# OMEGA-LIMIT SETS FOR SPIRAL MAPS

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Abstract. We investigate a class of homeomorphisms of a cylinder, with all trajectories convergent to the cylinder base and one fixed point in the base. Let $\mathcal{A}$ be a nonempty finite or countable family of sets, each of which can be a priori an $\omega$-limit set. Then there is a homeomorphism from our class, for which $\mathcal{A}$ is the family of all $\omega$-limit sets.

## 1. Introduction

For a simple map of a triangle, defined and investigated 50 years ago by P. Stein and S. Ulam in [10], the family of all $\omega$-limit sets has been recently identified by K. Barański and M. Misiurewicz [3]. However, the proofs in that paper include many pages of intricate estimates, used to show that certain curves are eventually stretched. Since there are other curves that are stretched for purely topological reasons, a question arises whether those estimates are necessary. Maybe the family of all $\omega$-limit sets is similar for a large class of homeomorphisms with very similar dynamics? In this paper we show that this is not the case.

The map investigated in [10] and [3] is a homeomorphism of the 2-dimensional simplex $\Delta = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x + y + z = 1\}$ onto itself, given by

$$f(x, y, z) = (x^2 + 2xy, y^2 + 2yz, z^2 + 2zx). \tag{1.1}$$

We will call it the *Stein-Ulam Spiral map.* The central point $(1/3, 1/3, 1/3)$ is fixed, as are the three vertices. The sides of $\Delta$ are invariant and the points there move under the action of the map in the clockwise direction towards the nearest vertex. The trajectories of all interior points (except the central fixed point) are unwinding "spirals" converging very fast to the boundary of $\Delta$ (see Figure 1). The trajectories spend most of time very close to the vertices of $\Delta$. This is not visible, because the points close to a given vertex cannot be distinguished from each other in the figure.

There are obvious symmetries of the Stein-Ulam Spiral map. Rotations of $\Delta$ by 120 and 240 degrees (that is, cyclic permutations of $x, y, z$) commute with it. When we divide the system by those symmetries, we get only one fixed point on the boundary; it is attracting from one side and repelling from the other. Moreover, the central fixed point is repelling, so the dynamics in its neighborhood is trivial. Thus, it makes sense to remove a small disk centered at the central fixed point and conjugate the system to a homeomorphism of a cylinder. This motivates the following definition of a class of cylinder homeomorphisms with the dynamics very similar to the one of the Stein-Ulam Spiral map.

We will consider a cylinder $\mathbb{A} = [0, 1) \times \mathbb{T}$ and its *base* $\mathbb{B} = \{0\} \times \mathbb{T}$.

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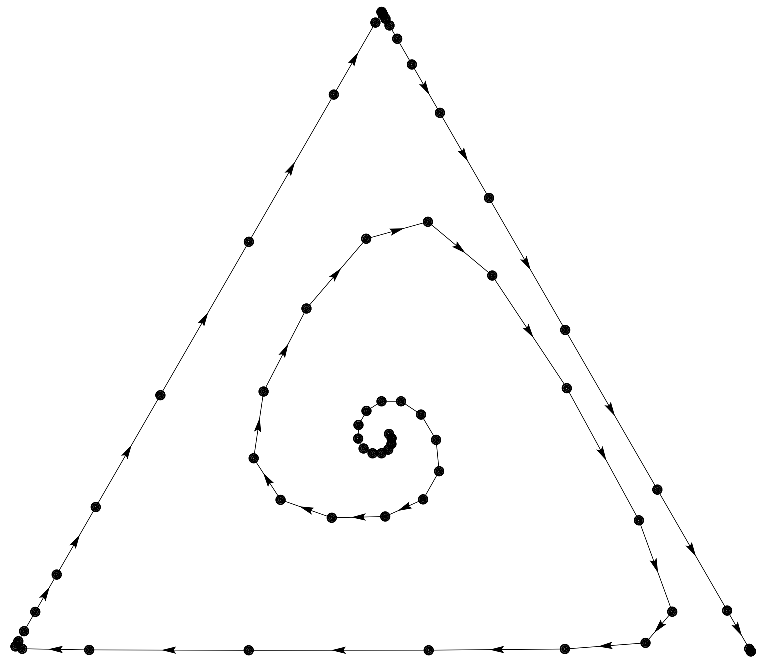

FIGURE 1. A piece of a trajectory of the Stein-Ulam Spiral map.

**Definition 1.1.** A map $f : \mathbb{A} \to \mathbb{A}$ will be called a *spiral cylinder map* if it satisfies the following conditions:

(c1) the map $f$ is a homeomorphism onto its image and the image is contained in $[0, c) \times \mathbb{T}$ for some $c < 1$,
(c2) the base $\mathbb{B}$ is invariant and the restriction of $f$ to $\mathbb{B}$, $f_0 : \mathbb{B} \to \mathbb{B}$ is an orientation preserving homeomorphism with exactly one fixed point,
(c3) the trajectory of every point converges to $\mathbb{B}$,
(c4) the map $f$ has a lifting $F$ to $[0, 1) \times \mathbb{R}$ such that $F_0 : \mathbb{R} \to \mathbb{R}$ (the restriction of $F$ to $\{0\} \times \mathbb{R}$) has fixed points and the second component of $F^n(x)$ goes to infinity for every $x \in (0, 1) \times \mathbb{R}$.

