OMEGA-LIMIT SETS FOR THE STEIN-ULAM SPIRAL MAP

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Dedicated to the centennial of Stanisław Ulam (1909–1984)

Abstract. In the late 1950's, using computers in the Los Alamos National Laboratory, Stanisław Ulam and Paul Stein performed a comprehensive research on a class of quadratic maps of the 2-dimensional simplex Δ to itself. Those maps arise in the theory of population genetics. One of them has the behavior much different than the 96 other ones. We call it the Stein-Ulam Spiral map. In 1972, S. Vallander asked whether the ω-limit set of any interior point of Δ, except its center, is equal to the boundary of Δ. We prove that this is the case for the points from a residual subset of Δ. On the other hand, we show that for any closed invariant subset E of the boundary of Δ intersecting all three sides of Δ, the set of points having E as the ω-limit set is relatively large.

1. Introduction

Stan Ulam and Paul Stein were pioneers in using computers for investigation of discrete dynamical systems. In the 1950's, using the computers in the Los Alamos National Laboratory, they performed a comprehensive research on a class of quadratic maps of the 2-dimensional simplex to itself. They called those maps *Binary Reaction Systems*. Programming was done by Mary Menzel, and the results were published in [6].

The Binary Reaction Systems can be described as follows. Let $\Delta = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x + y + z = 1\}$ be the 2-dimensional simplex. When we square $x + y + z$, we get 6 terms, which we can distribute among x', y' and z', with each of them receiving at least one term. Then the map given by $f(x, y, z) = (x', y', z')$ is a quadratic map of Δ to itself. This distribution of the six terms can be done in 540 different ways. However, up to the permutations of the coordinates, we get only 97 distinct maps (it is more than $540/6 = 90$ because some maps are invariant for some permutations).

Out of the 97 systems investigated, 95 were behaving in a similar way: the trajectory of a randomly chosen point was converging to an exponentially attracting periodic point of period 1, 2, or 3. In one system the attracting fixed point was attracting subexponentially. And then comes the most non-typical system, where there is no attracting periodic orbit. It is given by

$$f(x, y, z) = (x^2 + 2xy, y^2 + 2yz, z^2 + 2zx). \tag{1.1}$$

Date: April 6, 2009.

2000 *Mathematics Subject Classification.* 37E30, 37B99.

Key words and phrases. Population genetics, ω-limit set, Quadratic stochastic operator.

The first author was supported by Polish MNiSW Grant N N201 0234 33 and EU FP6 Marie Curie Programmes RTN CODY and ToK SPADE2.

This is the map that we are going to investigate in the paper. We will call it the Stein-Ulam Spiral map, or, in short, the SUS map.

Let us illustrate interpretations of Binary Reaction Systems given in [6] using the SUS map. In terms of population genetics, x, y, z represent fractions of the population of types 1, 2 and 3 respectively. Mating between types 1 and 1, or 1 and 2, produces type 1. Mating between types 2 and 2, or 2 and 3, produces type 2. Finally, mating between types 3 and 3, or 3 and 1, produces type 3. Instead of individuals mating to produce offspring, one can also consider physical particles with 3 possible characteristics which collide in pairs and produce through the collision a pair of particles with new characteristics. The name "Binary Reaction System" comes from the fact that the system associates with each pair $(i, j) \in \{1, 2, 3\}^2$ a unique result $k \in \{1, 2, 3\}$.

Let us note that various classes of quadratic maps to which f belongs are also called *quadratic stochastic operators* and *Volterra operators*.

Let us quote from [6]:

"The situation is illustrated in figure 7. The interior fixed point is nonattractive and the 3 corners of the triangle are attractive only along the boundaries in a clockwise direction. Under iteration, points will spiral out, approaching arbitrarily close to the boundaries but, e.g., as the transformation $\alpha' = 2\alpha(l - S)$ shows (for the bottom boundary), no point inside the triangle can ever reach the boundary. Thus a general point will continue to spiral indefinitely."

"Numerically, a spurious convergence was observed owing to the fact that the several random initial points chosen rapidly iterated to within a distance less than 10^{-8} from one or another boundary line."

"If one transforms the triangle into the unit circle in an appropriate manner, the situation can be viewed as follows: The center is a non-attractive fixed point, and all points lying in the circle spiral outwards towards the circumference. On the circumference itself, there are 3 fixed points located, say, at $\vartheta = 0, 2\pi/3, 4\pi/3$, which in turn define 3 arcs. Any point lying on one of these arcs will move under iteration in a clockwise direction, ultimately converging to the fixed point which constitutes the right-hand boundary of the arc in question. Interior points, however, can never reach the boundary. In general, the sequence of iterates of any interior point (excluding the center) does not converge."

This is a quite accurate description of the dynamics of the SUS map. Figure 7 from [6] (our Figure 1) presents the results of a computer simulation in a conveniently chosen coordinate system in Δ. One can think that using a primitive (by our standards) computer IBM 704, that carried out the computations, did not allow Ulam and Stein to get a much better picture, which would shed much more light on the nature of iterates of the SUS map. However, this is not the case. Let us compare two similar figures obtained a modern computer. Such computers are sophisticated by our standards, but we hope that they will be considered hopelessly primitive in 2059. The first figure shows a hundred iterates, the second figure ten million iterates. There is almost no difference between them! Thus, Stein and Ulam obtained a picture that is practically not worse than what we can do now.

The reason why Figure 3 does not differ much from Figure 2 is the speed with which the trajectory approaches the boundary. To produce those figures, we represented real numbers as the 80-bit *extended* type, that allows for numbers as small as 1.9×10^{-4932}.

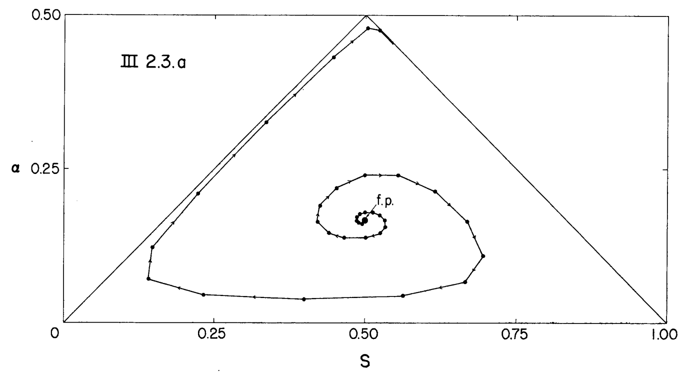

FIGURE 1. Figure 7 from [6].

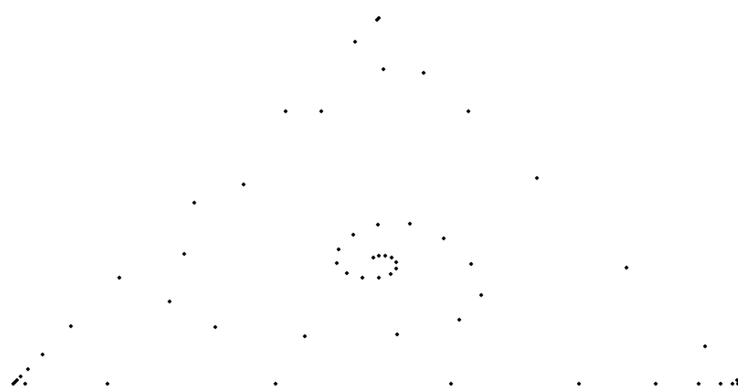

FIGURE 2. The trajectory of length 10^2 of the point $(0.32, 0.34, 0.34)$.

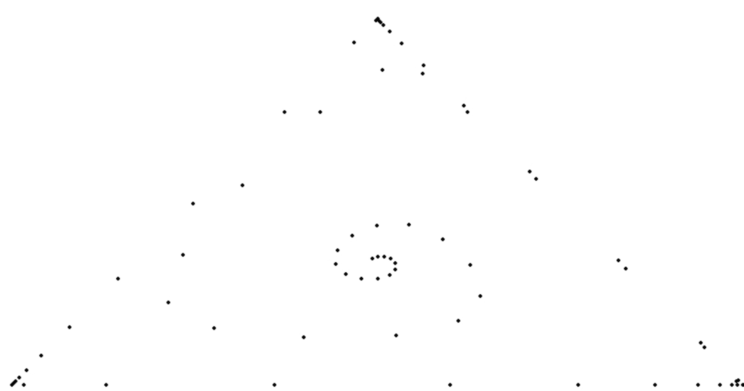

FIGURE 3. The trajectory of length 10^7 of the point $(0.32, 0.34, 0.34)$.

