

ROTATION THEORY

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ABSTRACT. Rotation Theory has its roots in the theory of rotation numbers for circle homeomorphisms, developed by Poincaré. It is particularly useful for the study and classification of periodic orbits of dynamical systems. It deals with ergodic averages and their limits, not only for almost all points, like in Ergodic Theory, but for all points. We present the general ideas of Rotation Theory and its applications to some classes of dynamical systems, like continuous circle maps homotopic to the identity, torus homeomorphisms homotopic to the identity, subshifts of finite type and continuous interval maps.

These notes present a brief survey of some aspects of Rotation Theory. Deeper treatment of the subject would require writing a book. Thus, in particular:

- Not all aspects of Rotation Theory are described here. A large part of it, very important and deserving a separate book, deals with homeomorphisms of an annulus, homotopic to the identity. Including this subject in the present notes would make them at least twice as long, so it is ignored, except for the bibliography.
- What is called “proofs” are usually only short ideas of the proofs. The reader is advised either to try to fill in the details himself/herself or to find the full proofs in the literature. More complicated proofs are omitted entirely.
- The stress is on the theory, not on the history. Therefore, references are not cited in the text. Instead, at the end of these notes there are lists of references dealing with the problems treated in various sections (or not treated at all).

1. CIRCLE MAPS

Consider the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the natural projection $\pi : \mathbb{R} \rightarrow \mathbb{T}$. If $f : \mathbb{T} \rightarrow \mathbb{T}$ is a continuous map, then there is a continuous map $F : \mathbb{R} \rightarrow \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{T} & \xrightarrow{f} & \mathbb{T} \end{array}$$

commutes. Such F is called a *lifting* of f . It is unique up to a translation by an integer ($\tilde{F}(x) = F(x) + k$, $k \in \mathbb{Z}$). There is an integer d such that $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$. It is called the *degree* of f and is independent of the choice of lifting. Denote by \mathcal{L}_1 the family of all liftings of continuous degree one maps of \mathbb{T} into itself.

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Let $F \in \mathcal{L}_1$. If $k \in \mathbb{Z}$ then $F(x+k) = F(x) + k$. All iterates of F also belong to \mathcal{L}_1 , so $F^n(x+k) = F^n(x) + k$.

We define *upper and lower rotation numbers* of $x \in \mathbb{R}$ for $F \in \mathcal{L}_1$ as

$$\bar{\varrho}_F(x) = \limsup_{n \rightarrow \infty} \frac{F^n(x) - x}{n}, \quad \underline{\varrho}_F(x) = \liminf_{n \rightarrow \infty} \frac{F^n(x) - x}{n}.$$

If $\underline{\varrho}_F(x) = \bar{\varrho}_F(x)$, we write $\varrho_F(x)$ and call it the *rotation number* of x for F .

If $F \in \mathcal{L}_1$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, then

$$\bar{\varrho}_F(x+k) = \bar{\varrho}_F(x), \quad \bar{\varrho}_{F^n-k}(x) = n \cdot \bar{\varrho}_F(x) - k$$

and similarly for $\underline{\varrho}$. If F is a lifting of f and $f^n(\pi(x)) = \pi(x)$ then $F^n(x) = x+k$ for some $k \in \mathbb{Z}$ and $\varrho_F(x) = k/n$.

Let \mathcal{L}'_1 be the space consisting of all nondecreasing elements of \mathcal{L}_1 .

Theorem 1.1. *If $F \in \mathcal{L}'_1$ is a lifting of a circle map f then $\varrho_F(x)$ exists for all $x \in \mathbb{R}$ and is independent of x . Moreover, it is rational if and only if f has a periodic point.*

We will call this number the *rotation number* of F and denote it by $\varrho(F)$.

Proof. Take a rational number k/n . If there is x such that $F^n(x) - k = x$ then $\varrho(F) = k/n$. If there is no such x then the whole graph of $F^n(x) - k$ lies either above the diagonal (and then $\underline{\varrho}_F(x) > k/n$ for all $x \in \mathbb{R}$) or below the diagonal (and then $\bar{\varrho}_F(x) < k/n$ for all $x \in \mathbb{R}$). This defines a Dedekind cut that corresponds to $\varrho(F)$. \square

The function $\varrho : \mathcal{L}'_1 \rightarrow \mathbb{R}$ is continuous.

For $F \in \mathcal{L}_1$ define $F_l, F_u \in \mathcal{L}'_1$ by “pouring water” from below and above respectively (see Figure 1):

$$F_l(x) = \inf\{F(y) : y \geq x\}, \quad F_u(x) = \sup\{F(y) : y \leq x\}.$$

Let $\text{Const}(G)$ be the union of all open intervals on which G is constant.

We list the basic properties of F_l and F_u :

