cK_s \quad \textit{for} \quad k = 1, 2, \ldots, n(c), \tag{4.22}$$

$$\left|\vartheta_{n(c)} - \left(\vartheta_0 - \tfrac{\pi}{2}\right)\right| \le cK_\vartheta, \tag{4.23}$$

$$\left|r_{n(c)} - \left(r_0 + cr_0\left(\frac{\pi}{4} - \frac{8}{3}r_0\right)\right)\right| \le c^2 K_r. \tag{4.24}$$

Moreover, there exist positive constants $K_{rr}, K_{r\vartheta}, K_{\vartheta r}, K_{\vartheta\vartheta}$ such that for all $c \in (0, K_c]$ and $(r_0, \vartheta_0) \in E_1$ we have

$$\left|\frac{\partial r_{n(c)}}{\partial r_0} - \left(1 + c\left(\frac{\pi}{4} - \frac{16}{3}r_0\right)\right)\right| \le c^2 K_{rr} \tag{4.25}$$

and

$$\left|\frac{\partial\vartheta_{n(c)}}{\partial r_0}\right| \le c^2 K_{\vartheta r}, \quad \left|\frac{\partial r_{n(c)}}{\partial\vartheta_0}\right| \le c^2 K_{r\vartheta}, \quad \left|\frac{\partial\vartheta_{n(c)}}{\partial\vartheta_0} - 1\right| \le c^2 K_{\vartheta\vartheta}. \tag{4.26}$$

In those formulas we include all one-sided derivatives when the derivatives do not exist.

Proof. Recall that $n(c) = \lfloor \pi/2c \rfloor$. By Lemma 4.4, for all $c \in [0, C]$ and $(r_0, \vartheta_0) \in E_2$

$$r_1 - r_0 = c^2\big(r_0\varphi_r(r_0, \vartheta_0) + c\xi_r(c, r_0, q(\vartheta_0))\big) \tag{4.27}$$

and

$$\vartheta_1 - (\vartheta_0 - c) = c^2\big(\varphi_t(r_0, \vartheta_0) + c\xi_t(c, r_0, q(\vartheta_0))\big). \tag{4.28}$$

This means there is a constant K_1 so that for all $c \in [0, C]$ and $(r_0, \vartheta_0) \in E_2$

$$|r_1 - r_0| \le c^2 K_1, \tag{4.29}$$

$$|\vartheta_1 - (\vartheta_0 - c)| \le c^2 K_1, \tag{4.30}$$

and

$$|r_1 - (r_0 + c^2 r_0 \varphi_r(r_0, \vartheta_0))| \le c^3 K_1. \tag{4.31}$$

Let $1/16 \le r_0 \le 1$ and $0 < c \le \min\{1/(16\pi K_1), C\}$. Then

$$|r_1 - r_0| \le c^2 K_1 \le \tfrac{1}{32}. \tag{4.32}$$

This means that $r_1 \le 2$ and (4.29) applies to $|r_2 - r_1|$. Repeating this argument inductively (note that if $k \le n(c)$ then $kc \le \pi/2$, so $kcK_1 \le 1/(32c)$), we get

$$|r_k - r_0| \le kc^2 K_1 \le \tfrac{1}{32} \tag{4.33}$$

for $k = 1, \dots, n(c)$. Now we can use (4.33) to get a better estimate:

$$|r_k - r_0| \le n(c) c^2 K_1 \le c \tfrac{\pi}{2} K_1 = cK_s, \tag{4.34}$$

where $K_s = \frac{\pi}{2} K_1$. This proves (4.22).

Keeping $1/16 \le r_0 \le 1$ and $0 < c \le \min\{1/(16\pi K_1), C\}$, and applying (4.30) k times gives us

$$|\vartheta_k - (\vartheta_0 - kc)| \le kc^2 K_1 \tag{4.35}$$

for $k = 1, \dots, n(c)$. Therefore

$$|\vartheta_{n(c)} - (\vartheta_0 - \tfrac{\pi}{2})| \le n(c) c^2 K_2 + (\tfrac{\pi}{2} - n(c)c) \le c\left(\tfrac{\pi}{2} K_2 + 1\right) = cK_t, \tag{4.36}$$

where $K_t = \frac{\pi}{2} K_1 + 1$. This proves (4.23).

The function φ_r is bounded and Lipschitz continuous for $0 \le c \le C$ and $(r, \vartheta) \in E_2$ with some bound M and some Lipschitz constant L.