However, one of the variables becomes smaller than that after 120 steps, and later it stays 0.

Estimates show that if the trajectory spends time n close to a vertex, it spends time of order $n^2/2$ close to the next vertex. Thus, if it is close to one vertex for 2^3

iterates, it is close to the next vertex for 2^5 iterates, then $2^9, 2^{17}, 2^{33}, 2^{65}, \ldots$ iterates. Since $2^{65} > 3 \times 10^{18}$, one cannot expect really better pictures even in 2059.

Let us continue our description of the history of the SUS map. Ulam in his book [7], published in 1960, stated the conjecture that for all 97 Binary Reaction Systems, including the Stein-Ulam Spiral, the ergodic averages of the identity function

$$\frac{1}{n}\sum_{k=0}^{n-1} f^n(x, y, z) \tag{1.2}$$

converge for almost every point $(x, y, z) \in \Delta$. This conjecture was cited in [3] and [4]. It was shown to be false by M. Zakharevich in 1978 [9]. He proved that for the SUS map, the averages (1.2) diverge for every interior point (x, y, z) of Δ, except the fixed point $(1/3, 1/3, 1/3)$. This is caused by the rapid increase of the times that trajectory spends close to the consecutive vertices of the triangle Δ, that we mentioned before (although Zakharevich was using weaker estimates).

In 1970, Kesten [2] proved that the function $\varphi(x, y, z) = xyz$ is a Lyapunov function for the SUS map, and consequently, for every point except the center of of Δ, the ω-limit set is contained in the boundary of Δ.

In 1972, S. Vallander [8] proved that the trajectory of each interior point of Δ, except the fixed point, spends almost all time close to one of the vertices of Δ, and that the ω-limit set for this trajectory is infinite. He asked the question whether this ω-limit set is the whole boundary of Δ. This is the question that we are answering affirmatively for typical (in the topological sense) points here (Theorem 6.3). This question has been repeated by Yuri Lyubich in his book [5] (the book contains also a lot of other interesting information on the SUS and similar maps). In fact, we learned about this question from Yuri Lyubich.

On the other hand, it turns out that for any closed invariant subset E of the boundary of Δ intersecting all three sides of Δ, the set of points having E as the ω-limit set is relatively large. Namely, its intersection with any non-empty open set has Hausdorff dimension at least 1 (Theorem 6.2).

Note that for the SUS map the boundary of Δ can be viewed as a heteroclinic cycle. However, there are two basic differences here. The first one is that normally heteroclinic cycles are defined for flows, while here we are considering discrete dynamical system (iterates of a map). For a flow the question about the ω-limit set would not be asked, because obviously it would be the boundary of Δ for all trajectories approaching it. Secondly, the vertices of Δ are kind of saddle points, but in the contracting direction the eigenvalue is 0 (so strictly speaking, our map is not a diffeomorphism on Δ). In particular, using just linear approximations in the neighborhoods of the vertices will not work.

The paper is organized as follows. In Section 2 we introduce the notation. In Sections 3, 4 and 5 we obtain necessary estimates. Finally, in Section 6 we prove a technical lemma and deduce from it the main results of the paper, Theorems 6.3 and 6.2.

We are grateful to Henk Bruin, who noticed that the method that we were using in an earlier version of the paper to prove Theorem 6.3 can be also used to find points whose ω-limit set is not the whole boundary of Δ.

2. Notation

As we already mentioned, we will investigate the Stein-Ulam Spiral map f on the simplex Δ, given by the equation (1.1). Clearly, it commutes with the cyclic permutation of the coordinates $\xi(x,y,z)=(z,x,y)$. We know that because xyz is a Lyapunov function, all trajectories (except the fixed center) converge to the boundary of Δ, so we may concentrate our attention on a small neighborhood of this boundary. Thus, we start where y and z is small, get to where x and z are small, and when y increases substantially, we apply ξ. This way, up to the permutations ξ and ξ^2, we are tracing the f-trajectory of the starting point.

The space Δ is 2-dimensional, so we do not need the third coordinate. When we switch to variables x, y, we get the map

$$F(x,y)=(X(x,y),Y(x,y))=(x^2+2xy,2y-2xy-y^2). \tag{2.1}$$

The coordinate permutation will now become the "transition map"

$$P(x,y)=(1-x-y,x).$$

We know that the trajectory of an interior point of Δ will never reach the boundary of Δ, so we consider only points with $x,y\in(0,1)$ and $x+y<1$.

Now we define "cycles" (they are not periodic orbits!) in the following way. We will start from a point (x_0,y_0) with $1/100<x_0<1/6$ and $y_0>0$ small, and apply F repeatedly. We will stop applying F when y gets sufficiently large. We will specify that moment later. Then we apply P once and begin the new cycle.

Let us compute the derivative of our map F:

$$DF(x,y)=2\begin{pmatrix} x+y & x \\ -y & 1-x-y \end{pmatrix}.$$

In order to perform estimates we have to introduce several new variables. If s is the reciprocal of the slope of a vector v then we denote by $S_{x,y}(s)$ the reciprocal of the slope of $DF(x,y)(v)$. We have

$$S_{x,y}(s)=\frac{xs+ys+x}{1-x-y-ys}\ .$$

We will investigate how the expression $t=sy/x$ changes. After one step, we get

$$\begin{aligned} T_{x,y}(t)=\frac{S_{x,y}(s)Y(x,y)}{X(x,y)} &= \frac{(xs+ys+x)(2-2x-y)y}{(1-x-y-ys)(x+2y)x} \\ &= \frac{(2-2x-y)y}{(1-x-y-tx)(x+2y)}\left(\frac{x+y}{y}\,t+1\right). \end{aligned}$$

If c is the second component of a vector v then we denote by $C_{x,y,s}(c)$ the second component of $DF(x,y)(v)$. We have

$$C_{x,y,s}(c)=2c(1-x-y-ys), \tag{2.2}$$

where s is the reciprocal of the slope of the vector v. We will always assume that $c>0$.

When y exceeds $5/6$ then we apply additionally the transition map P given by

$$P(x,y)=(1-x-y,x).$$

Its derivative is

$$DP(x,y) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that P commutes with F (as it should).

We will start from a point (x_0, y_0) with $1/100 < x_0 < 1/6$ and $y_0 > 0$ small, and apply F repeatedly. We will use notation $(x_n, y_n) = F^n(x_0, y_0)$.

Lemma 2.1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $1/100 < x_0 < 1/6$ and $0 < y_0 < \delta$ then there exists $N = N(x_0, y_0)$ such that*

(a) *for every $n \in \{0, 1, \dots, N-6\}$ we have $\min(x_n, y_n) < \varepsilon$ and $\max(x_n, y_n) < 1/6$;*
(b) *for every $n \in \{N-5, \dots, N\}$ we have $x_n < \varepsilon$,*
(c) *for every $n \in \{0, 1, \dots, N-6\}$ we have $x_{n+1} < x_n/2$ and $y_{n+1} > (3/2)y_n$,*
(d) *for every $n \in \{N-5, \dots, N-1\}$ we have $y_n \le 5/6$, and $5/6 < y_N \le 35/36$.*

Proof. Fix $\varepsilon > 0$. For a sufficiently small $\delta > 0$ if $1/100 < x_0 < 1/6$ and $0 < y_0 < \delta$ then $y_0 < \varepsilon$ and $\max(x_0, y_0) < 1/6$. If $\max(x_n, y_n) < 1/6$ then from the formulas for $X(x,y)$ and $Y(x,y)$ we get immediately that $x_{n+1} < x_n/2$ and $y_{n+1} > (3/2)y_n$. Therefore there must be $N' \ge 0$ such that $\min(x_n, y_n) < \varepsilon$ and $\max(x_n, y_n) < 1/6$ hold for $n = 0, 1, \dots, N'$, but not for $n = N' + 1$. In particular, (a) and (c) hold for N' replacing $N - 6$.

We know that $xy(1-x-y)$ is a Lyapunov function for F. Therefore for all n at least one of the numbers x_n, y_n, $1 - x_n - y_n$ is smaller than $\sqrt[3]{\delta}$, which is smaller than ε if δ is sufficiently small. For $n = N' + 1$, if $\min(x_n, y_n) \ge \varepsilon$ then $1 - x_n - y_n < \varepsilon$, which gives us $\max(x_n, y_n) \ge 1/6$ if $\varepsilon < 2/3$ (which we may assume). This proves that $\max(x_n, y_n) \ge 1/6$ for $n = N' + 1$. However, we know already that x_n decreases with n for $n \le N' + 1$, so since $x_0 < 1/6$, we get $x_n < 1/6$ for $n = N' + 1$. This means that $y_n \ge 1/6$ for $n = N' + 1$ and that $x_n < \varepsilon$ for $n = N'$.