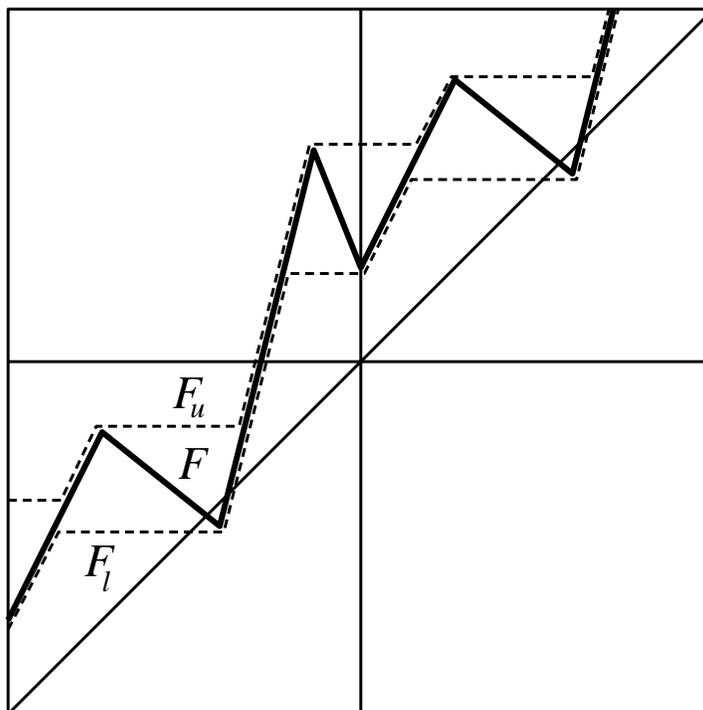
- (1) $F_l(x) \leq F(x) \leq F_u(x)$ for all $x \in \mathbb{R}$,
- (2) if $F \leq G$ then $F_l \leq G_l$ and $F_u \leq G_u$,
- (3) if $F \in \mathcal{L}'_1$ then $F_l = F_u = F$,
- (4) the maps $F \mapsto F_l$ and $F \mapsto F_u$ are Lipschitz continuous with constant 1 (in the sup norm),
- (5) the maps $F \mapsto \varrho(F_l)$ and $F \mapsto \varrho(F_u)$ are continuous,
- (6) if $F_l(x) \neq F(x)$ then $x \in \text{Const}(F_l)$; similarly for F_u ,
- (7) $\text{Const}(F) \subset \text{Const}(F_l) \cap \text{Const}(F_u)$.

For a map $f : X \rightarrow X$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, 2, \dots, n-1$, we call $\{x, f(x), \dots, f^{n-1}(x)\}$ a *cycle* (or a *periodic orbit*) of *period* n , or an *n -cycle*.

Lemma 1.2. *If $F \in \mathcal{L}'_1$ is a lifting of a circle map f and $\varrho(F) = k/n$ for coprime integers k, n then f has a n -cycle P such that $\pi^{-1}(P)$ is disjoint from $\text{Const}(F)$.*

Proof. Make a picture and observe how the graph of $F^n - k$ crosses the diagonal. \square

Let $F \in \mathcal{L}_1$. Then we define $F_\mu = (\min(F, F_l + \mu))_u$ for $\mu \in [0, \lambda]$, where $\lambda = \sup_{x \in \mathbb{R}} (F - F_l)(x)$ (see Figure 2). It has the following properties:


 FIGURE 1. Maps F_l and F_u

- (1) $F_\mu \in \mathcal{L}'_1$ for all μ ,
- (2) $F_0 = F_l$ and $F_\lambda = F_u$,
- (3) the map $\mu \mapsto F_\mu$ is Lipschitz continuous with constant 1,
- (4) the map $\mu \mapsto \varrho(F_\mu)$ is continuous,
- (5) if $\mu \leq \varkappa$ then $F_\mu \leq F_\varkappa$,
- (6) each F_μ coincides with F outside $\text{Const}(F_\mu)$,
- (7) $\text{Const}(F) \subset \text{Const}(F_\mu)$ for each μ .

Theorem 1.3. *Let $F \in \mathcal{L}_1$ be a lifting of a circle map f . Then the set of all rotation numbers of points is equal to $[\varrho(F_l), \varrho(F_u)]$. Moreover, for each rational a from this interval there is a point x such that $\pi(x)$ is periodic for f and $\varrho_F(x) = a$.*

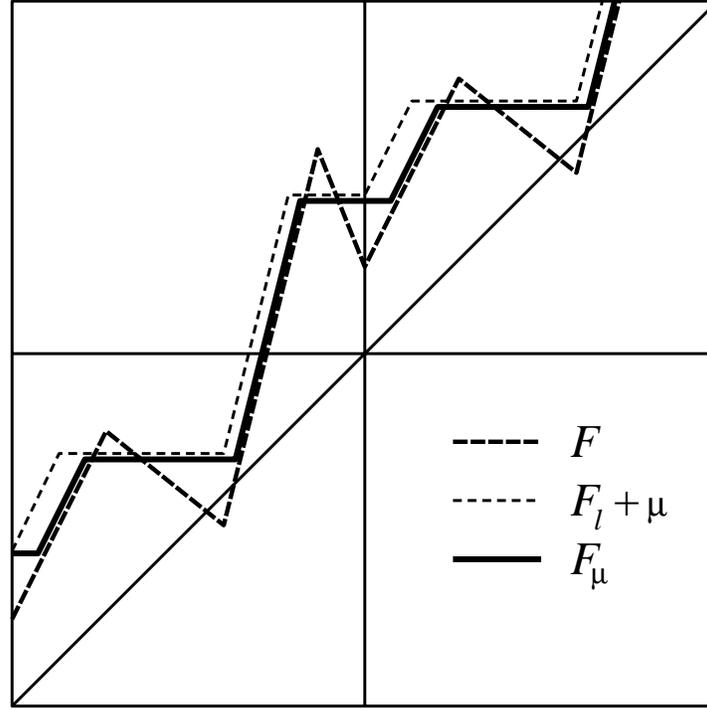
The interval $[\varrho(F_l), \varrho(F_u)]$ is called the *rotation interval* of F . We will denote it by $\text{Rot}(F)$.