Still keeping $1/16 \le r_0 \le 1$ and $0 < c \le \min\{1/(16\pi K_1), C\}$, and using (4.33) and (4.35), we get for $k = 1, \dots, n(c)$

$$\begin{aligned} |\varphi_r(r_k, \vartheta_k) - \varphi_r(r_0, \vartheta_0 - kc)| &\le L(|r_k - r_0| + |\vartheta_k - (\vartheta_0 - kc)|) \\ &\le L(kc^2 K_1 + kc^2 K_1) \le cK_2, \end{aligned} \tag{4.37}$$

where $K_2 = L\pi K_1$. Now we can obtain a good approximation to $r_{k+1} - r_k$ for k between 0 and $n(c)$. By (4.31), (4.22) and (4.37) we get

$$\begin{aligned} |(r_{k+1} - r_k) - c^2 r_0 \varphi_r(r_0, \vartheta_0 - kc)| &\le c^3 K_1 + |c^2 r_k \varphi_r(r_k, \vartheta_k) - c^2 r_0 \varphi_r(r_0, \vartheta_0 - kc)| \\ &\le c^3 K_1 + c^2 |r_k - r_0| \cdot |\varphi_r(r_k, \vartheta_k)| + c^2 r_0 |\varphi_r(r_k, \vartheta_k) - \varphi_r(r_0, \vartheta_0 - kc)| \\ &\le c^3 K_1 + c^3 K_s M + c^3 r_0 K_2 = c^3 K_3, \end{aligned} \tag{4.38}$$

where $K_3 = K_1 + K_s M + K_2$.

By (4.38) and the definition of $n(c)$ we have

$$\left| (r_{n(c)} - r_0) - \sum_{k=0}^{n(c)-1} c^2 r_0 \varphi_r(r_0, \vartheta_0 - kc) \right| \le n(c) c^3 K_3 \le c^2 \tfrac{\pi}{2} K_3. \tag{4.39}$$

We will approximate the sum above (divided by cr_0) by an integral. From the definition of the Riemann integral we have

$$\begin{aligned}\left|\sum_{k=0}^{n(c)-1} c\varphi_r(r_0,\vartheta_0-kc)-\int_0^{\pi/2}\varphi_r(r_0,\vartheta_0-t)\,dt\right| \\ \le \left|\sum_{k=0}^{n(c)-1} c\varphi_r(r_0,\vartheta_0-kc)-\int_0^{cn(c)}\varphi_r(r_0,\vartheta_0-t)\,dt\right| \\ +\left|\int_{cn(c)}^{\pi/2}\varphi_r(r_0,\vartheta_0-t)\,dt\right| \le n(c)\cdot c\cdot Lc+cM\le c\left(\tfrac{\pi}{2}L+M\right).\end{aligned}\tag{4.40}$$

By (4.39) and (4.40) together, we get

$$\left|(r_{n(c)}-r_0)-cr_0\int_0^{\pi/2}\varphi_r(r_0,\vartheta_0-t)\,dt\right|\le c^2\tfrac{\pi}{2}K_3+cr_0\cdot c\left(\tfrac{\pi}{2}L+M\right)\le c^2K_r,\tag{4.41}$$

where $K_r=\frac{\pi}{2}K_3+\frac{\pi}{2}L+M$.

The integral appearing in (4.41) can be easily evaluated, taking into account that φ_r is periodic in ϑ with period $\pi/2$. We have

$$\begin{aligned}\int_0^{\pi/2}\varphi_r(r_0,\vartheta_0-t)\,dt=\int_0^{\pi/2}\varphi_r(r_0,t)\,dt=\int_0^{\pi/2}\left(\tfrac12-4r_0\cos t\sin t(\cos t+\sin t)\right)\,dt \\ =\frac{\pi}{4}-4r_0\cdot 2\int_0^{\pi/2}\sin^2 t\cos t\,dt=\frac{\pi}{4}-8r_0\int_0^1 x^2\,dx=\frac{\pi}{4}-\frac{8}{3}r_0.\end{aligned}$$

Together with (4.41), this proves (4.24).

Now we deal with the derivatives. Let A_k be the matrix of the derivative of G_c at (r_k,ϑ_k) (for $k=0,1,\dots,n(c)$). Then, by Lemma 4.4, we have

$$A_k=\begin{pmatrix}1+c^2\psi_{rr}(r_k,\vartheta_k)+c^3\tau_{rr}(c,r_k,\vartheta_k) & c^2\psi_{rt}(r_k,\vartheta_k)+c^3\tau_{rt}(c,r_k,\vartheta_k)\\ c^2\psi_{tr}(r_k,\vartheta_k)+c^3\tau_{tr}(c,r_k,\vartheta_k) & 1+c^2\psi_{tt}(r_k,\vartheta_k)+c^3\tau_{tt}(c,r_k,\vartheta_k)\end{pmatrix}.\tag{4.42}$$

As we noted, if $1/16\le r_0\le 1$ (which we assumed, see (4.33)), then $1/32\le r_k\le 2$. Therefore, there exists a constant $K_4>0$ such that the matrix $|A_k|$, whose entries are the absolute values of the entries of A_k, is bounded (entry by entry) by the matrix

$$F=\begin{pmatrix}1+c^2K_4 & c^2K_4\\ c^2K_4 & 1+c^2K_4\end{pmatrix}.\tag{4.43}$$

Let

$$G=\begin{pmatrix}1&1\\1&1\end{pmatrix},\qquad H=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}.$$

Then $F=(1/2)((1+2c^2K_4)G+H)$ and $GH=HG=0$, $G^2=2G$, $H^2=2H$. Therefore by induction we get $F^k=(1/2)((1+2c^2K_4)^kG+H)$.