Set $\varphi(y) = Y(0, y) = y(2 - y)$. We have

$$\varphi\left(\frac{1}{6}\right) = \frac{11}{36} > \frac{1}{4},$$
$$\varphi\left(\frac{1}{4}\right) = \frac{7}{16} > \frac{3}{8},$$
$$\varphi\left(\frac{3}{8}\right) = \frac{39}{64} > \frac{9}{16},$$
$$\varphi\left(\frac{9}{16}\right) = \frac{207}{256} > \frac{3}{4},$$
$$\varphi\left(\frac{3}{4}\right) = \frac{15}{16} > \frac{5}{6}.$$

If x_n is sufficiently small, so are x_{n+i} for $i = 0, 1, 2, 3, 4$ (this follows from the formula for $X(x, y)$), and the above inequalities hold for $Y(x_n, y)$ replacing $\varphi(y)$. Since φ is increasing in the range where we use it, we see that if $y_n \ge 1/6$ then $y_{n+5} > 5/6$. Therefore $y_n > 5/6$ for $n = N' + 6$. Thus, there is $N \in \{N' + 1, \dots, N' + 6\}$ such that $y_n \le 5/6$ for $n \le N - 1$ and $y_N > 5/6$.

The function φ is increasing for the argument smaller than 1, so according to the formula for $Y(x,y)$ we get $y_N < \varphi(y_{N-1}) \le \varphi(5/6) = 35/36$. Now the lemma follows immediately from what we proved above. ☐

In the sequel, we will assume that $\varepsilon > 0$ is very small,

$$1/100 < x_0 < 1/6 \quad \text{and} \quad 0 < y_0 < \delta, \tag{2.3}$$

where δ is as in Lemma 2.1. We will use those assumptions without stating them. Moreover, sometimes we will use Lemma 2.1 without explicitly mentioning it.

We will stop applying F when $n = N$, then we apply P once. We will denote $(\widetilde{x}, \widetilde{y}) = P(x_N, y_N)$.

By Lemma 2.1, there is $M < N - 6$ such that $x_n > y_n$ for $n = 0, 1, \dots, M$ and $x_n \le y_n$ for $n = M+1, M+2, \dots, N$.

Additionally, we will look what happens under the action of the derivative of F with the vector $(s_0 c_0, c_0)^T$. We will denote the n-th image of this vector by $(c_n s_n, c_n)^T$. That is, $s_{n+1} = S_{x_n, y_n}(s_n)$ and $c_{n+1} = C_{x_n, y_n, s_n}(c_n)$. We will also denote $(\widetilde{sc}, \widetilde{c})^T = DP(x_N, y_N)(s_N c_N, c_N)^T$ and $\widetilde{t} = \widetilde{sy}/\widetilde{x}$.

3. Estimates for points

Here we will make estimates for the points of the trajectory of (x_0, y_0), under the assumption (2.3).

Lemma 3.1. *For $n = 0, 1, \dots, M$ we have $x_n < (1/2)^{2^n}$. Moreover, $2^M < -\log_2 y_0$.*

Proof. If $n \le M$ then $y_n < x_n$, so $x_{n+1} < 3x_n^2$. Since $x_0 < 1/6$, we get by induction $x_n < 3^{2^n - 1}(1/6)^{2^n}$, so $x_n < 3^{2^n}(1/6)^{2^n} = (1/2)^{2^n}$.

On the other hand, we know that $y_M > y_0$, so

$$y_0 < y_M < x_M < (1/2)^{2^M},$$

and therefore $2^M < -\log_2 y_0$. ☐

Lemma 3.2. *There exists a constant $K_1 > 0$, independent of x_0 and y_0, such that*

$$\prod_{n=0}^{M}(1 - 2x_n) > K_1 \quad \text{and} \quad \prod_{n=0}^{N-6}(1 - 3y_n) > K_1.$$

Proof. By Lemma 3.1, $x_n < (1/2)^{2^n}$ for $n \le M$, so

$$\prod_{n=0}^{M}(1 - 2x_n) > \prod_{n=0}^{M}(1 - 2(1/2)^{2^n}) > \prod_{n=0}^{\infty}(1 - 2(1/2)^{2^n}).$$

The series $\sum_{n=0}^{\infty}(1/2)^{2^n}$ is convergent, so the product $\prod_{n=0}^{\infty}(1 - 2(1/2)^{2^n})$ converges to some positive constant K_1'.

Similarly, by Lemma 2.1 (c), for $n \le N - 6$ we have $y_{N-6} > (3/2)^{N-6-n} y_n$, so $y_n < y_{N-6}(2/3)^{N-6-n}$. Therefore, taking into account that $y_{N-6} < 1/6$, we get

$$\prod_{n=0}^{N-6}(1 - 3y_n) > \prod_{n=0}^{N-6}\left(1 - 3y_{N-6}\left(\frac{2}{3}\right)^n\right) > \prod_{n=0}^{\infty}\left(1 - \frac{1}{2}\left(\frac{2}{3}\right)^n\right).$$

The series $\sum_{n=0}^{\infty}(1/2)(2/3)^n$ is convergent, so the product $\prod_{n=0}^{\infty}(1 - (1/2)(2/3)^n)$ converges to some positive constant K_1''.

Now we take $K_1 = \min(K_1', K_1'')$ and the proof is complete. □

Lemma 3.3. *There exist constants $K_2 > K_3 > 0$, independent of x_0 and y_0, such that*

$$K_3 < 2^N y_0 < K_2.$$

Proof. We have

$$y_{n+1} = 2y_n \left(1 - x_n - \frac{y_n}{2}\right),$$

so, taking into account Lemma 2.1 (d),

$$\frac{5}{6} \geq y_{N-5} = 2^{N-5} y_0 \prod_{n=0}^{N-6} \left(1 - x_n - \frac{y_n}{2}\right).$$

For $n = 0, 1, \dots, M$ we have $y_n < x_n$, so $1 - x_n - y_n/2 > 1 - 2x_n$, and by Lemma 3.2 we get

$$\prod_{n=0}^{M} \left(1 - x_n - \frac{y_n}{2}\right) > \prod_{n=0}^{M} (1 - 2x_n) > K_1.$$

Next, if $n > M$ then $y_n \geq x_n$, so $1 - x_n - y_n/2 > 1 - 3y_n$, and by Lemma 3.2 we get

$$\prod_{n=M+1}^{N-6} \left(1 - x_n - \frac{y_n}{2}\right) > \prod_{n=M+1}^{N-6} (1 - 3y_n) > K_1.$$

Thus, we get $5/6 > 2^{N-5} y_0 K_1^2$. Setting $K_2 = 80/(3K_1^2)$ we obtain $2^N y_0 < K_2$.

On the other hand, $y_{n+1} < 2y_n$, so $5/6 < y_N < 2^N y_0$. This completes the proof. □

Lemma 3.4. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y_0 < \delta$ then $M < \varepsilon N$, $x_N < y_0/2$ and $x_N < \varepsilon/N$.*

Proof. Fix $\varepsilon > 0$. By Lemmas 3.1 and 3.3 we have

$$\frac{M}{N} < \frac{\log_2(-\log_2 y_0)}{-\log_2 y_0 + \log_2 K_3}.$$

The right-hand side of this inequality goes to 0 as $y_0 \to 0$, so there exists $\delta > 0$ such that if $y_0 < \delta$ then $M < \varepsilon N$.

From the formula for $Y(x, y)$ we get $y_{n+1} < 2y_n$. From this and Lemma 2.1 (c) we get

$$x_{N-5} < 2^{-(N-M-6)} x_{M+1} \leq 2^{-(N-M-6)} y_{M+1} < 2^{-(N-2M-7)} y_0.$$

For $n = N-5, \dots, N-1$ we use the estimate $x_{n+1} < 3x_n$, which is valid because $x_n, y_n < 1$. Thus,

$$x_N < 2^{-(N-2M-7)} 3^5 y_0 < 2^{-(N-2M-7)} 3^5 \delta.$$

We may assume that $\varepsilon < 1/6$, so if $y_0 < \delta$ then $M < N/6$. Moreover, by taking a smaller δ if necessary, by Lemma 3.3 we may assume that $N > 42$. Then $N - 2M - 7 > 21$ and $N - 2M - 7 > N/2$. Thus, $3^5/2^{N-2M-7} < 1/2$, so $x_N < y_0/2$, and $x_N < 3^5\delta/2^{N/2}$. By taking a smaller δ again (if necessary), we get $3^5\delta < \varepsilon$, and since $2^{N/2} > N$, we obtain $x_n < \varepsilon/N$. □

4. Estimates for slopes

In this section we start with a vector $(s_0c_0, c_0)^T$ tangent to $\mathbb{R}^2$ at (x_0, y_0) and will estimate the reciprocal of the slope of its image $(s_nc_n, c_n)^T$ under the derivative of F^n, that is, s_n. More precisely, we will compare s_n with x_n/y_n, so we will estimate t_n.

