Math 153: Lecture Notes For Chapter 4

Section 4.1: Polynomial Functions of Degree Greater Than 2

It is very helpful to review Section 3.5 materials (shifting, reflecting, stretching, compressing).

Even Function:
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + c \]
- If \( n \) is even, the graph is symmetric with respect to the \( y \)-axis.
- \( a > 0 \), the graph rises to the left and to the right.
- \( a < 0 \), reflect the graph through the \( x \)-axis. The graph falls to the left and to the right.
- \( c > 0 \), shift upward
- \( c < 0 \), shift downward
- \( a > 1 \), Vertically Stretched
- \( a < 1 \), Vertically Compressed
- \( f(x) = f(-x) \) (see section 3.5 handout)

Odd Functions:
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + c \]
- If \( n \) is odd, the graph is symmetric with respect to the origin.
- \( a > 0 \), the graph falls to the left and rises to the right.
- \( a < 0 \), reflect the graph through the \( x \)-axis. The graph rises to the left and falls to the right.
- \( c > 0 \), shift upward
- \( c < 0 \), shift downward
- \( a > 1 \), Vertically Stretched
- \( a < 1 \), Vertically Compressed
- \( f(-x) = -f(x) \) (see section 3.5 handout)
Intermediate Value Theorem for Polynomials:

If \( f \) is a polynomial function and if \( f(a) \) and \( f(b) \) have opposite signs, then there is at least one value \( c \) between \( a \) and \( b \) for which \( f(c) = 0 \).

**Example:**
The following function has a zero between \( x = 1 \) and \( x = 2 \), and also between \( x = 3 \) and \( x = 4 \) because of the change in the function sign.

**Example 1:** Use the intermediate value theorem to show that \( f \) has a zero between \( a \) and \( b \).

\[
f(x) = x^3 + 2x^2 + 3x - 10 \quad ; \quad a = 1, \ b = 2
\]

Solution:

\[
f(1) = (1)^3 + 2(1)^2 + 3(1) - 10 = -4 \quad ; \quad f(2) = (2)^3 + 2(2)^2 + 3(2) - 10 = 12
\]

Since \( f(1) \) and \( f(2) \) have opposite sign, then there is at least one number \( c \) between \( a \) and \( b \) where \( f(c) = 0 \).

**Note:** for the following examples, it is helpful to review section 2.7 handout.

**Example 2:** Find all values of \( x \) such that \( f(x) > 0 \) and all \( x \) such that \( f(x) < 0 \), and then sketch the graph of \( f \).

\[
f(x) = x^3 - 3x^2 - 9x + 27
\]

Solution:

- \( f(x) = x^2(x - 3) - 9(x - 3) = (x^2 - 9)(x - 3) = (x + 3)(x - 3)(x - 3) \)
  or \( f(x) = (x + 3)(x - 3)^2 \)
- Find the intervals or regions when \( x = -3 \) and \( x = 3 \):

<table>
<thead>
<tr>
<th>Interval</th>
<th>((-\infty, -3))</th>
<th>((-3, 3))</th>
<th>((3, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 3))</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>((x - 3)^2)</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td><strong>Position of graph</strong></td>
<td><strong>Below x-axis</strong></td>
<td><strong>Above x-axis</strong></td>
<td><strong>Above x-axis</strong></td>
</tr>
</tbody>
</table>

- Below the x-axis if \( x < -3 \), above the x-axis if \( -3 < x < 3 \) or \( x > 3 \)

**Note:** \(-3 < x < 3 \) can be written as \( |x| < 3 \)
Example 3: 
\[ f(x) = 4x - x^3 \]
- \[ f(x) = x(4 - x^2) = x(2 - x)(2 + x) \]
- Find the intervals or regions when \( x = -2, x = 0 \) and \( x = 2 \):

<table>
<thead>
<tr>
<th>Interval</th>
<th>((-\infty, -2))</th>
<th>((-2, 0))</th>
<th>((0, 2))</th>
<th>((2, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( 2 - x )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( 2 + x )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Position of graph</td>
<td>+ or Above</td>
<td>- or Below</td>
<td>+ or Above</td>
<td>- or Below</td>
</tr>
</tbody>
</table>

Above the \( x \)-axis if \( x < -2 \) or \(-2 < x < 2\), below the \( x \)-axis if \(-2 < x < 0 \) or \( x > 2 \)

Example 4: 
\[ f(x) = x^4 + 3x^3 - 4x^2 \]
- \[ f(x) = x^2(x^2 + 3x - 4) = x^2(x + 4)(x - 1) \]
- Find the intervals or regions when \( x = -4, x = 0 \) and \( x = 1 \):

<table>
<thead>
<tr>
<th>Interval</th>
<th>((-\infty, -4))</th>
<th>((-4, 0))</th>
<th>((0, 1))</th>
<th>((1, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x + 4 )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x - 1 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Position of graph</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