If additionally the foliation by circles $\{a\} \times \mathbb{T}$ is invariant for $f$, then we will call it a *regular spiral cylinder map*.

If we want to extend a spiral cylinder map to a homeomorphism of a disk, we add a "lid" (a closed disk) on the top of the cylinder with a fixed repelling point in the center. Then the trajectories of every point except the fixed point at the center of the lid will converge to $\mathbb{B}$.

The main result of the paper is the existence of regular spiral cylinder maps for which the set of all $\omega$-limit sets of points from $\mathbb{A} \setminus \mathbb{B}$ is equal to a prescribed nonempty finite or countable family of admissible sets (Theorem 3.1). Here by "admissible" we mean a subset of the base that *a priori* can be an $\omega$-limit set. This result is proven in Section 3. The proof goes via a construction of maps restricted to the orbit of one circle of the invariant foliation, together with the base. Such a map can be considered as a nonautonomous system on the circle. This construction is made in Section 2 and it is the central part of the proof. Finally, in Section 4 we show that not all spiral cylinder maps are conjugate to regular ones. In fact, it is an open question whether the Stein-Ulam Spiral map is (after removing a neighborhood of the central fixed point and dividing by symmetries) conjugate to a regular cylinder spiral map.

There is a substantial literature on the structure of the $\omega$-limit sets, mainly in low dimensions, especially dimension 1 (see for instance [12] and the references therein; note the papers [6], [1], [2] and [5] as closely related to our research). While most of the papers are concerned with a question of which sets can be $\omega$-limit sets of continuous maps (that is, we fix a set and vary maps), there are some where the map is fixed and the sets vary, as in [3].

Especially interesting is the question for what spaces and what classes of maps the family of $\omega$-limit sets of all points is closed in the Hausdorff metric. Such a property has been proven first for continuous interval maps [4], then for continuous circle maps [11], and finally for continuous graph maps [9]. On the other hand, counterexamples have been found for continuous dendrite maps [8] and triangular square maps [5].

In Section 3 we get such a counterexample (in fact, a large family of counterexamples) for disk homeomorphisms "for free" as a by-product of our main theorem. Let us compare it to the known and trivial counterexamples for disk maps. By "known" we mean a triangular map from [5], and by "trivial" we mean a homeomorphism preserving concentric circles and rotating the circles with different speeds (then the $\omega$-limit sets will be finite on the circles with rational rotation numbers and the whole circle on the circles with irrational rotational number)[1]. Our counterexample is in a sense stronger than the triangular one – because our map is invertible – and stronger than the trivial one – because for our map all $\omega$-limit sets (except the singleton of the central fixed point) are contained in the boundary of the circle, so the system is almost one-dimensional.

Let us also use this opportunity to note that the trivial counterexample described above can be easily modified in order to get a triangular map with the identity in the base. In the fibers over a sequence convergent to 0 one can put transitive interval maps convergent uniformly to the identity (they can be easily constructed by a standard "zig-zag" method; see [7]), and in the fiber over 0 one can put the identity. Then in the fibers over the terms of our sequence the whole interval will be an $\omega$-limit set, while in the limit fiber, over 0, the only limit sets will be singletons.

## 2. Nonautonomous systems on the circle

In this section we will consider nonautonomous dynamical systems on a circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, that is, sequences of circle maps $(f_n)_{n=1}^{\infty}$. We will impose some special conditions on them. We assume that

(na1) maps $f_n$ are orientation preserving homeomorphisms,
(na2) maps $f_n$ converge uniformly as $n \to \infty$ to a limit homeomorphism $f_0$.

Since the system is nonautonomous, instead of iterating one map, we will compose the maps from the sequence. That is, $f^0$ is the identity and

$$f^n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1.$$

[1]Existence of this simple counterexample has been pointed out to us by Ľ. Snoha.

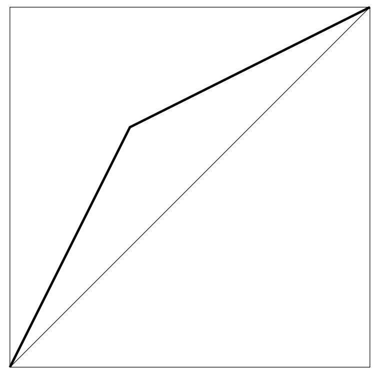

FIGURE 2. The graph of $F_0$ on $[0,1]$.