We will make the following assumption on y_0 and t_0:

$$-\frac{3333}{\sqrt{-\log_2 y_0}} < t_0 < 0. \tag{4.1}$$

Again, for most times we will not mention this assumption.

We will first investigate the sign of s_n (which is the same as the sign of t_n) and acquire some information about its size.

Lemma 4.1. *For every $\varepsilon > 0$ there is $\delta > 0$ such that if $y_0 < \delta$ then*

(a) $1 - x_n - y_n - y_ns_n > 0$ *for* $n = 0, 1, \dots, N-1$,
(b) $s_n < 1$ *for* $n = 0, 1, \dots, M$,
(c) $s_n < \varepsilon$ *for* $n = M+1, M+2, \dots, N$,
(d) *if* $s_k \geq 0$ *for some* $k \in \{1, 2, \dots, N-6\}$ *then* $s_n > 0$ *for* $n = k+1, k+2, \dots, N$.

Proof. If $s_n < 1$ and $x_n, y_n < 1/6$ then $x_ns_n + y_ns_n + x_n < 1/2$, while $1 - x_n - y_n - y_ns_n > 1/2$, so $s_{n+1} < 1$. Since $s_0 < 0 < 1$, by induction we get $s_n < 1$ for $n = 0, 1, \dots, N-5$. In particular, this proves (b). Moreover, this proves (a) for $n \leq N-5$.

Since $s_M < 1$, we have

$$s_{M+1} < \frac{2x_M + y_M}{1 - x_M - 2y_M} < 4x_M + 2y_M.$$

By Lemma 2.1 (a) there exists $\delta > 0$ such that $y_M < \varepsilon^2/36 < \varepsilon/6$ (we may assume that $\varepsilon < 6$) and $x_{M+1} < \varepsilon^2/36$. Then $x_M < \sqrt{x_{M+1}} < \varepsilon/6$, so $s_{M+1} < \varepsilon$. Further we use induction (up to $n = N-6$), taking into account that by Lemma 2.1 (a) we have $x_n < \varepsilon/6$. Assume that $s_n < \varepsilon$. If $s_n < 0$ then $s_{n+1} < 2x_n < \varepsilon$; if $s_n \geq 0$ then $s_{n+1} < (2/3)s_n + 2x_n < \varepsilon$. This proves (c) for $n = 0, 1, \dots, N-5$.

If ε is sufficiently small (which we may assume) then for the last 5 steps the denominator in the formula for $S_{x,y}(s)$ is uniformly bounded away from 0, while the numerator is less than $3\max(x, s)$. Thus a short (with 5 steps) induction shows that there is a constant α such that (c) holds with $\alpha\varepsilon$ instead of ε (we use Lemma 2.1 (b)). However, we could have taken ε/α from the very beginning, so (with a different $\delta > 0$) (c) holds. If ε is sufficiently small, this implies also that (a) holds. Finally, (d) follows from (a) and the formula for $S_{x,y}(s)$. □

In the sequel we will assume always that y_0 is smaller than the δ from the above lemma (for some ε, for instance $\varepsilon = 1$). Then by (a) we have

$$1 - x_n - y_n - x_nt_n = 1 - x_n - y_n - y_ns_n > 0$$

for $n = 0, 1, \dots, N-1$. We will use this and (d) without mentioning it explicitly.

Lemma 4.2. *There is a constant $K_4 > 0$ such that if $n \leq M+2$ and $t_n < 0$ then $|t_n| < K_4 \cdot 2^{M/2}$.*

Proof. If $t < 0$ and $T_{x,y}(t) < 0$ then

$$(4.2)\quad |T_{x,y}(t)| < \frac{2y}{(1-x-y)(x+2y)} \cdot \frac{x+y}{y} \cdot |t| = \frac{2|t|}{1-x-y} \cdot \frac{x+y}{x+2y} < \frac{2|t|}{1-x-y}.$$

Take $n \le M+1$ such that $t_n < 0$. By Lemma 4.1 (d), $t_k < 0$ for all $k \le n$. Therefore, by (4.1), (4.2) and Lemma 3.2

$$|t_n| < |t_0| \prod_{k=0}^{M} \frac{2}{1-2x_k} < \frac{3333}{\sqrt{-\log_2 y_0}} \cdot \frac{2^{M+1}}{K_1}.$$

Moreover, since $x_{M+1}, y_{M+1} < 1/6$, we have $2/(1-x_{M+1}-y_{M+1}) < 3$, so if $t_{M+2} < 0$ then, in view of (4.2), we get

$$|t_{M+2}| < 3 \cdot \frac{3333}{\sqrt{-\log_2 y_0}} \cdot \frac{2^{M+1}}{K_1}.$$

Now, applying Lemma 3.1, if $n \le M+2$ and $t_n < 0$, we obtain $|t_n| < K_4 \cdot 2^{M/2}$ with $K_4 = 6 \cdot 3333/K_1$. □

We will use the constant K_4 without mentioning its origin.

Let

$$M' = 2 + M + 2K_4 2^{M/2}.$$

Lemma 4.3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y_0 < \delta$ then*

$$M' < \varepsilon N.$$

Proof. By Lemmas 3.1 and 3.3,

$$\frac{M'}{N} < \frac{2 + \log_2(-\log_2 y_0) + 2K_4\sqrt{-\log_2 y_0}}{\log_2 K_3 - \log_2 y_0}.$$

The right hand side goes to 0 as $y_0 \to 0$, and the lemma follows. □

Lemma 4.4. *For every $\varepsilon > 0$ there is $\delta > 0$ such that if $n \le M'$, $t_n < 0$, and $y_0 < \delta$ then $y_n|t_{M+1}| < \varepsilon$ and $y_n|t_{M+2}| < \varepsilon$.*

Proof. By Lemmas 4.2 and 3.1, and since $y_{k+1} < 2y_k$ for every k, we get

$$y_n|t_{M+1}| < 2^{M'} y_0 K_4 2^{M/2} < 4(-\log_2 y_0) 2^{2K_4\sqrt{-\log_2 y_0}} y_0 K_4 \sqrt{-\log_2 y_0}.$$

Therefore

$$\log_2(y_n|t_{M+1}|) < 2 + \tfrac{3}{2}\log_2(-\log_2 y_0) + 2K_4\sqrt{-\log_2 y_0} + \log_2 y_0 + \log_2 K_4.$$

The expression at the right hand side of the above inequality is independent of n and goes to $-\infty$ as $y_0 \to 0$, so there exists $\delta > 0$ such that if $y_0 < \delta$ then $y_n|t_{M+1}| < \varepsilon$.

The same estimates hold if we replace $|t_{M+1}|$ by $|t_{M+2}|$. □

Lemma 4.5. *For every $\varepsilon \in (0, 1/12)$ there exists $\delta > 0$ such that if $y_0 < \delta$, $t_n < 0$ and $M \le n \le M'$ then*

$$(4.3)\qquad \left(2y_n + \frac{x_n}{y_n}\right)|t_n| < 6\varepsilon < \frac{1}{2},$$

and either $t_{n+1} \ge 0$ or

$$(4.4)\qquad |t_{n+1}| < |t_n| - \tfrac{1}{2}.$$

Consequently, if $M' < n \le N-5$ then $t_n \ge 0$.