Proof. Use the family F_μ and its properties. Additionally use Lemma 1.2 and its version for irrational rotation numbers. \square

Corollary 1.4. *The endpoints of $\text{Rot}(F)$ depend continuously on $F \in \mathcal{L}_1$.*

Sets of periods

Consider the following *Sharkovskiy's ordering* in the set \mathbb{N}_{Sh} of natural numbers together with 2^∞ : $3_{Sh} > 5_{Sh} > 7_{Sh} > \dots_{Sh} > 2 \cdot 3_{Sh} > 2 \cdot 5_{Sh} > 2 \cdot 7_{Sh} > \dots_{Sh} > 4 \cdot 3_{Sh} > 4 \cdot 5_{Sh} > 4 \cdot 7_{Sh} > \dots_{Sh} > 2^n \cdot 3_{Sh} > 2^n \cdot 5_{Sh} > 2^n \cdot 7_{Sh} > \dots_{Sh} > 2^\infty_{Sh} >$

FIGURE 2. Construction of the map F_μ

$\dots s_h > 2^n s_h > \dots s_h > 16 s_h > 8 s_h > 4 s_h > 2 s_h > 1$. For any $s \in \mathbb{N}_{s_h}$ in this ordering denote by $\text{Sh}(s)$ the set of all natural numbers n such that $s_{s_h} > n$, together with s (unless $s = 2^\infty$).

Theorem 1.5 (Sharkovsky). *If f is a continuous map of a closed interval into itself then there is $s \in \mathbb{N}_{s_h}$ such that the set of all periods of cycles of f is equal to $\text{Sh}(s)$. Conversely, every set $\text{Sh}(s)$ is the set of all periods of cycles of f for some interval map f .*

For $c \leq d$ denote by $M(c, d)$ the set of natural numbers n such that $c < k/n < d$ for some integer k . For $a \in \mathbb{R}$ and $s \in \mathbb{N}_{s_h}$ denote by $\text{Sh}(a, s)$ the empty set if a is irrational and the set $\{nq : q \in \text{Sh}(s)\}$ if $a = k/n$ with k, n coprime.

We say that a continuous map g of an interval or a real line into itself has a *horseshoe* if there exist closed intervals I, J , disjoint except perhaps a common endpoint, such that $I \cup J \subset g(I) \cap g(J)$. Existence of a horseshoe implies existence of cycles of all periods.

Lemma 1.6. *If $F \in \mathcal{L}_1$ is a lifting of f and $\varrho(F_l) < 0 < \varrho(F_u)$ then f has cycles of all periods.*

Proof. There are $x < y$ such that $\varrho_F(y) < 0 < \varrho_F(x)$. Look at the orbits of x and y . Produce a horseshoe (see Figure 3; t and t' belong to the orbit of y ; z and z' belong to the orbit of x). \square

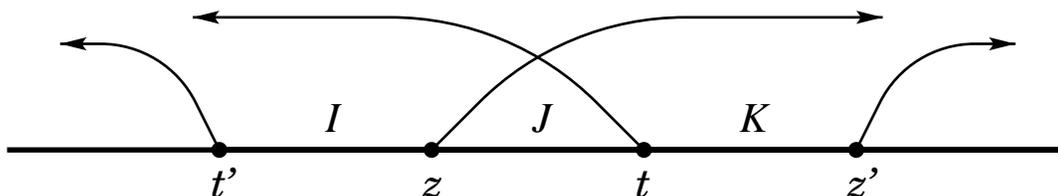


FIGURE 3. Situation from the proof of Lemma 1.6

Theorem 1.7. *Let $F \in \mathcal{L}_1$ be a lifting of a circle map f and let $\text{Rot}(F) = [c, d]$. Then there are $s_c, s_d \in \mathbb{N}_{Sh}$ such that the set of all periods of cycles of f is equal to*

$$\text{Sh}(c, s_c) \cup M(c, d) \cup \text{Sh}(d, s_d).$$

Conversely, for every set of this form there is f and its lifting $F \in \mathcal{L}_1$ with $\text{Rot}(F) = [c, d]$ and this set of periods.

Proof. Use Lemma 1.6 for maps of the form $F^n - k$ to show that the set of periods of cycles of f with rotation numbers from (c, d) is $M(c, d)$. Use Sharkovsky's Theorem for cycles with rotation numbers c and d . The converse part requires some extra work. \square

2. GENERAL FORMALISM

Another point of view on rotation numbers for degree one circle maps is the following. Let $F \in \mathcal{L}_1$ be a lifting of a circle map f ; define the *displacement function* $\varphi : \mathbb{T} \rightarrow \mathbb{R}$ by $\varphi(x) = F(y) - y$, where $\pi(y) = x$ (it is independent of the choice of $y \in \pi^{-1}(x)$). Then $F^n(y) - y = \sum_{j=0}^{n-1} \varphi(f^j(x))$. Thus,

$$\varrho_F(y) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)).$$

We can generalize it to the following situation. Let X be a compact metric space, $f : X \rightarrow X$ a continuous map and $\varphi : X \rightarrow \mathbb{R}^d$ a Borel bounded (usually continuous) function (an *observable*). If for $x \in X$ the limit

$$\varrho_{f,\varphi}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exist, we call it the *rotation vector* of x . The set $\text{Rot}_p(f, \varphi)$ of all rotation vectors of points of X is the *pointwise rotation set* of f for the observable φ . The *general rotation set* $\text{Rot}(f, \varphi)$ of f for the observable φ is the set of all limits of the sequences of the form

$$\frac{1}{n_i} \sum_{j=0}^{n_i-1} \varphi(f^j(x_i)),$$

where $x_i \in X$ and $n_i \rightarrow \infty$. For an ergodic invariant probability measure μ , its *rotation vector* $\varrho_{f,\varphi}(\mu)$ is the integral $\int \varphi d\mu$. The set $\text{Rot}_m(f, \varphi)$ of all rotation

vectors of ergodic invariant probability measures is the *measure rotation set* of f for the observable φ .