Then, as in the case of autonomous systems, we study the limit properties of the maps $f^n$ as $n$ goes to infinity. In particular, we can define the $\omega$-limit of $x \in \mathbb{T}$ (denoted by $\omega(x)$) as the set of accumulation points of the sequence $(f^n(x))_{n=0}^{\infty}$. By definition, it is closed. With our assumptions, it is also $f_0$-invariant. Indeed, if $n_k \to \infty$ and $f^{n_k}(x) \to y$ as $k \to \infty$, then

$$\lim_{k\to\infty} f^{n_k+1}(x) = \lim_{k\to\infty} (f_{n_k+1}(f^{n_k}(x))) = \lim_{k\to\infty} (f_0(f^{n_k}(x))) = f_0(y).$$

Similarly we get $\lim_{k\to\infty} f^{n_k-1}(x) = f_0^{-1}(y)$, so $\omega(x)$ is fully invariant for $f_0$ (that is, $f_0(\omega(x)) = \omega(x)$).

Now we make additional assumptions on the limit map and the entire system:

(na3) the map $f_0$ has exactly one fixed point,
(na4) the maps $f_n$, $n = 0, 1, 2, \dots$ have liftings to the real line $F_0, F_1, F_2, \dots$ such that $F_0$ has fixed points, the maps $F_n$ converge uniformly to $F_0$, and $F^n(x) \to \infty$ for every $x \in \mathbb{R}$ (where $F^n = F_n \circ F_{n-1} \circ \dots \circ F_2 \circ F_1$).

We will call a system satisfying conditions (na1)-(na4) a *spiral circle system*.

We will denote by $\pi : \mathbb{R} \to \mathbb{T}$ the natural projection. On the circle we will use the distance along the arcs and will denote it by dist. We will use small letters to denote maps of the circle and the corresponding capital letters to denote their liftings (not always explaining it). Moreover, we will write $F^{-n}$ for $(F^n)^{-1}$.

The aim of this section is to construct a spiral circle system with a prescribed $\omega$-limit set for all points. We start by fixing the limit map $f_0$, or more precisely, its lifting $F_0$. It is enough to define it on the interval $[0,1]$, since it has to have the property $F_0(x+1) = F_0(x) + 1$. We set

$$F_0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{x+1}{2} & \text{if } \frac{1}{3} \le x \le 1 \end{cases}$$

(see Figure 2). It is clearly a lifting of an orientation preserving circle homeomorphism, $f_0$, with one fixed point, so in particular condition (na3) is satisfied.

The idea of the construction is easily understood on the circle. The fixed point on the circle is $\pi(0)$. Let $(q_k)_{k=1}^{\infty}$ be a sequence of points converging to $\pi(0)$ from the right (the notions of "right" and "left" are inherited from $\mathbb{R}$). For each $q_k$, the sequence $(f_0^n(q_k))_{n=1}^{\infty}$ converges to $\pi(0)$ from the left. We construct a sequence of maps $(f_k)_{k=1}^{\infty}$. All but a few of the $f_k$ will be $f_0$ and the others will be perturbations of $f_0$. First iterate $f_0$ until $f_0^{k_1}(q_1)$ is close to $\pi(0)$ on the left. Define $f_i = f_0$ for $i = 1, \dots, k_1$. Define $f_{k_1+1}$ to be a perturbation of $f_0$ that is equal to $f_0$ away from 0 but takes $f_0^{k_1}(q_1)$ to $q_2$. Repeat this procedure using $q_2$ and $q_3$ getting $f_0^{k_2}(q_2)$ closer to $\pi(0)$ than $f_0^{k_1}(q_1)$. Continue this construction to get the desired sequence of maps. For this sequence of maps the trajectory of $q_1$ follows the $f_0$-trajectory of $q_1$ for a while, then jumps and follows the $f_0$-trajectory of $q_2$ for a longer time and so forth. Moreover, if the construction is done properly, the trajectory of *every* point on the circle will converge to the trajectory of $q_1$.

To make the actual construction we work with the liftings. First, fix a sequence of points $p_k \in (0,1)$ so that for $k = 1, 2, \dots$, $F_0^{-k}(1/3) \le p_k < F_0^{-k+1}(1/3)$ ($q_k = \pi(p_k)$ are the points mentioned in the preceding paragraph). Note that then $F_0^k(1/3) \le F_0^{2k}(p_k) < F_0^{k+1}(1/3)$. Using those sequences we will define liftings $G_k$ of orientation preserving circle homeomorphisms. We do it on a different interval of length 1 than in the definition of $F_0$, namely on the interval $[p_{k+1}, p_{k+1}+1]$. Set

$$b_k = F_0^{2k-1}(p_k), \quad c_k = F_0^{2k}(p_k) = F_0(b_k), \quad d_k = p_{k+1} + 1, \quad e_k = F_0(d_k).$$

Then $d_k - 1 < b_k < c_k < 1 < d_k$, so the points $b_k$ and $c_k$ divide the interval $[d_k - 1, d_k]$ into 3 subintervals. On the first of them, $[d_k - 1, b_k]$, we set $G_k = F_0$. In particular, $G_k(d_k - 1) = F_0(d_k - 1) = G_k(d_k) - 1$, $G_k(b_k) = c_k$, and $G_k(d_k) = e_k$. We set $G_k(c_k) = d_k$ and extend $G_k$ to the intervals $[b_k, c_k]$ and $[c_k, d_k]$ with $G_k([b_k, c_k]) = [c_k, d_k]$ and $G_k([c_k, d_k]) = [d_k, e_k)]$ (see Figure 3). We will specify later how we do it, but the main thing is that it should be a strong expansion immediately to the left of $c_k$ and a strong contraction immediately to the right of $c_k$. Note however, that any monotone extension will make $G_k$ the lifting of an orientation preserving circle homeomorphism such that

$$\|G_k - F_0\| \le e_k - c_k = F_0(p_{k+1}) + 1 - F_0^{2k}(p_k) < F_0^{-k}(1/3) + 1 - F_0^k(1/3), \tag{2.1}$$

where $\|\cdot\|$ is the supremum norm. Since $F_0^{-k}(1/3) \to 0$ and $F_0^k(1/3) \to 1$, the maps $G_k$ converge uniformly to $F_0$.