Proof. Observe first that if $t<0$, $T_{x,y}(t)<0$ and $x \le y < 1/3$ then

$$\begin{aligned}(1-x-y)(x+2y)(2-y)-(2-y-2x)y(2-2y) &= x(2-2x-3y-y^2+xy)\\ &> 2x(1-x-2y) \ge 0,\end{aligned}$$

so

$$\begin{aligned}|T_{x,y}(t)| &< \frac{2-y}{2-2y}\left(\frac{x+y}{y}|t|-1\right) = \left(1+\frac{y}{2-2y}\right)\left(\left(1+\frac{x}{y}\right)|t|-1\right)\\ &< \left(1+\frac{y}{2-2y}\right)\left(1+\frac{x}{y}\right)|t|-1 \le \left(1+\frac{2y}{2-2y}+\frac{x}{y}\right)|t|-1\\ &< \left(1+2y+\frac{x}{y}\right)|t|-1.\end{aligned} \tag{4.5}$$

Next note that if $x \le y < \varepsilon$ and $y|t| < \varepsilon$ then $y|s| \le y|t|$, so

$$|S_{x,y}(s)| = \left|\frac{xs+ys+x}{1-x-y-ys}\right| \le \frac{2y|s|+y}{1-2y-y|s|} < \frac{3\varepsilon}{1-3\varepsilon} < 4\varepsilon. \tag{4.6}$$

Let us assume now that $y_0 < \delta$ for $\delta > 0$ so small that we can use lemmas that we already proved with appropriately chosen values of ε. We will use induction. First notice that by (4.5), if (4.3) holds then either $t_{n+1} \ge 0$ or (4.4) holds.

Now, by Lemma 2.1 (a) we get $y_M < \varepsilon/2$, so $y_{M+1} < \varepsilon$. By Lemma 4.4, if $t_{M+1} < 0$ then $y_{M+1}|t_{M+1}| < \varepsilon$. Thus, we can use (4.6), and we get $|s_{M+2}| < 4\varepsilon$. Moreover, by Lemma 4.4, if $t_{M+2} < 0$ then $y_{M+2}|t_{M+2}| < \varepsilon$. Therefore

$$\left(2y_{M+2}+\frac{x_{M+2}}{y_{M+2}}\right)|t_{M+2}| = 2y_{M+2}|t_{M+2}| + |s_{M+2}| < 2\varepsilon + 4\varepsilon = 6\varepsilon,$$

so (4.3) holds for $n = M+2$.

Now assume that $M+2 \le n < M'$, (4.3) holds, and $t_{n+1} < 0$. By the inductive use of (4.4) we have $|t_n| \le |t_{M+2}|$. If $y_n \ge \varepsilon$ then by Lemma 4.4 $y_n|t_n| \le y_n|t_{M+2}| < \varepsilon/2$ (we can use that lemma for ε replaced by $\varepsilon/2$), so $\varepsilon|t_n| < \varepsilon/2$. Thus, $|t_n| < 1/2$, so by (4.4) $|t_{n+1}| < 0$, a contradiction. This proves that $y_n < \varepsilon$. From Lemma 4.4 we get $y_n|t_n| < \varepsilon$, so we can use (4.6). Therefore, we obtain (4.3) with $n+1$ replacing n in the same way as we did it in the preceding paragraph for $M+2$ replacing n (we use Lemma 4.4 once more to get $y_{n+1}|t_{n+1}| < \varepsilon$). This completes induction.

The last conclusion of the lemma follows from the inductive use of (4.4) and Lemma 4.2. □

Lemma 4.6. *There exists $\delta > 0$ such that if $y_0 < \delta$ then $t_N > N/33$.*

Proof. Observe first that if $x \le y$ and $2x + y < 1/4$ then $\sqrt{3} < 7/4 < 2-2x-y$, so $X(x,y) = x^2+2xy \le 3y^2 < (2-2x-y)^2y^2 = (Y(x,y))^2$. Moreover, if $t \ge 0$ and $x < y^2$ then $(2-2x-y)y > (1-x-y-tx)(x+2y)$, so $T_{x,y}(t) > t+1$. By Lemma 2.1 (a) with $\varepsilon = 1/24$, we get $2x_n + y_n < 1/4$ for $M' < n < N-5$, so the assumption from the first sentence of the proof is satisfied. Thus, by Lemmas 4.3 and 4.5, for a given $\eta > 0$ there is $\delta > 0$ such that if $y_0 < \delta$ then $t_{N-5} > (1-\eta)N$. In the last 5 steps we get

$$T_{x,y}(t) \geq (2-2x-y)\frac{x+y}{x+2y} \cdot t > (1-x-y/2) \cdot t > \frac{t}{2}$$

in view of Lemma 2.1 (b), (d). Thus, $t_N > (1-\eta)N/2^5 > N/33$ if η is sufficiently small. □

We need to know what happens when we complete a cycle, that is, we start with x_0, y_y, t_0 and end up with $\widetilde{x}, \widetilde{y}, \widetilde{t}$. Then we can repeat this cycle again, provided the assumption we made are satisfied by $\widetilde{x}, \widetilde{y}, \widetilde{t}$ instead of x_0, y_y, t_0. There are three such assumptions: $y_0 < \delta$ (for the smallest δ appearing in the preceding lemmas), $1/100 < x_0 < 1/6$ and (4.1).

According to the formulas for P, we have

$$\widetilde{y} = x_N, \quad \widetilde{x} = 1 - x_N - y_N, \quad \widetilde{t} = -\frac{x_N t_N + y_N}{t_N(1 - x_N - y_N)}\ . \tag{4.7}$$

Remark 4.7. The first equality above and Lemma 3.4 give us $\widetilde{y} < y_0/2$.

Lemma 4.8. *There is $\delta > 0$ such that if $y_0 < \delta$ then $1/100 < \widetilde{x} < 1/6$.*

Proof. If $\delta > 0$ is sufficiently small and $y_0 < \delta$ then by Lemma 2.1 (b) and (d) (for $\varepsilon = 1/100$) we get

$$\widetilde{x} = 1 - x_N - y_N > 1 - \frac{35}{36} - \frac{1}{100} > \frac{1}{100}$$

and $\widetilde{x} = 1 - x_N - y_N < 1 - 5/6 = 1/6$. □

Lemma 4.9. *There is $\delta > 0$ such that if $y_0 < \delta$ then $\widetilde{t} < 0$ and*

$$-\log_2 \widetilde{y} \cdot \widetilde{t}^2 < 3333^2$$

Proof. By Lemma 4.6, $t_N > 0$, so $\widetilde{t} < 0$. Moreover, we have

$$|\widetilde{t}| = \frac{x_N t_N + y_N}{t_N(1 - x_N - y_N)} = \frac{x_N}{1 - x_N - y_N} + \frac{y_N}{1 - x_N - y_N} \cdot \frac{1}{t_N}.$$

The right hand side is a decreasing function of (positive) t_N, so by Lemma 4.6 if y_0 is sufficiently small then

$$|\widetilde{t}| < \frac{x_N}{1 - x_N - y_N} + \frac{y_N}{1 - x_N - y_N} \cdot \frac{33}{N}.$$

Together with Lemmas 4.8 and 3.4 (with $\varepsilon = 0.33$), this gives us

$$|\widetilde{t}| < 100 x_N + \frac{3300}{N} < \frac{3333}{N}. \tag{4.8}$$

Now we have to estimate $-\log_2 \widetilde{y}$ from above. We have $x_{n+1} > 2x_n y_n$, so $x_N > x_0 \prod_{n=0}^{N-1}(2y_n)$. Therefore,

$$-\log_2 \widetilde{y} = -\log_2 x_N < -\log x_0 + \sum_{n=0}^{N-1}(-\log_2(2y_n)). \tag{4.9}$$

Since $y_{n+1} < 2y_n$ for each n, we get by induction $y_n > 2^{n-N} y_N$. Therefore

$$-\log_2(2y_n) < -\log_2 y_N + (N - n - 1).$$

Taking into account that $y_N > 5/6$, so $-\log_2 y_N < \log_2(6/5) < 1/2$, we get

$$\sum_{n=0}^{N-1}(-\log_2(2y_n)) < \frac{N}{2} + \frac{(N-1)N}{2} = \frac{N^2}{2}.$$

Together with (4.9) and since $x_0 > 1/100$, we obtain

$$-\log_2 \widetilde{y} < \log_2 100 + \frac{N^2}{2} < 7 + \frac{N^2}{2} < N^2,$$

because for δ small enough we get $N^2 > 14$. Together with (4.8) we get

$$-\log_2 \widetilde{y} \cdot \widetilde{t}^2 < N^2 \cdot \frac{3333^2}{N^2} = 3333^2.$$

This completes the proof. □

We can summarize the results of this section in the following proposition, that follows immediately from Lemmas 4.6, 4.8 and 4.9, and Remark 4.7.