Clearly, $\text{Rot}_p(f, \varphi) \subset \text{Rot}(f, \varphi)$. For an ergodic measure μ , by the Birkhoff Ergodic Theorem, $\varrho_{f, \varphi}(x) = \varrho_{f, \varphi}(\mu)$ for μ -almost every x , and thus $\text{Rot}_m(f, \varphi) \subset \text{Rot}_p(f, \varphi)$.

If P is an n -cycle of f then for $x \in P$

$$\varrho_{f, \varphi}(x) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu_P$$

for the probability measure μ_P equidistributed on P .

In order to use rotation theory efficiently for investigating cycles, we need *completeness*, that is we want the following properties to hold:

- (1) the rotation set is convex;
- (2) the rotation vector of an n -cycle is of the form $(m_1/n, \dots, m_d/n)$, where m_1, \dots, m_d are integers;
- (3) if $(m_1/n, \dots, m_d/n)$ belongs to the interior of the rotation set and the greatest common divisor of m_1, \dots, m_d, n is 1 then there exists an n -cycle with rotation vector $(m_1/n, \dots, m_d/n)$,
- (4) if $(m_1/n, \dots, m_d/n)$ belongs to the interior of the rotation set and there exists an n -cycle with rotation vector $(m_1/n, \dots, m_d/n)$ then there exists a kn -cycle with rotation vector $(m_1/n, \dots, m_d/n)$ for any positive integer k .

Condition (2) above is automatically satisfied if φ takes values from \mathbb{Z}^d . To this end, we often have to replace φ by another observable. Observables φ and ψ are *cohomologous* if there exists a bounded function u such that

$$\varphi - \psi = u \circ f - u.$$

Then

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) - \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k(x)) = \frac{1}{n} (u(f^n(x)) - u(x)),$$

so the rotation vectors for φ and ψ are the same.

For a circle map f we can take $\psi(x) = [F(y)] - [y]$ and $u(x) = y - [y]$, where $\pi(y) = x$ ($[\cdot]$ is the “floor” function). Then

$$\varphi(x) - \psi(x) = F(y) - y - [F(y)] + [y] = u(f(x)) - u(x).$$

Similar replacement can be made for torus maps with the displacement observable (see Section 3).

Symbolic dynamics

Let $\sigma : \Sigma \rightarrow \Sigma$ be a subshift of finite type (a topological Markov chain). Assume that an observable $\varphi : \Sigma \rightarrow \mathbb{R}^d$ is constant on cylinders of length 2 (depends only on the zeroth and first coordinates).

Let G be the directed graph of the subshift. The points of Σ can be identified with infinite paths in G . The function φ can be interpreted as assigning a vector from \mathbb{R}^d to each arrow in G (see Figure 4). To find the rotation vector of the path, we go along it, sum the vectors for the arrows we use, divide by the number of arrows, and pass to the limit (if the limit exists). Cycles correspond to the loops in G .

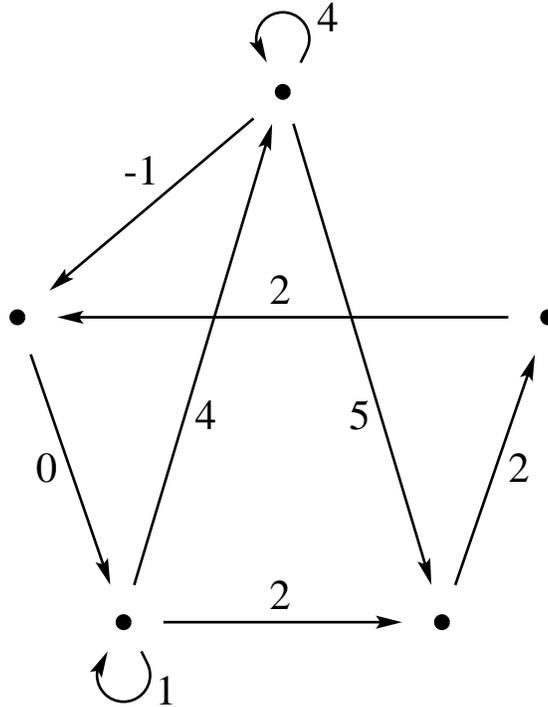


FIGURE 4. An example of a directed graph with elements of \mathbb{R}^1 assigned to arrows

A loop is *elementary* if it does not pass more than once through any vertex. Assume that the subshift is transitive (there is a path from every vertex to every vertex in G). Let τ_1, \dots, τ_s be all the elementary loops in G and let $\varrho_1, \dots, \varrho_s$ be their rotation vectors.

Theorem 2.1. *With the above assumptions and notation, the following properties hold.*

- (1) *The rotation set $\text{Rot}(\sigma, \varphi)$ is equal to the convex hull of $\varrho_1, \dots, \varrho_s$.*
- (2) *For every vector $\varrho \in \text{Rot}(\sigma, \varphi)$ there is a point $x \in \Sigma$ with rotation vector ϱ .*
- (3) *The set of rotation vectors of periodic points is dense in $\text{Rot}(\sigma, \varphi)$.*
- (4) *For every vector ϱ in the interior of $\text{Rot}(\sigma, \varphi)$ there is an ergodic invariant probability measure μ on Σ with rotation vector ϱ .*
- (5) *If φ takes values in \mathbb{Z}^d then for every vector $\varrho \in \mathbb{Q}^d$ which is contained in the interior of $\text{Rot}(\sigma, \varphi)$ there is a periodic point $x \in \Sigma$ with rotation vector ϱ .*

Proof. (1) One can rearrange the arrows of any finite path in order to get elementary loops and be left with the number of arrows less than the number of vertices of G .