Now we define the sequence $(F_n)_{n=1}^{\infty}$ as

$$(F_n) = (\underbrace{F_0, F_0}_{2 \text{ times}}, G_1, \underbrace{F_0, \dots, F_0}_{4 \text{ times}}, G_2, \underbrace{F_0, \dots, F_0}_{6 \text{ times}}, G_3, \dots). \tag{2.2}$$

Then all maps $F_n$ are liftings of orientation preserving circle homeomorphisms and converge uniformly to $F_0$, so conditions (na1) and (na2) and the first part of (na4) are satisfied.

To show the rest of (na4), let us trace first the trajectory of the special point $p_1$. This point is mapped to $F_0(p_1)$, and then to $F_0^2(p_1)$. Then $G_1$ is applied, so the next point on the trajectory is $G_1(F^2(p_1)) = p_2 + 1$. Then we apply $F_0$ 4 times, get to $F^4(p_2) + 1$ and the next point is $G_2(F^4(p_2)) + 1 = p_3 + 2$. This pattern repeats again and again. This means that on the circle we follow consecutively the trajectories of $\pi(p_k)$ for $2k + 1$ steps and in the liftings the trajectory of $p_1$ goes to the infinity. By monotonicity, if one trajectory goes to infinity, all of them do, so condition (na4) holds. This proves that our system is a spiral one.

In the next lemma we refine the construction of the maps $G_k$.

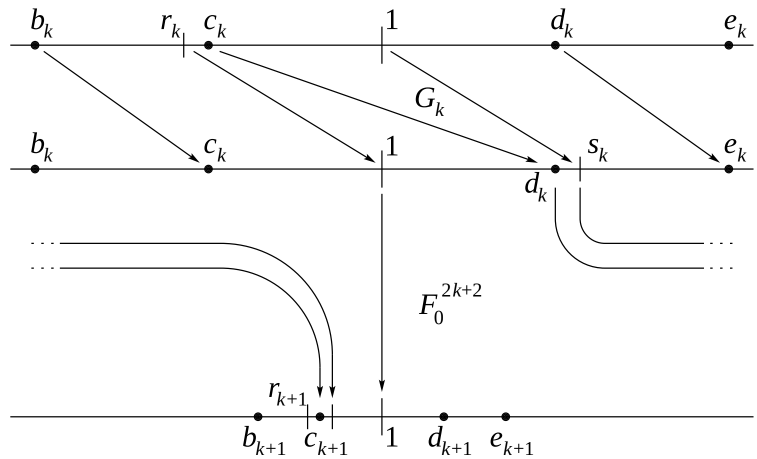


FIGURE 3. Construction of $G_k$ and the sequence $(F_n)_{n=1}^{\infty}$.

**Lemma 2.1.** *Extensions from the construction of $G_k$ can be done in such a way that*

$$\lim_{n\to\infty} \operatorname{dist}(f^n(\pi(p_1)), f^n(q)) = 0 \tag{2.3}$$

*for every $q \in \mathbb{T}$.*

*Proof.* We use the notation $b_k, c_k, d_k, e_k$ introduced earlier. We will add some conditions on the location of the points $r_k = G_k^{-1}(1) \in (b_k, c_k)$ and $s_k = G_k(1) \in (d_k, e_k)$. We advise the reader to look at Figure 3.

For every $k$ we specify $s_k \in [d_k, e_k]$ in such a way that

$$2^{2k+2}(s_k - d_k) < 1/k. \tag{2.4}$$

We can do it, because up to now we only require that $G_k$ maps $c_k$ to $d_k$ and $d_k$ to $e_k$, and is increasing in between.

In our construction we have $G_k = F_{\ell(k)+1}$ for $\ell(k) = k^2 + 2k$. For a given $y \in \mathbb{R}$, we want to compare $F^n(y)$ with $F^n(p_1)$. We do this first for $n = \ell(k)$. We have

$$F^{\ell(1)}(p_1) = F_0^2(p_1) = c_1.$$

Now we prove by induction that

$$F^{\ell(k)}(p_1) = c_k + k - 1 \tag{2.5}$$

for all $k$. Indeed, if it is true for some $k$, then

$$F^{\ell(k+1)}(p_1) = F_0^{2k+2}(G_k(c_k + k - 1)) = F_0^{2k+2}(d_k + k - 1) = c_{k+1} + k.$$