Consider $(1+2c^2K_4)^{\pi/(2c)}$ as a function of c. Since its limit as $c\to 0$ is 1, we can use L'Hôpital's Rule to compute

$$\lim_{c\to 0}\frac{(1+2c^2K_4)^{\pi/(2c)}-1}{c}=\pi K_4.$$

Therefore there is a constant $K_5 > 0$ such that $(1+2c^2K_4)^k - 1 \le cK_5$ for $c \in (0, C]$ and $k = 1, \dots n(c)$.

Let B_k be the matrix of the derivative of G_c^k at (r_0, ϑ_0). In particular, $B_0 = I$ and $B_{k+1} = A_k B_k$. For any matrix B we denote by $\|B\|$ the sum of the moduli of its entries. We have

$$|B_k| \le F^k \le \begin{pmatrix} 1+cK_5 & cK_5 \\ cK_5 & 1+cK_5 \end{pmatrix} \tag{4.44}$$

for $k = 1, \dots n(c)$.

By (4.43) and (4.44), we get

$$\|(B_{k+1} - B_k) - (A_k - I)\| = \|(A_k - I)(B_k - I)\| \le 8c^3 K_4 K_5.$$

By this, (4.42), and boundedness of the functions τ_{ij} $(ij = rr, rt, tr, tt)$, we get

$$\left\| (B_{k+1} - B_k) - \begin{pmatrix} c^2\psi_{rr}(r_k, \vartheta_k) & c^2\psi_{rt}(r_k, \vartheta_k) \\ c^2\psi_{tr}(r_k, \vartheta_k) & c^2\psi_{tt}(r_k, \vartheta_k) \end{pmatrix} \right\| \le c^3 K_6$$

for some constant K_6. Thus, taking into account the definition of $n(c)$, we see that

$$\left\| (B_{n(c)} - I) - c^2 \sum_{k=0}^{n(c)-1} \begin{pmatrix} \psi_{rr}(r_k, \vartheta_k) & \psi_{rt}(r_k, \vartheta_k) \\ \psi_{tr}(r_k, \vartheta_k) & \psi_{tt}(r_k, \vartheta_k) \end{pmatrix} \right\| \le c^2 K_6 \frac{\pi}{2}. \tag{4.45}$$

In the same way as (4.37), we get for $ij = rr, rt, tr, tt$

$$|\psi_{ij}(r_k, \vartheta_k) - \psi_{ij}(r_0, \vartheta_0 - kc)| \le cK_7 \tag{4.46}$$

for some constant K_7. The only difference is that ψ_{tr} and ψ_{rt} have discontinuities at $\theta = m\pi/2$. However, by (4.30), such discontinuity can occur between ϑ_k and $\vartheta_0 - kc$ for at most one k, provided $c^2 K_1 < c/2$ (we made even a stronger assumption long ago). Thus, (4.46) holds for all $k = 0, 1, \dots, n(c)$, except at most one value of k. Since ψ_{ij} is bounded in E_2, together with (4.46) we get that

$$\sum_{k=0}^{n(c)-1} |\psi_{ij}(r_k, \vartheta_k) - \psi_{ij}(r_0, \vartheta_0 - kc)|$$

is bounded by a constant. From this and (4.45) we get

$$\left\| (B_{n(c)} - I) - c^2 \sum_{k=0}^{n(c)-1} \begin{pmatrix} \psi_{rr}(r_0, \vartheta_0 - kc) & \psi_{rt}(r_0, \vartheta_0 - kc) \\ \psi_{tr}(r_0, \vartheta_0 - kc) & \psi_{tt}(r_0, \vartheta_0 - kc) \end{pmatrix} \right\| \le c^2 K_8 \tag{4.47}$$

for some constant K_8.

Again in the same way as for φ_r, we approximate the sum by the integral, and we get

$$\left| c \sum_{k=0}^{n(c)-1} \psi_{ij}(r_0, \vartheta_0 - kc) - \int_0^{\pi/2} \psi_{ij}(r_0, \vartheta_0 - t)\, dt \right| \le cK_9$$

for some constant K_9. Note that a possible discontinuity of ψ_{ij} will add a constant times c, so the estimate is correct. Because each function ψ_{ij} is periodic in ϑ with period $\pi/2$, we have

$$\int_0^{\pi/2} \psi_{ij}(r_0, \vartheta_0 - t)\, dt = \int_0^{\pi/2} \psi_{ij}(r_0, t)\, dt.$$

Hence,

$$\left| c^2 \sum_{k=0}^{n(c)-1} \psi_{ij}(r_0, \vartheta_0 - kc) - c \int_0^{\pi/2} \psi_{ij}(r_0, t)\, dt \right| \le c^2 K_9 \tag{4.48}$$

for $ij = rr, rt, tr, tt$.