Proposition 4.10. *There exists a constant $\delta > 0$ such that if $0 < y_0 < \delta$, $1/100 < x_0 < 1/6$ and (4.1) holds, then $0 < \widetilde{y} < \delta/2$, $1/100 < \widetilde{x} < 1/6$, (4.1) holds with y_0, t_0 replaced by $\widetilde{y}, \widetilde{t}$, and $t_N > N/33$.*

5. Estimates for stretching

Now we investigate how c/y varies along the trajectory.

Lemma 5.1. *There are constants $K_5 > 0$ and $\delta > 0$ such that if $y_0 < \delta$ then*

$$\frac{c_N}{y_N} > K_5 \frac{c_0}{y_0}.$$

Proof. Note first that

$$\frac{C_{x,y,s}(c)}{Y(x,y)} = \frac{c}{y} \cdot \frac{2-2x-2y-2ys}{2-2x-y} = \frac{c}{y}\left(1 - \frac{y+2ys}{2-2x-y}\right).$$

Therefore, if $2x + y < 1$ and $s \le 0$ then

$$\frac{C_{x,y,s}(c)}{Y(x,y)} > \frac{c}{y}(1-y),$$

and if $2x + 1 < 1$ and $s > 0$ then

$$\frac{C_{x,y,s}(c)}{Y(x,y)} > \frac{c}{y}(1-y-2ys).$$

By Lemma 2.1, we have $2x_n + y_n < 1$ for $0 \le n < N$. Therefore we get two different estimates from below for the ratio $(c_{n+1}/y_{n+1})/(c_n/y_n)$ depending on the sign of s_n. The first one is straightforward, $1 - y_n$. The second one is $1 - y_n - 2y_n s_n$. However, by Lemma 4.1 $s_n < 1$, so $1 - y_n - 2y_n s_n > 1 - 3y$ for $n = 0, 1, \ldots, N-6$. Thus, we get

$$\frac{c_{n+1}}{y_{n+1}} > \frac{c_n}{y_n}(1-3y_n)$$

for $n = 0, 1, \dots, N-6$. By Lemmas 4.1 (c) and 2.1 (d) there is $\delta > 0$ such that if $y_0 < \delta$ then $1 - y_n - 2y_n s_n > 1/10$ for $n = N-5, \dots, N-1$. Hence, in view of Lemma 3.2,

$$\frac{c_N}{y_N} > 10^{-5}\frac{c_0}{y_0}\prod_{n=0}^{N-6}(1-3y_n) > 10^{-5}K_1\frac{c_0}{y_0}.$$

Taking $K_5 = 10^{-5}K_1$ we get the desired inequality. □

The next lemma contains estimates relating c_N and c_{N-1}.

Lemma 5.2. *There is $\delta > 0$ such that if $y_0 < \delta$ then $c_{N-1}/4 < c_N < 2c_{N-1}$.*

Proof. By Lemma 4.5, we have $s_{N-1} > 0$, so from (2.2) it follows that $c_N < 2c_{N-1}$. Now, if δ is sufficiently small and $y_0 < \delta$ then by Lemma 2.1 (b) and (d) and Lemma 4.1 (c) we get $1 - x_{N-1} - y_{N-1} - y_{N-1}s_{N-1} > 1/8$. From this and (2.2) it follows that $c_{N-1}/4 < c_N$. □

Now we make many cycles that we defined in Section 2. We will use the old notation adding the argument j (time). For instance $c_{N(j)}(j)$ means c_N where we consider j-th cycle (we begin with $j = 0$). In particular, $\widetilde{x}(j) = x_0(j+1)$, etc.

Theorem 5.3. *There exist $K_6 > 0$ and $\delta > 0$ such that if $0 < y_0 < \delta$, $1/100 < x_0 < 1/6$ and (4.1) holds then*

$$c_{N(j+1)}(j+1) > -K_6 \log_2 y_0(j) \cdot c_{N(j)}(j) > 2c_{N(j)}(j) \tag{5.1}$$

for every $j \geq 0$.

Proof. By Proposition 4.10 used inductively, there is $\delta > 0$ such that if $0 < y_0 < \delta$, $1/100 < x_0 < 1/6$ and (4.1) holds then

$$\frac{s_{N(j)}(j)}{x_{N(j)}(j)} > \frac{N(j)}{33} \quad \text{and} \quad 0 < y_0(j) < \delta$$

for every j. Thus, by Lemma 3.3 there exists $\delta > 0$ such that if $y_0 < \delta$ then

$$\frac{s_{N(j)}(j)}{x_{N(j)}(j)} > \frac{\log_2(K_3/y_0(j))}{33} \tag{5.2}$$

for every j.

We have

$$y_0(j+1) = x_{N(j)}(j), \qquad c_0(j+1) = c_{N(j)}(j) \cdot s_{N(j)}(j).$$

Therefore, by Lemma 5.1 and (5.2),

$$\frac{c_{N(j+1)}(j+1)}{y_{N(j+1)}(j+1)} > K_5\frac{c_0(j+1)}{y_0(j+1)} = K_5\frac{c_{N(j)}(j)s_{N(j)}(j)}{x_{N(j)}(j)} > \frac{K_5}{33}\log_2\frac{K_3}{y_0(j)} \cdot c_{N(j)}(j).$$

Since by Lemma 2.1 (d) $y_{N(j+1)}(j+1) > 5/6$, we get

$$c_{N(j+1)}(j+1) > -K_6 \log_2 y_0(j) \cdot c_{N(j)}(j)$$

with $K_6 = K_5/40$, provided δ is sufficiently small.

Taking once more δ sufficiently small, we get $-K_6 \log_2 y_0(j) > -K_6 \log_2 \delta > 2$, which shows (5.1). □

Thus, $c_{N(j)}(j)$ undergoes stretching as j grows. This stretching is exponential in j (the number of cycles), but not exponential in the number of steps we make (the number of times we iterate F).

6. Theorems

Now we can prove the main results of the paper. We start with a technical lemma.

A part of the lemma cannot be stated in a concise way that is both rigorous and understandable, so let us start with some explanations. We have a sequence of points $(p_0, p_1, p_2, \dots)$ from the consecutive (clockwise) sides of Δ, so that

$$p_j = \begin{cases} (0, v_j, 1 - v_j) & \text{if } j \equiv 0 \pmod 3, \\ (v_j, 1 - v_j, 0) & \text{if } j \equiv 1 \pmod 3, \\ (1 - v_j, 0, v_j) & \text{if } j \equiv 2 \pmod 3, \end{cases} \tag{6.1}$$

with $v_j \in [3/4, 15/16)$. We look for a point p whose trajectory passing along consecutive sides gets very close to the corresponding points p_j. Since we use the results of the preceding sections, first we have to make several iterates to get $f^k(p) = (x_0, y_0, z_0)$ with $0 < y_0 < \delta$ and $1/100 < x_0 < 1/6$. This is possible, since the trajectory of p approaches the boundary of Δ and the above inequality for x_0 defines a segment on one of the sides of Δ that strictly contains a fundamental domain on that side. Then we can work in the coordinates (x, y) and apply all the formalism of the preceding sections. Since $y_{N(j)-1}(j) \le 5/6 < y_{N(j)}(j)$ and $3/4 < 5/6 < 15/16$, we can expect that either $y_{N(j)-1}(j)$ or $y_{N(j)}(j)$ will be close to v_j. To get this value of y starting from (x_0, y_0) we make $\sum_{i=0}^{j} N(i) - 1$ or $\sum_{i=0}^{j} N(i)$ iterates. In other words, we expect the point $f^{k+n_j}(p)$ to be close to p_j, where

$$\sum_{i=0}^{j} N(i) \in \{n_j, n_j + 1\}. \tag{6.2}$$

Now we are ready to state the lemma.

Lemma 6.1. *Let $(v_0, v_1, v_2, \dots)$ be a sequence of numbers from the interval $[3/4, 15/16)$ and let V be a non-empty open subset of Δ. Then there exists a subset $A \subset V$ of Hausdorff dimension at least 1 with the following property. For every $p \in A$ there is $k \ge 0$ such that $f^k(p) = (x_0, y_0, z_0)$ with $0 < y_0 < \delta$ and $1/100 < x_0 < 1/6$, with δ equal to the minimum of δ's from the preceding sections, and*

$$\lim_{j \to \infty} \| f^{k+n_j}(p) - p_j \| = 0, \tag{6.3}$$

where $\| \cdot \|$ is the Euclidean norm, p_j is defined by (6.1), and the numbers n_j satisfy (6.2).

Proof. We work in coordinates (x, y), and consider the maps F, P defined in Section 2. However, sometimes we will use the original coordinates (x, y, z). Although we will try to distinguish between the two, there will be cases when this would lead to an overly complicated notation, so incidentally they may be identified.