We have

$$F^{\ell(k+1)}(F^{-\ell(k)}([c_k, r_k{+}1])) = (F_0^{2k+2}(G_k([c_k, r_k{+}1])) = [c_{k+1}{+}1, 2] \subset [c_{k+1}{+}1, r_{k+1}{+}2].$$

Therefore by induction we get

$$F^{\ell(k+i)}(F^{-\ell(k)}([c_k, r_k + 1])) \subset [c_{k+i} + i, i + 1]$$

for all $i > 0$. Hence, since

$$\ell(k + i + 1) - \ell(k + i) = (k + i + 1)^2 - (k + i)^2 + 2 = 2(k + i) + 3, \tag{2.6}$$

we have

$$F^{\ell(k+i)+j}(F^{-\ell(k)}([c_k, r_k+1])) \subset F_0^{j-1}(G_{k+i}([c_{k+i}+i, i+1])) = F_0^{j-1}([d_{k+i}+i, s_{k+i}+i])$$

for $j = 1, 2, \dots, 2(k+i)+3$. By (2.4) (applied to $k+i$ instead of $k$), the length of $F_0^{j-1}([d_{k+i}+i, s_{k+i}+i])$ is less than $1/(k+i)$. In view of (2.6), every $n > \ell(k+1)$ can be written as $\ell(k+i)+j$ with $0 < j \leq 2(k+i)+3$, so we proved that the lengths of the intervals $F^n(F^{-\ell(k)}([c_k, r_k+1]))$ go to 0 as $n \to \infty$.

By projecting everything onto the circle and taking into account (2.5), we get the following result. If there exists $k$ such that $f^{\ell(k)}(q) \in \pi([c_k, r_k+1])$, then both points $\pi(p_1)$ and $q$ belong to the arc $\pi(F^{-\ell(k)}([c_k, r_k+1])) = f^{-\ell(k)}(\pi([c_k, r_k+1]))$. Since the lengths of the images of this arc go to 0, (2.3) holds. Now our aim is to show that with the right choice of the points $r_k$ this happens for every $q \in \mathbb{T}$.

Thus, we want to define the points $r_k$ in such a way that for every $q \in \mathbb{T}$ there is $k$ such that $q \in \pi([c_k, r_k+1])$. Suppose that there is $q \in \mathbb{T}$ for which this is not the case. Take $y \in \pi^{-1}(q)$. Then for every $k$ there is $m$ such that $F^{\ell(k)}(y) \in (r_k+m, c_k+m)$. We have

$$F^{\ell(k+1)}(F^{-\ell(k)}((r_k, c_k))) = F_0^{2k+2}(G_k((r_k, c_k))) = (1, c_{k+1}+1) \supset (r_{k+1}+1, c_{k+1}+1).$$

Thus, we may assume that $F^{\ell(1)}(y) \in (r_1, c_1)$ and then by induction we get

$$F^{\ell(k)}(y) \in (r_k+k-1, c_k+k-1) = (r_k+k-1, F^{\ell(k)}(p_1)),$$

that is,

$$y \in (F^{-\ell(k)}(r_k+k-1), p_1) \tag{2.7}$$

for any $k$.

We define the maps $G_k$ one by one. Thus, at the moment when we are constructing $G_k$, the map $F^{\ell(k)}$ is already defined. This means that we can choose $r_k \in (b_k, c_k)$ in such a way that $p_1 - F^{-\ell(k)}(r_k+k-1) < 1/k$. Then there cannot be $y$ satisfying (2.7) for all $k$. This completes the proof. □

Observe that all maps that can serve as the limit map $f_0$ of a spiral circle system are conjugate to one another. Indeed, it is enough to specify this conjugacy on a fundamental domain of the first map, so that it sends it in a continuous increasing way to a fundamental domain of the second map, and extend it via iterates of both maps to the whole circle (and of course, send the fixed point to the fixed point). Therefore for all spiral circle systems that we consider, we may fix $f_0$ as the map whose lifting is $F_0$ specified earlier (see Figure 2).

We will call a set $A \subset \mathbb{T}$ *admissible* if it is closed, $f_0$-invariant, and contains more than one point. Clearly, for a spiral circle system the $\omega$-limit set of every point is admissible.

**Theorem 2.2.** *For every admissible set $A \subset \mathbb{T}$ there exists a spiral circle system with the $\omega$-limit set of every point equal to $A$.*

*Proof.* Take a dense countable (or finite) set $B \subset A$ that does not contain the fixed point of $f_0$. Produce a sequence $(a_n)_{n=1}^{\infty}$ of points of $(0,1)$ such that each $\pi(a_n)$ belongs to $B$ and for each element of $x \in B$ there are infinitely many terms $a_n$ with $\pi(a_n) = x$. Now for any $n$ take a positive integer $i_n$ such that $F_0^{-n}(1/3) \leq p_n = F_0^{-i_n}(a_n) < F_0^{-n+1}(1/3)$. Then make the construction described earlier in this section. The trajectory of the projection of $p_1$ to the circle is contained in $B$ and passes through every point of $B$ infinitely many times. Therefore the $\omega$-limit of this point is equal to $A$. Hence, by Lemma 2.1, the $\omega$-limit set of every point of the circle is equal to $A$. □

