LEE-YANG-FISHER ZEROS FOR DHL AND 2D RATIONAL DYNAMICS,

II. GLOBAL PLURIPOTENTIAL INTERPRETATION.

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Abstract. In a classical work of the 1950's, Lee and Yang proved that for fixed nonnegative temperature, the zeros of the partition functions of a ferromagnetic Ising model always lie on the unit circle in the complex magnetic field. Zeros of the partition function in the complex temperature were then considered by Fisher, when the magnetic field is set to zero. Limiting distributions of Lee-Yang and of Fisher zeros are physically important as they control phase transitions in the model. One can also consider the zeros of the partition function simultaneously in both complex magnetic field and complex temperature. They form an algebraic curve called the Lee-Yang-Fisher (LYF) zeros. In this paper we study their limiting distribution for the Diamond Hierarchical Lattice (DHL). In this case, it can be described in terms of the dynamics of an explicit rational function R in two variables (the Migdal-Kadanoff renormalization transformation). We prove that the Lee-Yang-Fisher zeros are equidistributed with respect to a dynamical $(1,1)$-current in the projective space. The free energy of the lattice gets interpreted as the pluripotential of this current. We also describe some of the properties of the Fatou and Julia sets of the renormalization transformation.

Contents

1. Introduction

1.1. **Lee-Yang-Fisher zeros.** We will begin with providing a brief background on the Lee-Yang-Fisher zeros that continues the discussion in Part I [BLR1].

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We consider the Ising model on a finite graph Γ and its partition function Z_Γ, which is a Laurent polynomial in two variables (z,t), where z is a "field-like" variable and t is "temperature-like" one.

For a fixed $t \in [0,1]$, the complex zeros of $\mathsf{Z}(z,t)$ in z are called the Lee-Yang zeros. The Lee-Yang Theorem [YL, LY] asserts that for the ferromagnetic Ising model on any graph, *the zeros of the partition function lie on the unit circle* $\mathbb{T}$ in the complex plane.

If we have a hierarchy of graphs Γ_n of increasing size, then under fairly general conditions, zeros of the partition functions $\mathsf{Z}_n = \mathsf{Z}_{\Gamma_n}$ will have a limiting distribution μ_t on the unit circle. This distribution captures phase transitions in the model.

Instead of freezing temperature, one can freeze the external field, and study zeros of $\mathsf{Z}(z,t)$ in the t-variable. They are called *Fisher zeros* as they were first studied by Fisher for the regular two-dimensional lattice, see [F, BK]. Similarly to the Lee-Yang zeros, asymptotic distribution of the Fisher zeros is supported on the singularities of the magnetic observables, and is thus related to phase transitions in the model. However, Fisher zeros do not lie on the unit circle any more. For instance, for the regular 2D lattice at zero field (corresponding to $z=1$), the asymptotic distribution lies on the union of two *Fisher circles* depicted on Figure 1.1.

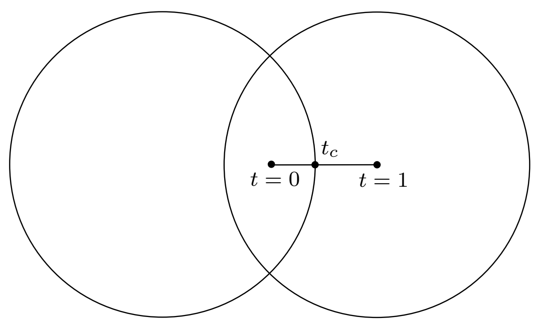

FIGURE 1.1. The Fisher circles: $|t \pm 1| = \sqrt{2}$.

More generally, one can study the distribution of zeros for $\mathsf{Z}(z,t)$ on other complex lines in $\mathbb{C}^2$. In order to organize the limiting distributions over all such lines into a single object, we use the theory of currents; see [dR, Le].

A $(1,1)$-current ν on $\mathbb{C}^2$ is a linear functional on the space of $(1,1)$-forms that have compact support (see Appendix A.5). A basic example is the current $[X]$ of integration over an algebraic curve X. A plurisubharmonic function G is called a *pluripotential* of ν if $\dfrac{i}{\pi}\partial\bar{\partial}G = \nu$, in the sense of distributions. (Informally, this means that $\dfrac{1}{2\pi}\Delta(G|\,L) = \nu|\,L$ for almost any complex line L, so $G|\,L$ is an electrostatic potential of the charge distribution $\nu|\,L$.)