Let us note that F is a homeomorphism of the triangle with vertices $(1, 0)$, $(0, 0)$ and $(0, 1)$. Moreover, P permutes cyclically the three sides of the triangle and commutes with F.

Denote by I_1 the vertical side $\{(0,y) : y \in [0,1]\}$ and let $I_1' = \{(0,y) : y \in [3/4, 15/16)\}$. The map F on I_1 has the form $F(0,y) = (0, 2y - y^2)$, which can be written as $F(0, 1-v) = (0, 1-v^2)$, $v \in [0,1]$. Hence, I_1' is a fundamental domain of F restricted to I_1.

Similarly, $I_2' = \{(x,0) : x \in [1/49, 1/7)\}$ is a fundamental domain of F restricted to the horizontal side $I_2 = \{(x,0) : x \in [0,1]\}$. We know that the forward trajectory of every point from the interior of the triangle except $(1/3, 1/3)$ converges to the boundary, and its ω-limit set is infinite and contains points from all three (open) sides of the triangle (see [8]). Hence, it has to contain a point from I_2'. This shows that there is $k \geq 0$ and a point

$$(\vartheta_0, \eta_0) \in F^k(\widetilde{V}),$$

(where $\widetilde{V}$ is the set V in the coordinates (x,y)), such that $\vartheta_0 \in (1/100, 1/6)$ and $\eta_0 \in (0, \delta)$.

Define a curve

$$\gamma(\vartheta) = (\vartheta_0, \eta_0) + (\vartheta - \vartheta_0)(1, -1) = (\vartheta, \eta_0 + \vartheta_0 - \vartheta)$$

for $\vartheta \in [\vartheta_-, \vartheta_+]$, where $\vartheta_- < \vartheta_0 < \vartheta_+$ are so close to ϑ_0, that

$$\gamma \subset F^k(\widetilde{V})$$

and for every $(x,y) \in \gamma$ we have $1/100 < x < 1/6$, $0 < y < \delta$. We will use the same symbol γ for the curve treated as a map and for the curve treated as a subset of the triangle. Note that for every $\vartheta \in (\vartheta_-, \vartheta_+)$ the tangent vector $-\gamma'(\vartheta)$ has the form $(-1, 1)$ which can be written as $(s_0 c_0, c_0)$ for $s_0 = -1$, $c_0 = 1$. It is easy to check that $s_0 = -1$ satisfies condition (4.1) for $(x_0, y_0) \in \gamma$ if δ is sufficiently small. Thus, we can apply the results of the preceding sections for the starting points $(x_0, y_0) \in \gamma$ (note that here we are using the notation (x_0, y_0) in a generic sense of the preceding sections, not in the sense of the statement of the lemma).

Extending notation from Theorem 5.3, we will denote by $(x_n^\vartheta(j), y_n^\vartheta(j))$ the point $(x_n(j), y_n(j))$ for the starting point $(x_0, y_0) = \gamma(\vartheta)$ and use a similar convention for $c_n(j), N(j)$. Remember that we enumerate j beginning from 0, so that the cycle starting at (x_0, y_0) is the 0-th cycle. We will also denote by y_n^ϑ (without an argument) the second coordinate of the image of the starting point $\gamma(\vartheta)$ under the n-th iterate of our map, i.e.

$$y_n^\vartheta = y^\vartheta_{n - \sum_{i=0}^{j} N^\vartheta(i)}(j) \quad \text{for} \quad \sum_{i=0}^{j} N^\vartheta(i) < n \leq \sum_{i=0}^{j+1} N^\vartheta(i).$$

We will construct inductively a nested sequence of intervals $(\alpha_j, \beta_j) \subset [\vartheta_-, \vartheta_+]$ and a sequence of integers n_j for large j, such that $f^{k+n_j}(\gamma((\alpha_{j+1}, \beta_{j+1})))$ is closer and closer to p_j as j increases. We will require that

$$\sum_{i=0}^{j} N^\vartheta(i) \in \{n_j, n_j + 1\} \tag{6.4}$$

for every $\vartheta \in (\alpha_j, \beta_j)$ and

$$y^{\alpha_j}_{n_j - 1} = y^{\beta_j}_{n_j + 1} = \frac{5}{6}. \tag{6.5}$$

Making the first step requires some work.

For an arbitrary $\vartheta \in [\vartheta_-, \vartheta_+]$, consider the point $(x_0, y_0) = \gamma(\vartheta)$. We have $1/100 < x_0 < 1/6$ and $0 < y_0 < \delta$. By Lemma 5.1, assuming that $\delta < 5/6$ (which we can do), we get $c^\vartheta_{N(0)}(0) > K_5$, where K_5 is the constant from that lemma. Now, by Proposition 4.10, we can apply Theorem 5.3 inductively, and we get

$$c^\vartheta_{N(j)}(j) > K_5 2^j \tag{6.6}$$

for every $j \geq 0$. This and Lemma 5.2 give us the next estimate,

$$c^\vartheta_{N(j)-1}(j) > K_5 2^{j-1}. \tag{6.7}$$

Fix j_0 divisible by 3 and so large that

$$K_5 2^{j_0-1} \min(\vartheta_+ - \vartheta_0, \vartheta_0 - \vartheta_-) > 1 \tag{6.8}$$

and set

$$n_0 = \sum_{i=0}^{j_0} N^{\vartheta_0}(i) - 1. \tag{6.9}$$

This means that starting from $(x_0, y_0) = \gamma(\vartheta_0)$ and making n_0 steps, we complete j_0 full cycles and then $y^{\vartheta_0}_{n_0} \leq 5/6$ but $y^{\vartheta_0}_{n_0+1} > 5/6$. Thus, for all ϑ sufficiently close to ϑ_0, we have

$$\sum_{i=0}^{j_0} N^\vartheta(i) \in \{n_0, n_0+1\}. \tag{6.10}$$

Let (α, β) be the maximal interval containing ϑ_0 on which (6.10) holds. Suppose that $\alpha = \vartheta_-$ or $\beta = \vartheta_+$. The derivative of $y^\vartheta_{n_0}$ with respect to ϑ is equal to $-c^\vartheta_{n_0}$, that is, either $-c^\vartheta_{N^\vartheta(j_0)}(j_0)$ or $-c^\vartheta_{N^\vartheta(j_0)-1}(j_0)$. Therefore, in view of (6.6) and (6.7),

$$y^\alpha_{n_0} - y^\beta_{n_0} > \int_\beta^\alpha (-K_5 2^{j_0-1})\, d\vartheta = K_5 2^{j_0-1}(\beta - \alpha). \tag{6.11}$$

By this and (6.8) we get a contradiction. Thus, $\vartheta_- < \alpha < \vartheta_0 < \beta < \vartheta_+$.

Let us consider what properties of α and β make the interval (α, β) be the maximal one. When $\vartheta \in (\alpha, \beta)$ then starting from $\gamma(\vartheta)$ and making n_0 steps, we complete j_0 full cycles and then either $y^\vartheta_{n_0} \leq 5/6$ but $y^\vartheta_{n_0+1} > 5/6$ or $y^\vartheta_{n_0-1} \leq 5/6$ but $y^\vartheta_{n_0} > 5/6$. When $\vartheta = \alpha$ or $\vartheta = \beta$, this begins to break, so when starting from $\gamma(\vartheta)$ and making n_0 steps, we complete j_0 full cycles and then either $y^\vartheta_{n_0-1} = 5/6$ or $y^\vartheta_{n_0+1} = 5/6$. Since the derivative of $y^\vartheta_{n_0}$ with respect to ϑ is equal to $-c^\vartheta_{n_0}$, which is negative, we see that $y^\alpha_{n_0-1} = 5/6$ and $y^\beta_{n_0+1} = 5/6$. This means that if we set $\alpha_{j_0} = \alpha$, $\beta_{j_0} = \beta$ and $n_{j_0} = n_0$, then (6.4) and (6.5) will be satisfied for $j = j_0$. This completes the first step of the induction.