## 3. Cylinder maps

Now we include the systems constructed in the preceding section in a regular spiral cylinder map (see Definition 1.1). Such a map $f$ projects onto the first coordinate, so it has a factor $\widetilde{f} : [0, 1) \to [0, 1)$ there. It is a homeomorphism of $[0, 1)$ into itself, with $\widetilde{f}([0, 1)) \subset [0, c)$ and with 0 fixed and globally attracting. All such maps are conjugate to each other, so we may assume that $\widetilde{f}(x) = x/2$ for all $x$. Similarly, we may assume that the map $f_0$ is the same as $f_0$ considered in the preceding section (so this is also the case for $F_0$). Now, for each $t \in (0, 1)$, the map $f$ restricted to $\{t, \widetilde{f}(t), \widetilde{f}^2(t), \dots\} \times \mathbb{T}$ can be viewed as a spiral circle system. Conversely, we can "embed" spiral circle systems in this way into a regular spiral cylinder map, controlling this way the set of $\omega$-limit sets.

**Theorem 3.1.** *For every nonempty finite or countable family $\mathcal{A}$ of admissible sets there exists a regular spiral cylinder map whose set of all $\omega$-limit sets of points from $\mathbb{A} \setminus \mathbb{B}$ is equal to $\mathcal{A}$.*

*Proof.* Assume first that $\mathcal{A} = A_1, \dots, A_s$ is finite. Denote by $g_{n,i}$ the $n$-th map of a spiral circle system satisfying Theorem 2.2 for $A = A_i$. Set $a = \lim_{t\to 1} \widetilde{f}(t)$. Divide the interval $[a, 1)$ into consecutive intervals (closed from the left, open from the right) $I_i$, $i = 1, 2, \dots, s$. In each interval $I_i$ choose an ascending sequence of closed intervals $(I_{i,n})_{n=1}^{\infty}$ whose union is $I_i$. Now put

$$f(t, x) = (\widetilde{f}(t), g_{n,i}(x)) \tag{3.1}$$

if $t \in \widetilde{f}^n(I_{i,n})$. This way we defined $f(t, x)$ for $t$ in a countable family of pairwise disjoint closed intervals, whose only accumulation point is 0. By (2.1) and (2.2) $f$ restricted to the circles $t = c$ converges uniformly to $f_0$ as $c \to 0$. Therefore if we fill the gaps between consecutive closed cylinders on which $f$ is already defined in a linear (in the lifting) way, the resulting map $f$ will be continuous. Here by "linear" we mean that if $f$ is defined for $t = t_0, t_1$ with $t_0 < t_1$ and not defined for $t \in (t_0, t_1)$ then we set in the lifting

$$F(st_0 + (1 - s)t_1), x) = (\widetilde{f}(st_0 + (1 - s)t_1), sF(t_0, x) + (1 - s)F(t_1, x))$$

for $s \in (0, 1)$. This clearly preserves the foliation by circles. Moreover, since a convex combination of strictly increasing functions is strictly increasing, it is a homeomorphism onto its image.

Let us take any point $(t, x) \in \mathbb{A} \setminus \mathbb{B}$. Then there is $k \geq 0$ such that $(t, x) = f^k(s, y)$ with $s \in [a, 1)$. Therefore there are $m$ and $i$ such that $y \in I_{i,n}$ for all $n \geq m$. This means that starting at some moment the second iterate of the trajectory of $(s, y)$ coincides with the trajectory of some point of a spiral circle system satisfying Theorem 2.2 for $A = A_i$. Thus, its $\omega$-limit set is $A_i$. Taking into account that all intervals $I_{j,n}$ are nonempty, this proves that for $f$ the set of all $\omega$-limit sets of points from $\mathbb{A} \setminus \mathbb{B}$ is equal to $\mathcal{A}$.

Let us try to make the same construction with $\mathcal{A} = A_1, A_2, A_3, \ldots$ countable. Everything works the same way, except that the family of closed intervals in variable $t$ for which we define $f$ in the first step, has additional points of accumulation, namely the points $\widetilde{f}^n(a)$. This may make $f$ discontinuous. To avoid this, when we define $f$ by (3.1), we do it only for $i \le n$. With this modification, the rest of the proof is the same as for $\mathcal{A}$ finite. □

Since we can include in $\mathcal{A}$ some convergent sequence, but exclude its limit, as a corollary we get immediately the following result.

**Corollary 3.2.** *There exists a regular spiral cylinder map for which the set of all $\omega$-limit sets of points from $\mathbb{A} \setminus \mathbb{B}$ is not closed in the space of all admissible sets with the Hausdorff metric.*

As we noticed in the Introduction, a spiral cylinder map can be extended to a disk homeomorphism with the only an additional $\omega$-limit set equal to the set consisting of the fixed point in the added disk. Observe also that for all points from $\mathbb{B}$ their $\omega$-limit set consists of the fixed point in $\mathbb{B}$. In such a way we get a corollary to the corollary.