(2) Use (1) to build an infinite path with the given rotation vector.

(3) Look at the proof of (1).

(4) Make the construction from the proof of (2) carefully, so that every sufficiently long piece of the path has rotation vector close to the desired one. Then the closure of the orbit of the point (path) we get will be a compact invariant set with the same rotation vector at every point. Then take an ergodic measure supported on this set.

(5) The rotation vector of any loop is in this case rational. We can get finitely many loops passing through a common vertex such that ϱ is a convex combination of their rotation vectors with rational coefficients. Then the appropriate concatenation of the repetitions of those loops will have rotation vector ϱ . \square

Note that this does not prove completeness.

We can use symbolic dynamics for circle maps: Take a cycle of a circle map f ; note which interval between adjacent points of the cycle is mapped over other intervals of this type, code to get a subshift of finite type. The observable measures how far forward in the lifting we move.

Center of mass

If $\varphi(x) = x$ then the rotation number (vector) of a point is the center of mass of its orbit.

Let $f_a(x) = axe^{-x}$ for $a > e$ and $x \in [a^2e^{-1-a/e}, a/e]$. If $f_a^n(x) = x$ and $x_i = f_a^i(x)$ then

$$\prod_{i=0}^{n-1} x_i = a^n \prod_{i=0}^{n-1} x_i \exp\left(-\sum_{i=0}^{n-1} x_i\right),$$

so $\exp(\sum_{i=0}^{n-1} x_i) = a^n$. Therefore $\varrho_{f_a, \varphi}(x) = \ln a$. Since any orbit of those maps can be approximated by cycles, rotation numbers of all points are $\ln a$. Are there any other smooth interval maps with this property?

Another example is the tent map. For a cycle with a given itinerary it is easy to compute its center of mass. For the mirror itinerary the center of mass is the same. Can there be 3 different cycles with the same center of mass?

3. TORUS MAPS

Let $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be a continuous map, homotopic to the identity. Then for a lifting $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ we have $F(x+k) = F(x) + k$ if $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$. Therefore the displacement function $\varphi : \mathbb{T}^d \rightarrow \mathbb{R}^d$ is well defined by $\varphi(x) = F(y) - y$, where $\pi(y) = x$ (here $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$ is the natural projection).

Unfortunately, the rotation theory is “reasonable” only if $d = 2$ and f is a homeomorphism. Let us give examples of what can go wrong in other cases.

Example 3.1. A continuous map $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, homotopic to the identity, with bad properties.

Choose two disjoint balls of radius $1/10$, $U, V \subset \mathbb{T}^2$. In each of them choose a segment, J and K . Construct f in such a way that (see Figure 5):

- (1) the lifting F of f is equal to the identity outside $U \cup V$,
- (2) F does not move any point of $\pi^{-1}(U)$ more than $1/4$ to the left or down and more than by $5/4$ to the right or up, and does not move any point of $\pi^{-1}(V)$ more than by $5/4$ to the left or down and more than by $1/4$ to the right or up,
- (3) $f(U) \cap V = f(V) \cap U = \emptyset$,
- (4) F stretches a component \hat{J} of $\pi^{-1}(J)$ and \hat{K} of $\pi^{-1}(K)$, mapping them through \hat{J} , $\hat{J} + (0, 1)$, $\hat{J} + (1, 0)$ and \hat{K} , $\hat{K} - (0, 1)$, $\hat{K} - (1, 0)$ respectively.

An example of a point which belongs
to $\text{Conv}(\rho(F))$ but not to $\rho(F)$

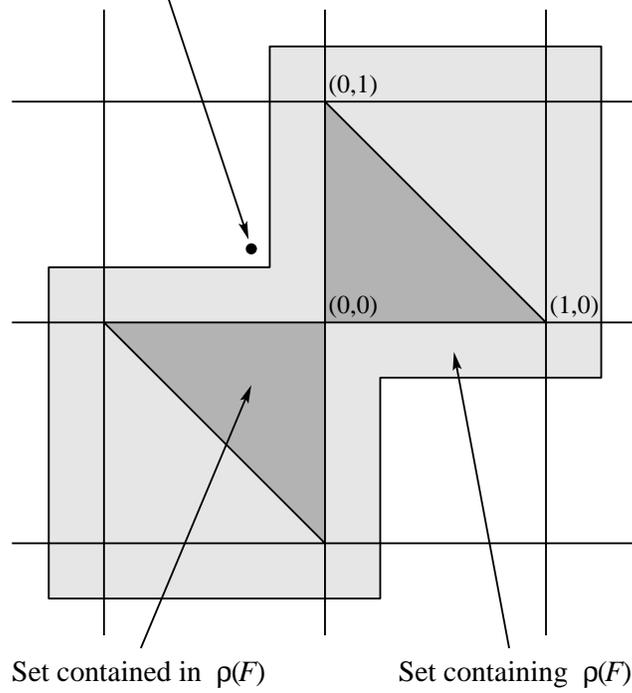


FIGURE 6. The rotation set in Example 3.1 is not convex

- (2) If $\text{Rot}(f, \varphi)$ has a nonempty interior, P is a finite subset of $\bigcup \varrho_{f, \varphi}(x)$, where the union is taken over all periodic points x of f , and C is a compact connected subset of $\text{Conv}(P)$, then there exists $y \in \mathbb{T}^2$ with $\varrho_{f, \varphi}(y) = C$. In particular (since C can consist of one point), for every $v \in \text{int}(\text{Rot}(f, \varphi))$ there exists $y \in \mathbb{T}^2$ with $\varrho_{f, \varphi}(y) = \{v\}$.
- (3) The set $\text{Rot}(f, \varphi)$ is convex.
- (4) $\text{Rot}(f, \varphi) = \text{Conv}(\text{Rot}_m(f, \varphi)) = \text{Conv}(\text{Rot}_p(f, \varphi))$.
- (5) $\text{int}(\text{Rot}(f, \varphi)) \subset \text{Rot}_m(f, \varphi)$.