Now we define inductively (for $j > j_0$) numbers n_j and intervals (α_j, β_j). Suppose they are already defined for some j. By (6.5), we have

$$y^{\alpha_j}_{n_j} > \frac{35}{36} - \varepsilon \quad \text{and} \quad y^{\beta_j}_{n_j} < 1 - \frac{\sqrt{6}}{6} + \varepsilon \tag{6.12}$$

for a small $\varepsilon > 0$. Since

$$1 - \sqrt{6}/6 < 3/4 \quad \text{and} \quad 15/16 < 35/36, \tag{6.13}$$

this implies that there exists (a unique) $\vartheta_j \in (\alpha_j, \beta_j)$ such that

$$y_{n_j}^{\vartheta_j} = v_j. \tag{6.14}$$

Let

$$n_{j+1} = \sum_{i=0}^{j+1} N^{\vartheta_j}(i) - 1.$$

Then, repeating our previous arguments, we see that

$$\sum_{i=0}^{j+1} N^{\vartheta}(i) \in \{n_{j+1}, n_{j+1} + 1\} \tag{6.15}$$

for ϑ close to ϑ_j, so we can define $(\alpha_{j+1}, \beta_{j+1}) \subset (\alpha_j, \beta_j)$ to be the maximal interval containing ϑ_j, such that (6.15) holds for every $\vartheta \in (\alpha_{j+1}, \beta_{j+1})$, in particular the condition (6.4) holds for $j+1$ instead of j. Then the derivative of $y_{n_{j+1}}^{\vartheta}$ with respect to ϑ on $(\alpha_{j+1}, \beta_{j+1})$ is either $-c_{N(j+1)}^{\vartheta}(j+1)$ or $-c_{N(j+1)-1}^{\vartheta}(j+1)$. Since $(\alpha_{j+1}, \beta_{j+1}) \subset (\alpha_j, \beta_j)$, the derivative of $y_{n_j}^{\vartheta}$ with respect to ϑ on $(\alpha_{j+1}, \beta_{j+1})$ is either $-c_{N(j)}^{\vartheta}(j)$ or $-c_{N(j)-1}^{\vartheta}(j)$. Moreover, by Lemma 5.2,

$$c_{N(j)}^{\vartheta}(j) > c_{N(j)-1}^{\vartheta}(j)/4, \qquad c_{N(j+1)-1}^{\vartheta}(j+1) > c_{N(j+1)}^{\vartheta}(j+1)/2,$$

and by Proposition 4.10, $y_0^{\vartheta}(j) < \delta/2^j$. Using these facts together with Theorem 5.3, we get

$$\begin{aligned} 1 > y_{n_{j+1}}^{\alpha_{j+1}} - y_{n_{j+1}}^{\beta_{j+1}} &\geq \frac{1}{2} \int_{\alpha_{j+1}}^{\beta_{j+1}} c_{N(j+1)}^{\vartheta}(j+1)\, d\vartheta \\ &> -\frac{K_6}{2} \log_2 \frac{\delta}{2^j} \int_{\alpha_{j+1}}^{\beta_{j+1}} c_{N(j)}^{\vartheta}(j)\, d\vartheta \geq \frac{K_6}{8}(j - \log_2 \delta)(y_{n_j}^{\alpha_{j+1}} - y_{n_j}^{\beta_{j+1}}), \end{aligned}$$

which can be written as

$$y_{n_j}^{\alpha_{j+1}} - y_{n_j}^{\beta_{j+1}} < \delta_j \quad \text{for} \quad \delta_j = \frac{8}{K_6(j - \log_2 \delta)}. \tag{6.16}$$

Note that δ_j tends to 0 as $j \to \infty$. Since $y_{n_j}^{\vartheta}$ is a decreasing function of ϑ on $(\alpha_{j+1}, \beta_{j+1})$, (6.14) and (6.16) give

$$|y_{n_j}^{\vartheta} - v_j| < \delta_j \quad \text{for} \quad \vartheta \in (\alpha_{j+1}, \beta_{j+1}). \tag{6.17}$$

By (6.12), (6.13) and (6.14), we see that $\alpha_j < \alpha_{j+1} < \beta_{j+1} < \beta_j$ if j is sufficiently large (which we can assume enlarging j_0 if necessary). Similarly as before, this implies $y_{n_{j+1}-1}^{\alpha_{j+1}} = y_{n_{j+1}+1}^{\beta_{j+1}} = 5/6$. Since this is the condition (6.5) with j replaced by $j+1$, the inductive step of the construction is complete, and we additionally get (6.17).

As $\alpha_j < \alpha_{j+1} < \beta_{j+1} < \beta_j$, the intersection $\bigcap_{j=j_0}^{\infty} (\alpha_j, \beta_j)$ is non-empty. Take a point $\widehat{\vartheta}$ from this intersection and set

$$(x_0, y_0) = \gamma(\widehat{\vartheta}), \qquad p = f^{-k}(x_0, y_0, 1 - x_0 - y_0).$$

Since $\gamma \subset F^k(\widetilde{V})$, we have $p \in V$. Since we know that the trajectory of p approaches the boundary of Δ, the number j_0 is divisible by 3, the transition map P commutes with F and P^3 is the identity, by (6.1) and (6.17) we get (6.3).

In this way we proved existence of one point $p \in V$ satisfying (6.3). However, the choice of a point $(\vartheta_0, \eta_0) \in F^k(\widetilde{V})$ at the beginning of the proof was arbitrary to some degree. If we slightly vary η_0, the same proof will work. Since the segments γ for different choices of η_0 are disjoint from one another, we get distinct points $p \in V$ fulfilling (6.3) with $f^k(p)$ from each of those segments. Hence, if A is the set of all these points p, the projection of $f^k(A)$ to a line perpendicular to the segments γ contains an interval, and therefore the Hausdorff dimension of $f^k(A)$ is at least 1 (see, e.g., [1], Chapter 6, page 90). Since f is a diffeomorphism on the interior of Δ, the Hausdorff dimension of A is also at least 1. □

Now we are in a position to prove the main results of the paper. For a set E contained in the boundary of Δ denote by $\Lambda(E)$ the set of those points of Δ whose ω-limit set is E. We know that for every interior point p of Δ, except its center, the ω-limit set of p is a closed fully invariant (equal to its image) subset of the boundary of Δ, containing all three vertices and intersecting all three (open) sides. Call such sets *admissible*. The first theorem shows that all admissible sets are ω-limit sets for a rather large subset of Δ.

Theorem 6.2. *For any admissible set E and any non-empty open set $V \subset \Delta$, the Hausdorff dimension of $\Lambda(E) \cap V$ is at least* 1.

Proof. Choose a countable dense subset E' of E that does not contain the vertices of Δ. Take a sequence $(q_0, q_1, q_2, \dots)$ with terms from E' and such that each element of E' appears in it infinitely many times and the points q_{3i} are from the side $x = 0$, the points q_{3i+1} from the side $z = 0$ and the points q_{3i+2} from the side $y = 0$. Now replace each point q_j by a point p_j on its trajectory (forward or backward) of the form (6.1) with $v_j \in [3/4, 15/16)$. This is possible since on each side of Δ points of this form constitute a fundamental domain of f. Now apply Lemma 6.1 to get a set A. From (6.3) and the fact that each p_j appears infinitely many times it follows that the ω-limit set of any $p \in A$ contains all points p_j. By the invariance of the ω-limit sets, it contains all points q_j, and then, since ω-limit sets are closed, it contains E. We will show the reverse inclusion.

On the other hand, the trajectory of $p \in A$ passes close to each of the three fundamental domains mentioned above only once in three cycles, and all those passes are accounted for in (6.3) (in fact, there can be two close passes if p is an endpoint of the fundamental domain that we are considering, but then we use (6.3) and the continuity of f). Thus, the only ω-limit points of p are limits of the points p_j or their images and preimages under the iterates of f. However, all such points belong to E, so this proves that the ω-limit set of p is contained in E.

This proves that the ω-limit set of every point $p \in A$ is E, so by Lemma 6.1 the Hausdorff dimension of $\Lambda(E) \cap V$ is at least 1. □

Among the sets $\Lambda(E)$ one is much bigger than the other ones, namely the one with E equal to the whole boundary of Δ.

Theorem 6.3. *When E is the whole boundary of Δ then $\Lambda(E)$ is residual.*

Proof. Let E be the boundary of Δ. Choose a countable subset $H \subset E$. Let $\mathcal{U}$ be the family of all balls centered at the elements of H with rational radii. For $U \in \mathcal{U}$ denote $W(U) = \bigcup_{n=0}^{\infty} f^{-n}(U)$. By Theorem 6.2, $\Lambda(E)$ is dense in Δ, so the set $W(U)$

is also dense. Moreover, it is open. Thus $W = \bigcap_{U \in \mathcal{U}} W(U)$ is a dense G_δ set. Clearly, the ω-limit set of every $p \in W$ is E, so $W \subset \Lambda(E)$. This proves that the set $\Lambda(E)$ is residual. □

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