**Corollary 3.3.** *There exists a homeomorphism of a closed disk for which the set of all $\omega$-limit sets of points is not closed in the space of all invariant closed nonempty subsets of the disk with the Hausdorff metric. It can be chosen such that all $\omega$-limit sets, except the singleton of one interior fixed point, are contained in the boundary of the disk.*

Of course Theorem 3.1 does not mean that all sets of $\omega$-limit sets for a regular spiral cylinder map are always of the form described there. We are not going to construct a separate example, since we know from [3] that the Stein-Ulam Spiral map has as the set of all $\omega$-limit sets the space of all admissible sets. We can make from it a spiral cylinder map by factorizing by the group of rotations by 0, $2\pi/3$ and $4\pi/3$ and removing some neighborhood of the central fixed point. While we do not know whether the spiral cylinder map obtained this way is conjugate to a regular one, we do not think that a construction of a regular example, in which a proof very similar to the one from [3] works, is worth the effort.

## 4. Non-regular spiral cylinder map

Considerations from the last paragraph of the preceding section bring an interesting question: does there exist a spiral cylinder map that is not conjugate to a regular one? We will show that the answer is “yes.”

To construct an example we need some property of regular spiral cylinder maps that is not satisfied by all spiral cylinder maps. To this end we will define for each point in $(0, 1) \times \mathbb{R}$ a sequence of *transition times*. One can think about it as the reciprocal of the rotation number, but since the true rotation number would have to go to 0 as we approach the base of the cylinder, we get only a sequence here.

We will use the notation from the definition of the spiral cylinder maps. Additionally, we will always assume that the lifting of the map restricted to the base ($F_0$ in the definition) has fixed points at integers. This can be thought of as choosing where on the circle the special point 0 is; we can always place it at the fixed point. We will denote by $P$ the projection of $[0,1)\times\mathbb{R}$ to the second coordinate.

Now we define the *n-th transition time* of a point $x\in(0,1)\times\mathbb{R}$ for $f$ as the minimal non-negative integer $k$ such that $P(F^k(x))\ge n+\frac{1}{2}$. We will denote it by $\tau_n(f,x)$. We will investigate how this sequence varies with $f$ and $x$ in certain situations.

**Lemma 4.1.** *If $H$ is a conjugacy between the liftings $F$ and $G$ of spiral cylinder maps $f$ and $g$ (and is a lifting of a conjugacy between $f$ and $g$) then for every $x\in(0,1)\times\mathbb{R}$ there exists an integer $r$ such that $\tau_{n-r}(g,H(x))\le\tau_n(f,x)$ for all sufficiently large $n$.*

*Proof.* The distance of $F^{\tau_n(f,x)}(x)$ from the segment $\{0\}\times\left[n+\frac{1}{2},F_0\left(n+\frac{1}{2}\right)\right]$ goes to 0 as $n\to\infty$. Therefore the distance of $G^{\tau_n(f,x)}(H(x))$ from the segment $\{0\}\times\left[H\left(n+\frac{1}{2}\right),G_0\left(H\left(n+\frac{1}{2}\right)\right)\right]$ also goes to 0. There is an integer $r$ such that

$$\left[H\left(n+\tfrac{1}{2}\right),G_0\left(H\left(n+\tfrac{1}{2}\right)\right)\right]\subset(n-r+1,n-r+2)$$

for every $n$. Thus, $P(G^{\tau_n(f,x)}(H(x)))>n-r+\frac{1}{2}$ for sufficiently large $n$. This completes the proof. ☐

**Lemma 4.2.** *Assume that $f$ is a regular spiral cylinder map with lifting $F$ and $x,y\in(0,1)\times\mathbb{R}$ have the same first component. Then there exists an integer $r$ such that $\tau_{n-r}(f,y)\le\tau_n(f,x)$ for all $n\ge\max(0,r)$.*

*Proof.* There is an integer $r$ such that $P(x)\le P(y+(0,r))$. Then for each $k$ we have $P(F^k(x))\le P(F^k(y+(0,r)))$. Thus, if $P(F^k(x))\ge n+\frac{1}{2}$, then $P(F^k(y+(0,r)))\ge n+\frac{1}{2}$, so $P(F^k(y))\ge n-r+\frac{1}{2}$. This happens in particular when $k=\tau_n(f,x)$, and therefore $\tau_{n-r}(f,y)\le\tau_n(f,x)$. ☐

**Lemma 4.3.** *Assume that a lifting $F$ of a spiral cylinder map $f$ is the time-one map of a flow $\Phi$ (that is, $F=\Phi^1$) given by a vector field whose second component is strictly positive except at the fixed points of $F$. Assume that $x,y\in(0,1)\times\mathbb{R}$ are on the same trajectory of $\Phi$ (that is, $y=\Phi^s(x)$ for some $s\in\mathbb{R}$). Then there exists an integer $j$ such that $\tau_n(f,y)\le\tau_n(f,x)+j$ for all $n$.*