Proof. (1) Look at $F^n - (p, q)$. Since $v \in \text{int}(\text{Rot}(f, \varphi))$, there are 4 points with trajectories moving in directions of different vectors, with the origin in the interior of the convex hull of those vectors. They can be chosen in the same δ -transitive component of the chain recurrent set (use Conley Theory and Nielsen-Thurston Theory). Use them to create a δ -chain. Deduce that $F^n - (p, q)$ has a fixed point.

(2) Remove finitely many periodic orbits with rotation vectors in P from the torus. Use Nielsen-Thurston theory on the resulting punctured torus. Show that F is homotopic to a pseudo-Anosov homeomorphism. Use symbolic dynamics.

(3) We have

$$\text{Rot}(f, \varphi) = \bigcap_{n \geq 1} \overline{\bigcup_{j \geq n} K_j(F)},$$

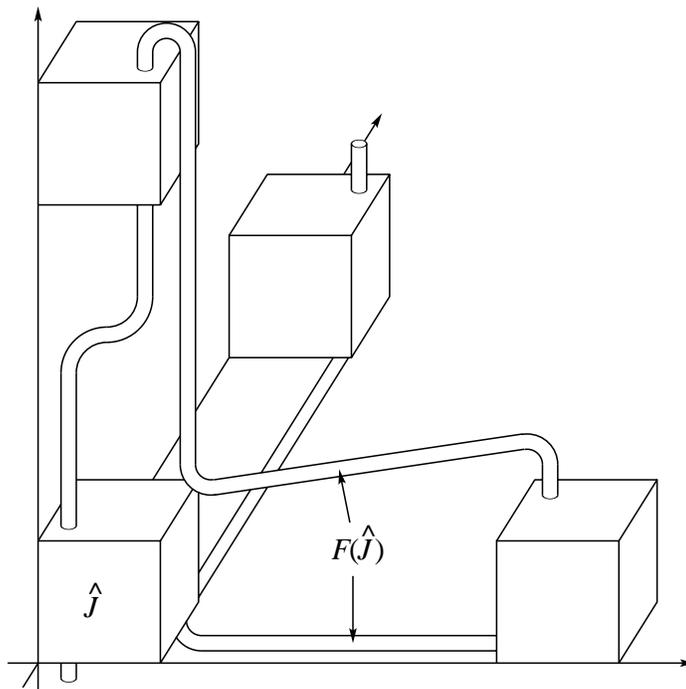


FIGURE 7. Construction from Example 3.2

where

$$K_j(F) = \left\{ \frac{F^j(x) - x}{j} : x \in \mathbb{R}^2 \right\} = \left\{ \frac{F^j(x) - x}{j} : x \in [0, 1]^2 \right\}.$$

Using this representation, prove that $\text{Conv}(F^n([0, 1]^2))$ is contained in the $\sqrt{2}$ -neighborhood of $F^n([0, 1]^2)$ (by examining the images of simple curves in \mathbb{R}^2).

(4) Use (2) and prove that the extremal points of the set $\text{Conv}(\text{Rot}(f, \varphi))$ belong to $\text{Rot}_m(f, \varphi)$ (take a weak-* limit of measures concentrated on relevant pieces of orbits and then look at its ergodic components).

(5) Use the same method as in the proof of (2). □

Theorem 3.4. *The function ϱ from \mathcal{H}_2 into the space of all subsets of \mathbb{R}^2 is upper semi-continuous, i.e. if $f \in \mathcal{H}_2$ and U is a neighborhood of $\text{Rot}(f, \varphi)$ in \mathbb{R}^2 then there exists a neighborhood V of f in \mathcal{H}_2 with the topology of uniform convergence such that if $g \in V$ then $\varrho(g, \psi) \subset U$ (where ψ is a displacement function for g , corresponding to the lifting of g closest to the lifting of f corresponding to φ).*

At all f with $\text{int}(\text{Rot}(f, \varphi)) \neq \emptyset$ it is continuous.

Proof. Use the characterization of $\text{Rot}(f, \varphi)$ via the sets $K_n(F)$. To get continuity when $\text{int}(\text{Rot}(f, \varphi)) \neq \emptyset$, use the fact that the periodic points obtained from the pseudo-Anosov homeomorphism have non-zero indices, so they cannot be removed by a small perturbation. □

It is not so difficult to construct a homeomorphism from \mathcal{H}_2 with the rotation set equal to the prescribed convex polygon with rational vertices. With more work,

this can be done for a convex “polygon” with infinitely many vertices (all of them rational). From Theorem 3.4 it follows that there must be other rotation sets. Can one realize every compact convex set with a nonempty interior as the rotation set of a homeomorphism from \mathcal{H}_2 ?

4. INTERVAL MAPS

Denote the family of all continuous maps of the interval $\mathbb{I} = [0, 1]$ into itself by \mathcal{I} . Assume first that $f \in \mathcal{I}$ has a unique fixed point, a . Then we can introduce rotation numbers which for a cycle count how many of its points are to the right of a , compared to the total number of points. We can do it in a straightforward way, by taking the observable φ_a equal 1 to the right of a and 0 to the left of a . Since we want it to be defined on the whole interval, we set $\varphi_a(a) = 1/2$.