Now we have to evaluate the integrals appearing in (4.48). The first one, with ψ_{rr}, is very similar to the integral of φ_r, and it is equal to $\pi/4-(16/3)r_0$. Together with (4.47) and (4.48), this proves (4.25) with $K_{rr} = K_8 + K_9$. The integrals from (4.48) with ψ_{rt} and ψ_{tr} are zero, because $\psi_{rt}(r_0, \pi/2 - t) = -\psi_{rt}(r_0, t)$ and $\psi_{tr}(r_0, \pi/2 - t) = -\psi_{tr}(r_0, t)$. Together with (4.47) and (4.48), this proves the first two inequalities of (4.26). It remains to evaluate the integral with ψ_{tt}. It should be 0, because basically we are integrating the derivative of a function whose values at the endpoints are equal. We have

$$\begin{aligned}\int_0^{\pi/2} (\cos t + \sin t)(3 \cos t \sin t - 1)\, dt = 6 \int_0^{\pi/2} \sin^2 t \cos t\, dt - 2 \int_0^{\pi/2} \cos t\, dt \\ = 6 \int_0^1 x^2\, dx - 2 = 0,\end{aligned}$$

so indeed this integral is 0. Together with (4.47) and (4.48), this proves the third inequality of (4.26). □

5. More estimates

Now we can investigate more closely the dynamics of $G_c^{n(c)}$.

Lemma 5.1. *For every $\varepsilon > 0$ there is $C_1 \in (0, K_c]$ such that if $0 < c < C_1$ and $1/16 \le r_0 \le 3\pi/32 - \varepsilon$ then $r_{n(c)} \ge r_0 + c\varepsilon/12$, and if $0 < c < C_1$ and $3\pi/32 + \varepsilon \le r_0 \le 1$ then $r_{n(c)} \le r_0 - c\varepsilon/12$.*

Proof. If $r_0 \le 3\pi/32 - \varepsilon$ and $c < K_c$ then

$$\frac{\pi}{4} - \frac{8}{3} r_0 \ge \frac{\pi}{4} - \frac{8}{3}\left(\frac{3\pi}{32} - \varepsilon\right) = \frac{8}{3}\varepsilon,$$

so by Lemma 4.5 (formula (4.24)),

$$r_{n(c)} \ge r_0 + c r_0 \left(\frac{\pi}{4} - \frac{8}{3} r_0\right) - c^2 K_r \ge r_0 + \frac{8}{3}\varepsilon c r_0 - c^2 K_r = r_0 + c\left(\frac{8}{3}\varepsilon r_0 - c K_r\right). \tag{5.1}$$

Set $C_1 = \min(\varepsilon/(12 K_r), K_c)$. If $r_0 \ge 1/16$ and $0 < c < C_1$ then

$$\frac{8}{3}\varepsilon r_0 - c K_r \ge \frac{\varepsilon}{6} - \frac{\varepsilon}{12} = \frac{\varepsilon}{12}, \tag{5.2}$$

so by (5.1), $r_{n(c)} \ge r_0 + c\varepsilon/12$.

If $r_0 \geq 3\pi/32 + \varepsilon$ then similarly,

$$\frac{\pi}{4} - \frac{8}{3}r_0 \leq \frac{\pi}{4} - \frac{8}{3}\left(\frac{3\pi}{32} + \varepsilon\right) = -\frac{8}{3}\varepsilon,$$

so by Lemma 4.5,

$$r_{n(c)} \leq r_0 + cr_0\left(\frac{\pi}{4} - \frac{8}{3}r_0\right) + c^2K_r \leq r_0 - \frac{8}{3}\varepsilon cr_0 + c^2K_r = r_0 - c\left(\frac{8}{3}\varepsilon r_0 - cK_r\right). \tag{5.3}$$

Since $r_0 \geq 3\pi/32 + \varepsilon > 1/16$, (5.2) holds, so if $0 < c < C_1$ then we get by (5.3), $r_{n(c)} \leq r_0 - c\varepsilon/12$. ☐

For a given $\eta \in (0, 3\pi/32)$, denote by H_η the annulus given by $3\pi/32 - \eta \leq r \leq 3\pi/32 + \eta$. Denote also by D_1 the unit disk in $\mathbb{R}^2$.

Proposition 5.2. *For every $\varepsilon \in (0, \pi/32 - 1/16)$ there is $C_2 \in (0, C_1]$ such that if $0 < c < C_2$ then:*

(a) *the annulus $H_{2\varepsilon}$ is invariant for $G_c^{n(c)}$,*
(b) *all images of $H_{2\varepsilon}$ under the iterates of G_c are contained in $H_{3\varepsilon}$,*
(c) *G_c-trajectories of all points of D_1, except the origin, enter H_ε,*
(d) *G_c-trajectories of all points of D_1, except the origin, eventually stay in $H_{2\varepsilon}$.*

Proof. Fix $\varepsilon \in (0, \pi/32 - 1/16)$ and let C_1 be the corresponding constant from Lemma 5.1. Let K_s be the constant from Lemma 4.5. Set $C_2 = \min(C_1, \varepsilon/K_s)$. Assume that $0 < c < C_2$. Note that $K_s c < \varepsilon$.