*Proof.* Take any integer $-j\le s$. Since $\Phi$ is given by a vector field whose second component is strictly positive, $P(\Phi^s(x))$ grows with $s$. Therefore

$$P(F^k(y))=P(\Phi^{k+s}(x))\ge P(F^{k-j}(x))=P(\Phi^{k-j}(x))$$

Hence, if $P(F^{k-j}(x))\ge n+\frac{1}{2}$, then $P(F^k(y))\ge n+\frac{1}{2}$. This happens in particular when $k-j=\tau_n(f,x)$, and therefore $\tau_n(f,y)\le k=\tau_n(f,x)+j$. ☐

**Theorem 4.4.** *There exists a spiral cylinder map which is not conjugate to a regular one.*

*Proof.* We start with a a vector field $V_1(t,z)=(-t,1)$ on $[0,1)\times\mathbb{R}$. It gives us a flow $\Psi_1^s(t,z)=(te^{-s},z+s)$, whose trajectories projected to the cylinder $\mathbb{A}$ are spirals approaching the base $\mathbb{B}$ (except the trajectory in the base, which is a circle). If we multiply the projection of $V_1$ to the cylinder by a function which is positive everywhere except at one point in the base, where it is 0, then the trajectories will be only reparametrized (except in the base, where we will get one fixed point), so the time-one map of the corresponding flow will be a spiral cylinder map.

In the lifting this multiplication is a multiplication by a function which is periodic of period 1 with respect to the second coordinate, and is positive everywhere except at $(0, n)$, $n \in \mathbb{Z}$, where it is 0. Let us choose such function, $\eta_1(t, z) = t + 1 - \cos(2\pi x)$. In such a way we get a vector field $V_2 = \eta_1 V_1$ with the projection $v_2$ on the cylinder $\mathbb{A}$ that generates a flow $\psi$ on $\mathbb{A}$. Take two disjoint trajectories $\gamma_0$ and $\gamma_1$ of this flow and points $x_0$, $x_1$ on their liftings to $[0, 1) \times \mathbb{R}$. Let $\widehat{f}$ be the time-one map of $\psi$. Consider the sequence of transition times $\tau_n(\widehat{f}, x_0)$. Then choose a sequence of small open sets converging to the fixed point of $\psi$, intersecting $\gamma_1$ at consecutive times when it passes close to the fixed point, and disjoint from $\gamma_0$. Take a positive (except at the fixed point of $\psi$) function $\eta_2 : \mathbb{A} \to \mathbb{R}$ which is 1 outside those sets and very close to 0 at some parts of their intersections with $\gamma_1$. In this way we can slow down as much as we want the flow on $\gamma_1$ without affecting it on $\gamma_0$. In particular, we can do it in such a way that if $\varphi$ is the flow given by the vector field $v_3 = \eta_2 v_2$ and $f$ is its time-one map, then

$$\tau_n(f, x_1) > n\tau_{2n}(f, x_0) \tag{4.1}$$

for all sufficiently large $n$.

By the construction, $f$ is a spiral cylinder map. Suppose that it is conjugate to a regular one, say $g$, and denote the conjugacy by $h$. Use the corresponding capital letters for the liftings of those three maps, and let $\Gamma_i$ be the liftings of $\gamma_i$, $i = 0, 1$. The curves $H(\Gamma_i)$ are converging to the base $\{0\} \times \mathbb{R}$ and going to infinity in the direction of the second coordinate. Therefore there are points $y_i \in \Gamma_i$ such that the first components of $H(y_0)$ and $H(y_1)$ are the same.

At this point we should alert the reader that the notation in the following paragraph does not necessarily correspond to the notation from Lemmas 4.1, 4.2 and 4.3.

Assume that $n$ is sufficiently large. By Lemma 4.3 there is an integer $j_1$ such that $\tau_n(f, x_1) \leq \tau_n(f, y_1) + j_1$. By Lemma 4.1 there is an integer $r_1$ such that $\tau_n(f, y_1) \leq \tau_{n+r_1}(g, H(y_1))$. By Lemma 4.2 there is an integer $r_2$ such that $\tau_{n+r_1}(g, H(y_1)) \leq \tau_{n+r_1+r_2}(g, H(y_0))$. By Lemma 4.1 there is an integer $r_3$ such that $\tau_{n+r_1+r_2}(g, H(y_0)) \leq \tau_{n+r_1+r_2+r_3}(f, y_0)$. By Lemma 4.3 there is an integer $j_2$ such that $\tau_{n+r_1+r_2+r_3}(f, y_0) \leq \tau_{n+r_1+r_2+r_3}(f, x_0) + j_2$. Thus, for any sufficiently large $n$ we get

$$\tau_n(f, x_1) \leq \tau_{n+r_1+r_2+r_3}(f, x_0) + j_1 + j_2.$$

Together with (4.1), we get

$$n\tau_{2n}(f, x_0) < \tau_{n+r_1+r_2+r_3}(f, x_0) + j_1 + j_2$$

for all sufficiently large $n$. However, the sequence of transition times is non-decreasing. Thus, if $n > r_1 + r_2 + r_3$ then we get

$$n\tau_{2n}(f, x_0) < \tau_{2n}(f, x_0) + j_1 + j_2,$$

for all sufficiently large $n$, a contradiction. This completes the proof. □

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