Observe that if $x < a$ then $f(x) > x$ and if $x > a$ then $f(x) < x$. Therefore $\varphi_a = \varphi$, where φ is defined by $\varphi(x) = 1$ if $f(x) < x$, $\varphi(x) = 0$ if $f(x) > x$ and $\varphi(x) = 1/2$ if $f(x) = x$. This definition works equally well if f has more than one fixed point.

There is another way to get similar rotation numbers. Set $\varphi_o(x) = 1/2$ if $(f^2(x) - f(x))(f(x) - x) \leq 0$ and 0 otherwise. If there is only one fixed point a , instead of counting how many times the point of an orbit is to the right of a , we count how many times it is to the right of a and its image is to the left of a .

To see this, take the function u defined by $u(x) = 1/2$ if $x \geq a$ and 0 otherwise, and note that $\psi = \varphi_o + u - u \circ f$ is defined by $\psi(x) = 1$ if $x > a$ and $f(x) < a$, $\psi(x) = 1/2$ if $x \geq a$ and $f(x) = a$, and $\psi(x) = 0$ otherwise. When we use the observable φ_o , we will refer to the rotation numbers as *over-rotation numbers* (when we use the observable φ , we will speak simply of rotation numbers).

In general, rotation and over-rotation numbers are different even for cycles. When there is a unique fixed point a and all points to the right of a are mapped to the left of a (or all points to the left of a are mapped to the right of a) then the rotation and over-rotation numbers coincide. This is the case for unimodal maps.

Theorem 4.1. *Rotation and over-rotation theories for interval maps are complete (except that the rotation number for a fixed point is $1/2$). In particular, the rotation and over-rotation sets are intervals.*

Proof. Although formally the over-rotation number of an n -cycle is of the form $k/(2n)$, the number k is even.

Look at the cycles of the map. If a given cycle forces two different fixed points (that is, any continuous interval map with this cycle has to have at least 2 fixed points), use symbolic dynamics to show that all possible (over-)rotation numbers (and cycles with all possible (over-)rotation numbers and compatible periods) occur. Otherwise, use symbolic dynamics by looking at the intervals joining points of the cycle with the fixed point. When returning from the symbolic setting, show that the obtained trajectories avoid the fixed point (essentially, the fixed point is repelling). \square

The (over-)rotation interval always contains $1/2$. On the other hand, the rotation interval is always contained in $[0, 1]$, while the over-rotation interval is contained in $[0, 1/2]$.

Assume that an interval map f has a cycle P of an odd period $n \geq 3$ and that $k > n$ is an odd integer. Since the rotation number of P is of the form p/n for some integer

p (the over-rotation number is of the same form) and there is a rational number with denominator k between p/n and $1/2$, there must be a cycle of f of period k . Thus, from the completeness of the rotation theory (as well as from the completeness of the over-rotation theory) for interval maps we get the essential part of the Sharkovsky's Theorem.

Let us concentrate on the over-rotation theory. For a cycle P its *over-rotation pair* is the pair (p, q) , where p/q is the over-rotation number of P and q the period of P . Because of completeness and the fact that $1/2$ is always an endpoint of the over-rotation interval, we can introduce a *forcing relation* among the over-rotation pairs. This means that if (p, q) forces (r, s) and a continuous interval map has a cycle with over-rotation pair (p, q) , then it has to have a cycle with over-rotation pair (r, s) . Here, a pair (p, q) forces (r, s) if either $p/q < r/s$ or $p/q = r/s = m/n$ with m and n coprime and $p/m_{Sh} > r/m$ (notice that $p/m, r/m \in \mathbb{N}$).

This relation is a linear ordering. Therefore the theorem characterizing all possible sets of over-rotation pairs for interval maps will be stated in a similar way to the Sharkovsky's Theorem. The definition of those sets of over-rotation pairs will be similar as in the case of circle maps of degree one.

Let \mathcal{M} be the set consisting of $0, 1/2$, all irrational numbers between 0 and $1/2$, and all pairs (α, n) , where α is a rational number from $(0, 1/2]$ and $n \in \mathbb{N}_{Sh}$. Then for $\eta \in \mathcal{M}$ the set $O(\eta)$ is equal to the following. If η is an irrational number, 0 , or $1/2$, then $O(\eta)$ is the set of all pairs (p, q) with $\eta < p/q \leq 1/2$. If $\eta = (r/s, n)$ with r, s coprime, then $O(\eta)$ is the union of the set of all pairs (p, q) with $r/s < p/q \leq 1/2$ and the set of all pairs (mr, ms) with $m \in \text{Sh}(n)$.

Theorem 4.2. *For every $f \in \mathcal{I}$ there exists $\eta \in \mathcal{M}$ such that the set of all over-rotation pairs of f is equal to $O(\eta)$. Conversely, for every $\eta \in \mathcal{M}$ there exists a map $f \in \mathcal{I}$ such that the set of all over-rotation pairs of f is equal to $O(\eta)$.*

Proof. The first part basically follows from Theorem 4.1 and the linearity of the forcing order in \mathcal{M} . To prove the second part, use the family of truncated tent maps. □

Minimal topological entropy

Topological entropy measures the chaoticity of the system. For a piecewise monotone interval or circle map f it is equal to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log c_n(f),$$

where $c_n(f)$ is the number of laps (monotonicity pieces) of f^n . Often estimates of entropy are added to the theorems on the sets of periods or the rotation sets. Here are some of them. They are sharp.