By Lemma 5.1 and Lemma 4.5 (formula (4.22)), if $1/16 \leq r_0 \leq 3\pi/32 - \varepsilon$ then $r_0 + c\varepsilon/12 \leq r_{n(c)} \leq r_0 + \varepsilon$, if $r_0 \geq 3\pi/32 + \varepsilon$ then $r_0 - \varepsilon \leq r_{n(c)} \leq r_0 - c\varepsilon/12$, and always $r_0 - \varepsilon \leq r_{n(c)} \leq r_0 + \varepsilon$. The first two of these properties guarantee that if $r_0 \in H_{2\varepsilon} \setminus H_\varepsilon$ then $r_{n(c)} \in H_{2\varepsilon}$, the third one guarantees that if $r_0 \in H_\varepsilon$ then $r_{n(c)} \in H_{2\varepsilon}$. This proves (a).

Now (b) follows from (a) and Lemma 4.5 (formula (4.22)). To prove (c), observe that by Lemmas 4.1 and 5.1, the numbers r_i (with i incremented by 1 or $n(c)$) move towards $3\pi/32$, until they get within ε of it, and the change of r_i cannot exceed ε by (4.22).

To prove (d), use (c) to find a point p of the trajectory that belongs to H_ε. Note that if $k < n(c)$ then by (4.22) $G_c^k(p) \in H_{2\varepsilon}$. Finally, use induction on i to show that $G_c^{in(c)+k}(p) \in H_{2\varepsilon}$. Indeed, if $G_c^{in(c)+k}(p) \in H_\varepsilon$ then $G_c^{(i+1)n(c)+k}(p) \in H_{2\varepsilon}$ by (4.22); if $G_c^{in(c)+k}(p) \in H_{2\varepsilon} \setminus H_\varepsilon$ then $G_c^{(i+1)n(c)+k}(p) \in H_{2\varepsilon}$ by Lemma 5.1. ☐

Lemma 5.3. *If $0 < \varepsilon < 0.03$ then there exists $C_3 \in (0, C_2]$ such that if $0 < c < C_3$ and $(r_0, \vartheta_0) \in H_{2\varepsilon}$ then $0 < \partial r_{n(c)}/\partial r_0 < 1 - \pi c/16$, $|\partial r_{n(c)}/\partial\vartheta_0| < c\varepsilon$, $|\partial\vartheta_{n(c)}/\partial r_0| < c\varepsilon$ and $|\partial\vartheta_{n(c)}/\partial\vartheta_0 - 1| < c\varepsilon$.*

Proof. Assume that $0 < \varepsilon < 0.03$ (note that $0.03 < 3\pi/256$ and $0.03 < \pi/32 - 1/16$) and $(r_0, \vartheta_0) \in H_{2\varepsilon}$. Then

$$\frac{\pi}{4} - \frac{16}{3}r_0 \leq \frac{\pi}{4} - \frac{16}{3}\left(\frac{3\pi}{32} - 2\varepsilon\right) < \frac{\pi}{4} - \frac{\pi}{2} + \frac{32}{3}\cdot\frac{3\pi}{256} = -\frac{\pi}{8}$$

and

$$\frac{\pi}{4} - \frac{16}{3}r_0 \geq \frac{\pi}{4} - \frac{16}{3}\left(\frac{3\pi}{32} + 2\varepsilon\right) > \frac{\pi}{4} - \frac{\pi}{2} - \frac{32}{3}\cdot\frac{3\pi}{256} = -\frac{3\pi}{8}.$$

By those inequalities and Lemma 4.5 (formula (4.25)), we get

$$1-\frac{3\pi}{8}c-c^2K_{rr}<\frac{\partial r_{n(c)}}{\partial r_0}<1-\frac{\pi}{8}c+c^2K_{rr}.$$

If $c<\pi/(16K_{rr})$, this yields

$$1-\frac{7\pi}{16}c<\frac{\partial r_{n(c)}}{\partial r_0}<1-\frac{\pi}{16}c.$$

If additionally $c<16/(7\pi)$, we get $0<\partial r_{n(c)}/\partial r_0<1-\pi c/16$.

By Lemma 4.5 (formula (4.26)), for the other three inequalities to hold, we need only $cK_{\vartheta r}, cK_{r\vartheta}, cK_{\vartheta\vartheta}$ to be smaller than ε. Thus, all four inequalities that we want to obtain hold if we set

$$C_3=\min\left(C_2,\frac{\pi}{16K_{rr}},\frac{16}{7\pi},\frac{\varepsilon}{K_{\vartheta r}},\frac{\varepsilon}{K_{r\vartheta}},\frac{\varepsilon}{K_{\vartheta\vartheta}}\right).$$

□

Denote by $M_{\lambda,\eta}$ the set of all 2×2 matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ for which $0<\alpha<\lambda$, $|\beta|<\eta$, $|\gamma|<\eta$ and $|\delta-1|<\eta$.