- \( f(x) > 0 \) (above) when \( x < -4 \) or \( x > 1 \), \( f(x) < 0 \) (below) when \(-4 < x < 0 \) or \( 0 < x < 1 \)

The following examples are from the book:

34.: If \( f(x) = kx^3 + x^2 - kx + 2 \), find a number \( k \) such that the graph of \( f \) contains the point \((2, 12)\)

36.: If one zero of \( f(x) = x^3 - 3x^2 - kx + 12 \) is -2, find two other zeros
Section 4.2: Properties of Division

To divide 11 by 4:

\[
\begin{array}{c|ccc}
& 2 & \text{Quotient} \\
\hline
4 & 11 & \text{Divisor} \\
\hline
-8 & & \\
3 & \text{Dividend} \\
\end{array}
\]

The answer is = 2 + 3/4

Check: 11 = (4)(2) + 3

Division Algorithm: \( f(x) = p(x)q(x) + r(x) \)

Long Division:

Example 1: Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \):

\[ f(x) = 4x^4 + 6x^3 - 3x - 2 \quad ; \quad p(x) = 2x^2 - 1 \]

Example 2: a) Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \)

b) Find \( f(1) \)

\[ f(x) = x^3 - 2x^2 + 3x + 1 \quad ; \quad p(x) = x - 1 \]

Example 3: a) Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \)

b) Find \( f(-2) \)

\[ f(x) = x^3 - 7x - 6 \quad ; \quad p(x) = x + 2 \]

- **Remainder Theorem:** If \( f(x) \) is divided by \( (x - c) \), then the remainder is \( f(c) \)
- **Factor Theorem:** If \( f(c) = 0 \), then \( (x - c) \) is a factor of \( f(x) \).

To understand the theorem, use the solution of example 3: \( p(x) = x + 2 \) or \( p(x) = x - c \) where \( c = -2 \)

- **Remainder Theorem:**
  Dividing \( f(x) \) by \( p(x) \) where \( c = -2 \) has a remainder = 0, then \( f(-2) = 0 \)
- **Factor Theorem:**
  Since \( f(-2) = 0 \), then \( (x + 2) \) is a factor of \( f(x) \)
**Example 4:** Use the remainder theorem to find \( f(c) \) when:
\[
f(x) = 2x^3 - 7x^2 + 5, \quad c = 3
\]
Solution: Divide \( f(x) = 2x^3 - 7x^2 + 5 \) by \((x - 3)\), and you will find the remainder = -4, then \( f(3) = -4 \)

**Example 5:** Use the factor to show that \((x - c)\) is a factor of \( f(c) \) when:
\[
f(x) = x^3 - 7x + 6, \quad c = -3
\]
Solution: Divide \( f(x) = x^3 - 7x + 6 \) by \((x + 3)\), and the remainder must be = 0

**Example 6:** Find a polynomial \( f(x) \) with leading coefficient =1 and having the given degree and zeros:

<table>
<thead>
<tr>
<th>Degree</th>
<th>Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-1, 1, 2</td>
</tr>
<tr>
<td>4</td>
<td>-2, 2, 0, 5</td>
</tr>
</tbody>
</table>

Answers: a) \((x - 1)(x + 1)(x - 2)\) b) \(x(x - 2)(x + 2)(x - 5)\)

**Synthetic Division, Division by \((x - c)\):**

**Example 7:** Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \) (same as example 2)
\[
f(x) = x^3 - 2x^2 + 3x + 1; \quad p(x) = x - 1
\]

**Example 8:** Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \) (same as example 1)
\[
f(x) = 4x^4 + 6x^3 + 3x - 1; \quad p(x) = 2x^2 - 1
\]
Not possible, Synthetic Division can only be used when dividing by \((x - c)\).

**Example 9:** Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \) (same as example 3)
\[
f(x) = x^3 - 7x - 6; \quad p(x) = x + 2
\]

**Example 10:** Find the quotient and remainder if \( f(x) \) is divided by \( p(x) \) (same as example 3)
\[
f(x) = 2x^3 + 3x^2 - 2x + 1; \quad p(x) = x - \frac{1}{2}
\]

**Example 11:** Use the synthetic division to find \( f(c) \) when:
\[
f(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3; \quad c = -2
\]

**Example 12:** Use the synthetic division to show that \( c \) is a zero of \( f(x) \) when:
\[
f(x) = 2x^3 + 7x^2 + 6x - 5; \quad c = \frac{1}{2}
\]

**Example 13:** Find all values of \( k \) such that \( f(x) \) is divisible by the given function:
\[
f(x) = k^2x^3 - 4kx + 3; \quad (x - 1)
\]

**Example 14:** Use the synthetic division to decide whether \((x + 3)\) is a factor, and if it is, find all other factors:
\[
f(x) = x^3 + 5x^2 - 2x - 24
\]

**Example 15:** Use the synthetic division to decide whether \((x - 2)\) is a factor, and if it is, find all other factors:
\[
f(x) = x^3 - 7x + 6
\]