If a continuous interval map has a cycle of period $n \cdot 2^k$, where n is odd, then its entropy is at least $(1/2^k) \log \lambda_n$, where $\lambda_1 = 1$ and λ_n is the largest zero of the polynomial $x^n - 2x^{n-2} - 1$ for $n \geq 3$ odd.

If a continuous circle map of degree 1 has rotation interval $[c, d]$ with $c < d$, then its entropy is at least $\log \beta_{c,d}$, where $\beta_{c,d}$ is the largest root of the equation $\sum t^{-q} = 1/2$. Here the sum is taken over all pairs of integers (p, q) such that $q > 0$ and $c < p/q < d$.

If a continuous interval map over-rotation interval $[c, 1/2]$ with $c < 1/2$, then its entropy is at least $\log \gamma_c$, where γ_c is the largest root of the equation $\sum t^{-q} = 1$. Here the sum is taken over all pairs of integers (p, q) such that $q > 0$ and $c < p/q < 1 - c$. In particular, if a continuous interval map has a cycle with rotation pair (p, q) then its entropy is at least $\log \gamma_{p/q}$ if $p/q < 1/2$. If $p/q = 1/2$ then the entropy is at least $(1/2^k) \log \lambda_n$, where $q = n \cdot 2^k$ and n is odd.

5. FLOWS

Rotation theory can be applied not only to systems with discrete time, but also to systems with continuous time. Then, if X is a compact metric space, instead of a map $f : X \rightarrow X$ we have a *continuous semiflow* on X , that is, a continuous function $T : [0, \infty) \times X \rightarrow X$, such that $T(0, x) = x$ and $T(s + t, x) = T(t, T(s, x))$ for every $x \in X$, $s, t \in [0, \infty)$. We will often write $T^t(x)$ instead of $T(t, x)$. Let ξ be an *time-Lipschitz continuous observable cocycle* for (X, T) with values in \mathbb{R}^m , that is, a continuous function $\xi : [0, \infty) \times X \rightarrow \mathbb{R}^m$ such that $\xi(s + t, x) = \xi(s, T^t(x)) + \xi(t, x)$ and $\|\xi(t, x)\| \leq Lt$ for some constant L independent of t and x .

The *rotation set* R of (X, T, ξ) is the set of all limits

$$\lim_{n \rightarrow \infty} \frac{\xi(t_n, x_n)}{t_n}, \quad \text{where } \lim_{n \rightarrow \infty} t_n = \infty.$$

By the definition, R is closed, and is contained in the closed ball in \mathbb{R}^m , centered at the origin, of radius L . In particular, R is compact. It is easy to show that it is also connected.

When we work with invariant measures, we have to use a slightly different formalism. Namely, the observable cocycle ξ has to be the integral of the *observable function* ζ along an orbit piece. That is, $\zeta : X \rightarrow \mathbb{R}^m$ is a bounded Borel function, integrable along the orbits, and

$$\xi(t, x) = \int_0^t \zeta(T^s(x)) ds.$$

Assume that T is a continuous semiflow. Then, if μ is a probability measure, invariant and ergodic with respect to T , then by the Birkhoff Ergodic Theorem, for μ -almost every point $x \in X$ the *rotation vector*

$$\lim_{t \rightarrow \infty} \frac{\xi(t, x)}{t}$$

of x exists and is equal to $\int_X \zeta(x) d\mu(x)$.

We do not have to assume that ξ is continuous. A reasonable assumption is that the set of its discontinuity points has measure zero for every T -invariant probability measure.

The simplest (non-trivial) example is a flow on a two-dimensional torus, with the displacement cocycle.

Theorem 5.1. *If T is a continuous flow on \mathbb{T}^2 and ξ the displacement cocycle, then the rotation set of (T, ξ) is one of the following:*

- (1) *a set consisting of a single point of \mathbb{R}^2 ;*
- (2) *a segment of a line passing through the origin and some other point of \mathbb{Q}^2 (the segment need not contain the origin);*

(3) *a line segment with one end at the origin and having irrational slope.*

Another example is a billiard on the m -dimensional torus with one or more obstacles, with the displacement cocycle. If there is only one obstacle, which is strictly convex and small, the rotation set contains a subset with nonempty interior, with properties similar to those listed in Theorem 2.1.

6. BIBLIOGRAPHY

The list of references is quite long, but nevertheless it does not contain all relevant positions. The reader wanting to know everything about Rotation Theory can try to look at the papers cited in the papers listed below, and look for the newer papers, that are perhaps omitted here.

Sometimes the papers span several areas of the Rotation Theory, so it is difficult to say to which area they belong. However, we can divide them approximately as follows.

- Circle maps (Section 1): [1], [2], [3], [5], [9], [10], [11], [12], [13], [26], [32], [44], [45], [56], [58], [59], [60], [61], [66], [76], [79], [83], [85], [88], [89], [91], [92], [93]. Some discontinuous circle maps (for instance Lorenz-like maps) can also be studied using Rotation Theory: [4], [50], [69], [77], [80], [84], [90].
- General formalism (Section 2): [19], [31], [51], [62], [63], [64], [65], [94].
- Torus maps (Section 3): [46], [48], [54], [70], [71], [73], [78], [86], [87].
- Interval maps (Section 4): [18], [20], [21], [22], [23], [24], [27], [28], [29].
- Flows (Section 5): [25], [49].
- Annulus maps: [6], [7], [8], [14], [15], [16], [17], [30], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [47], [52], [53], [55], [57], [67], [68], [72], [74], [75], [81], [82].

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