Lemma 5.4. *Assume that $0<\varepsilon<0.03$, C_3 is the constant from Lemma 5.3, $0<c<C_3$ and $(r_0,\vartheta_0),(s_0,\sigma_0)\in H_{2\varepsilon}$. Then there is a matrix $A\in M_{1-\pi c/16,c\varepsilon}$ such that*

$$A\begin{pmatrix}s_0-r_0\\ \sigma_0-\vartheta_0\end{pmatrix}=\begin{pmatrix}s_{c(n)}-r_{c(n)}\\ \sigma_{c(n)}-\vartheta_{c(n)}\end{pmatrix}.$$

Proof. Consider the segment joining (r_0,ϑ_0) with (s_0,σ_0), parametrized by

$$\mathcal{S}(t)=(ts_0+(1-t)r_0,t\sigma_0+(1-t)\vartheta_0),\quad t\in[0,1]$$

(we are now working in the (r,ϑ)-plane, not in the (x,y)-plane) and its image under $G_c^{n(c)}$. We have

$$\begin{pmatrix}s_{n(c))}-r_{n(c))}\\ \sigma_{n(c)}-\vartheta_{n(c))}\end{pmatrix}=\int_0^1\left(DG_c^{n(c)}(ts_0+(1-t)r_0,t\sigma_0+(1-t)\vartheta_0)\right)\begin{pmatrix}s_0-r_0\\ \sigma_0-\vartheta_0\end{pmatrix}dt. \tag{5.4}$$

The point $(ts_0+(1-t)r_0,t\sigma_0+(1-t)\vartheta_0)$ belongs to $H_{2\varepsilon}$, so the derivative of $G_c^{n(c)}$ at this point belongs to $M_{1-\pi c/16,c\varepsilon}$ by Lemma 5.3. Thus, in (5.4) we are integrating from 0 to 1 some matrix valued function $A(t)$ times a constant vector, and all $A(t)$ belong to $M_{1-\pi c/16,c\varepsilon}$. The set $M_{1-\pi c/16,c\varepsilon}$ is convex, and therefore the integral is of the form some matrix $A\in M_{1-\pi c/16,c\varepsilon}$ times the vector $(s_0-r_0,\sigma_0-\vartheta_0)^T$. This is what we wanted to prove. □

For positive constants ε and L, let $\mathcal{L}_{\varepsilon,L}$ be the space of all functions from the unit circle S^1 to $[3\pi/32-2\varepsilon,3\pi/32+2\varepsilon]$ which are Lipschitz continuous with constant L. The graph of such function is a subset of $H_{2\varepsilon}$.

Lemma 5.5. *For every $L>0$ there exists $\varepsilon_0\in(0,0.03]$ such that for every $\varepsilon\in(0,\varepsilon_0)$ if C_3 is the constant from Lemma 5.3, $0<c<C_3$, and Γ is a graph of a function from $\mathcal{L}_{\varepsilon,L}$, then $G_c^{n(c)}(\Gamma)$ is also a graph of a function from $\mathcal{L}_{c,\varepsilon,L}$.*

Proof. Under the assumptions of the lemma, the set Γ is a subset of $H_{2\varepsilon}$, so by Proposition 5.2, $G_c^{n(c)}(\Gamma)$ is also a subset of $H_{2\varepsilon}$. Thus, we have to show only that $G_c^{n(c)}(\Gamma)$ is a graph of a function from $\mathcal{L}_{\varepsilon,L}$. By Lemmas 5.3 and 5.4, to prove this, it is sufficient to show that if a vector v belongs to the cone (in coordinates r, ϑ)

$$V_L = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : |a| \le L|b| \right\}$$

and a matrix A belongs to $M_{1-\pi c/16, c\varepsilon}$ then $Av \in V_L$.

Thus, let us take

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{1-\pi c/16, c\varepsilon} \quad \text{and} \quad v = \begin{pmatrix} a \\ b \end{pmatrix} \in V_L.$$

Then $Av = (\alpha a + \beta b, \gamma a + \delta b)^T$. We know that $|a| \le L|b|$ and want to prove that

$$|\alpha a + \beta b| \le L|\gamma a + \delta b|. \tag{5.5}$$

By the definition of $M_{1-\pi c/16, c\varepsilon}$, we have $0 < \alpha < 1 - \pi c/16$, $|\beta| < c\varepsilon$, $|\gamma| < c\varepsilon$ and $|\delta - 1| < c\varepsilon$. We get

$$|\alpha a + \beta b| \le (1 - \tfrac{\pi}{16}c)|a| + c\varepsilon|b| \le \big((1 - \tfrac{\pi}{16}c)L + c\varepsilon\big)|b|$$

and

$$L|\gamma a + \delta b| \ge L\big((1 - c\varepsilon)|b| - c\varepsilon|a|\big) \ge L\big((1 - c\varepsilon)|b| - c\varepsilon L|b|\big) = (L - c\varepsilon L - c\varepsilon L^2)|b|.$$

Therefore (5.5) will hold if

$$(1 - \tfrac{\pi}{16}c)L + c\varepsilon \le L - c\varepsilon L - c\varepsilon L^2.$$

This inequality is equivalent to

$$\varepsilon \le \frac{\pi}{16} \cdot \frac{L}{1 + L + L^2}.$$

Thus, the lemma holds if we take

$$\varepsilon_0 = \min\left(0.03, \frac{\pi}{16} \cdot \frac{L}{1 + L + L^2}\right).$$

□

The space $\mathcal{L}_{\varepsilon,L}$ is equipped in a natural way with the C^0-metric

$$d(\chi, \omega) = \sup_{\vartheta \in S^1} |\chi(\vartheta) - \omega(\vartheta)|.$$

With this metric it is a complete space. Under the conditions listed in Lemma 5.5, there is an operator $\Phi : \mathcal{L}_{\varepsilon,L} \to \mathcal{L}_{\varepsilon,L}$, such that the graph of $\Phi(\omega)$ is the image under $G_c^{n(c)}$ of the graph of ω. We will show that with some additional restrictions on c, ε, L, this operator is a contraction.

Lemma 5.6. *For every $L > 0$ there exists $\varepsilon_1 \in (0, \varepsilon_0]$ (where ε_0 is the constant from Lemma 5.5) such that for every $\varepsilon \in (0, \varepsilon_1)$ there exists $C_4 \in (0, C_3]$ (where C_3 is the constant from Lemma 5.3) such that if $0 < c < C_4$, then the operator $\Phi : \mathcal{L}_{\varepsilon,L} \to \mathcal{L}_{\varepsilon,L}$ is Lipschitz continuous with some constant $\lambda < 1$.*

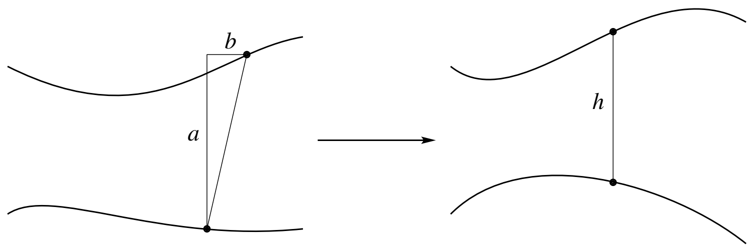

FIGURE 4. Points from the proof of Lemma 5.6

Proof. Assume that $0c < C_3$. Take $\omega, \chi \in \mathcal{L}_{\varepsilon,L}$ and estimate $d(\omega,\chi)$. To do this, fix ϑ and set $h = (\Phi(\chi))(\vartheta) - (\Phi(\omega))(\vartheta)$. By our assumptions, $((\Phi(\chi))(\vartheta),\vartheta) = G_c^{n(c)}(\chi(\vartheta_\chi),\vartheta_\chi)$ and $((\Phi(\omega))(\vartheta),\vartheta) = G_c^{n(c)}(\omega(\vartheta_\omega),\vartheta_\omega)$ for some ϑ_χ and ϑ_ω. Set $a = \chi(\vartheta_\chi) - \omega(\vartheta_\omega)$ and $b = \vartheta_\chi - \vartheta_\omega$ (see Figure 4). Then by Lemma 5.4, $A(a,b)^T = (h,0)^T$ for some matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{1-\pi c/16, c\varepsilon}.$$

We have $\alpha a + \beta b = h$ and $\gamma a + \delta b = 0$, so

$$|b| = \frac{|\gamma|}{|\delta|}|a| \le \frac{c\varepsilon}{1-c\varepsilon}|a|$$

and

$$|h| = |\alpha a + \beta b| \le (1 - \tfrac{\pi}{16}c)|a| + c\varepsilon|b| \le \left(1 - \frac{\pi}{16}c + \frac{c^2\varepsilon^2}{1-c\varepsilon}\right)|a|. \tag{5.6}$$

We have

$$|a| \le d(\chi,\omega) + L|b| \le d(\chi,\omega) + L\frac{c\varepsilon}{1-c\varepsilon}|a|,$$

which yields

$$|a| \le \frac{1-c\varepsilon}{1-(1+L)c\varepsilon} \cdot d(\chi,\omega).$$

Together with (5.6), this gives us

$$|h| \le \left(1 - \frac{\pi}{16}c + \frac{c^2\varepsilon^2}{1-c\varepsilon}\right) \cdot \frac{1-c\varepsilon}{1-(1+L)c\varepsilon} \cdot d(\chi,\omega). \tag{5.7}$$

The distance between $\Phi(\chi)$ and $\Phi(\omega)$ is equal to the supremum of the modulus of $h = h(\vartheta)$ as ϑ runs over S^1. Therefore it is smaller than or equal to the right-hand side of (5.7). Hence, it remains to find $\lambda < 1$ for which

$$\left(1 - \frac{\pi}{16}c + \frac{c^2\varepsilon^2}{1-c\varepsilon}\right) \cdot \frac{1-c\varepsilon}{1-(1+L)c\varepsilon} \le \lambda.$$

Such λ exists if

$$1 - \frac{\pi}{16}c + \frac{c^2\varepsilon^2}{1-c\varepsilon} < \frac{1-(1+L)c\varepsilon}{1-c\varepsilon}.$$

This inequality is equivalent to

$$c\varepsilon\left(\varepsilon + \tfrac{\pi}{16}\right) < \tfrac{\pi}{16} - L\varepsilon. \tag{5.8}$$

Now we set $\varepsilon_1 = \min(\varepsilon_0, \pi/(16L))$, fix $\varepsilon \in (0, \varepsilon_1)$ and set

$$C_4 = \min\left(C_3, \frac{\frac{\pi}{16} - L\varepsilon}{\varepsilon(\varepsilon + \frac{\pi}{16})}\right).$$

If $0 < c < C_4$ then (5.8) holds, so Φ is Lipschitz continuous with some constant $\lambda < 1$. □

6. Main result

Theorem 6.1. *For every $L, \varepsilon > 0$ there exists $C_{L,\varepsilon} > 0$ such that for every $c \in (0, C_{L,\varepsilon})$ there exists a function $\omega \in \mathcal{L}_{\varepsilon,L}$ whose graph Γ is G_c-invariant in the strong sense ($G_c(\Gamma) = \Gamma$) and attracts exponentially G_c-trajectories of all points from D_1 except the origin.*

Proof. Fix $L, \varepsilon > 0$. We may make ε smaller, since $\mathcal{L}_{\varepsilon',L} \subset \mathcal{L}_{\varepsilon,L}$ if $\varepsilon' \le \varepsilon$. Thus, we may assume that $\varepsilon < \varepsilon_1$, where ε_1 is the constant from Lemma 5.6. Then we set $C_{L,\varepsilon} = C_4$, where C_4 is the constant from Lemma 5.6. Fix $c \in (0, C_4)$. By Lemma 5.6, the operator $\Phi : \mathcal{L}_{\varepsilon,L} \to \mathcal{L}_{\varepsilon,L}$ is a contraction, and since the space $\mathcal{L}_{\varepsilon,L}$ is complete, there exists a function $\omega \in \mathcal{L}_{\varepsilon,L}$ which is a fixed point of Φ and attracts exponentially $G_c^{n(c)}$-trajectories of all functions from $\mathcal{L}_{\varepsilon,L}$. Since ω is a fixed point of $G_c^{n(c)}$ and $G_c^{n(c)}$ is a homeomorphism, for the graph Γ of ω we have $G_c^{n(c)}(\Gamma) = \Gamma$.

Consider two constant functions in $\mathcal{L}_{\varepsilon,L}$, $\chi_- \equiv 3\pi/32 - 2\varepsilon$ and $\chi_+ \equiv 3\pi/32 + 2\varepsilon$. By Lemma 5.6 there exist constants $N > 0$ and $\lambda \in (0, 1)$ such that $d(\Phi^{kn(c)}(\chi_\pm), \omega) \le N\lambda^k$ for all $k \ge 0$. Since G_c is a homeomorphism, the set $G_c^{kn(c)}(H_{2\varepsilon})$ is bounded by the graphs of $\Phi^{kn(c)}(\chi_\pm)$. Hence, for every $p \in H_{2\varepsilon}$ the distance along the rays from the origin, and therefore also the Euclidean distance, of $G_c^{kn(c)}(p)$ from Γ, is bounded by $N\lambda^k$. In particular, the intersection of the images of $H_{2\varepsilon}$ under the iterates of $G_c^{n(c)}$ is Γ.

Take now any point $p \in D_1$ which is not the origin. By Proposition 5.2 (d), its G_c-trajectory eventually stays in $H_{2\varepsilon}$. Therefore there is i such that the points $G_c^i(p), G_c^{i+1}(p), \dots, G_c^{i+n(c)-1}(p)$ belong to $H_{2\varepsilon}$. By the preceding paragraph, $G_c^{n(c)}$-trajectories of all those points are attracted exponentially to Γ. Hence, the G_c-trajectory of p is attracted exponentially to Γ.

Look at $G_c(\Gamma)$. By Proposition 5.2 (b) it is contained in $H_{3\varepsilon}$. By Lemma 5.1 and Proposition 5.2 (a) there is j such that $G_c^{jn(c)}(H_{3\varepsilon}) \subset H_{2\varepsilon}$. Since $G_c^{n(c)}(\Gamma) = \Gamma$, we have also $G_c^{n(c)}(G_c(\Gamma)) = G_c(\Gamma)$. Therefore $G_c(\Gamma) \subset H_{2\varepsilon}$, and thus $G_c(\Gamma)$ is contained in the intersection of the images of $H_{2\varepsilon}$ under the iterates of $G_c^{n(c)}$, which is Γ. The set Γ is homeomorphic to a circle, and G_c is a homeomorphism. Therefore $G_c(\Gamma)$ is also homeomorphic to a circle, so since it is contained in Γ, it must be equal to Γ. This proves that Γ is G_c-invariant in the strong sense. □

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