For each n, the zero locus of $\mathsf{Z}_n(z,t)$ is an algebraic curve $\mathcal{S}_n^c \subset \mathbb{C}^2$, which we call the *Lee-Yang-Fisher* (LYF) zeros. Let d_n be the degree of $\mathcal{S}_n^c$. It is natural to ask whether there exists a $(1,1)$-current $\mathcal{S}^c$ so that

$$\frac{1}{d_n}[\mathcal{S}_n^c] \to \mathcal{S}^c.$$

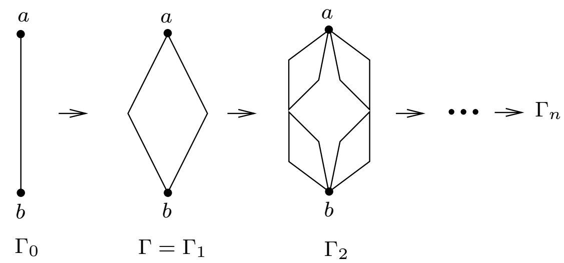

FIGURE 1.2. Diamond hierarchical lattice.

It will describe the limiting distribution of Lee-Yang-Fisher zeros. Within almost any complex line L, the limiting distribution of zeros can be obtained as the restriction $\mathcal{S}^c|L$.

In order to justify existence of $\mathcal{S}^c$, one considers the sequence of "free energies"

$$F_n^{\#}(z,t) := \log|\check{\mathsf{Z}}_n(z,t)|,$$

where $\check{\mathsf{Z}}_n(z,t)$ is the polynomial obtained by clearing the denominators of Z_n. We will say that the sequence of graphs Γ_n has a *global thermodynamic limit* if

$$\frac{1}{d_n}F_n^{\#}(z,t) \to F^{\#}(z,t)$$

in $L^1_{\text{loc}}(\mathbb{C}^2)$. In Proposition 2.1 we will show that this is sufficient for the limiting current $\mathcal{S}^c$ to exist.

The support of $\mathcal{S}^c$ consists of the singularities of the magnetic observables of the model, thus describing global phase transitions in $\mathbb{C}^2$. Connected components of $\mathbb{C}^2 \setminus \operatorname{supp}\mathcal{S}^c$ describe the distinct (complex) phases of the system.

1.2. **Diamond hierarchical model.** The *diamond hierarchical lattice* (DHL) is a sequence of graphs Γ_n illustrated on Figure 1.2. Part I [BLR1] and the present paper are both fully devoted to study of this lattice.

The Migdal-Kadanoff renorm-group RG equations for the DHL have the form:

$$(z_{n+1}, t_{n+1}) = \left(\frac{z_n^2 + t_n^2}{z_n^{-2} + t_n^2},\ \frac{z_n^2 + z_n^{-2} + 2}{z_n^2 + z_n^{-2} + t_n^2 + t_n^{-2}}\right) := \mathcal{R}(z_n, t_n), \tag{1.1}$$

where z_n and t_n are the renormalized field-like and temperature-like variables on Γ_n. The map $\mathcal{R}$ that relates these quantities is also called the *renormalization transformation.*

To study the Fisher zeros, we consider the line $\mathcal{L}_{\text{inv}} = \{z = 1\}$ in $\mathbb{CP}^2$. This line is invariant under $\mathcal{R}$, and $\mathcal{R} : \mathcal{L}_{\text{inv}} \to \mathcal{L}_{\text{inv}}$ reduces to a fairly simple one-dimensional rational map

$$\mathcal{R} : t \mapsto \left(\frac{2t}{t^2+1}\right)^2.$$

The Fisher zeros at level n are obtained by pulling back the point $t = -1$ under $\mathcal{R}^n$. As shown in [BL], the limiting distribution of the Fisher zeros in this case exists and it coincides with the measure of maximal entropy (see [Br, Ly]) of $\mathcal{R}|\,L$. The limiting support for this measure is the Julia set for $\mathcal{R}|\mathcal{L}_{\text{inv}}$, which is shown in Figure 1.3. It was studied by [DDI, DIL, BL, Ish] and others.

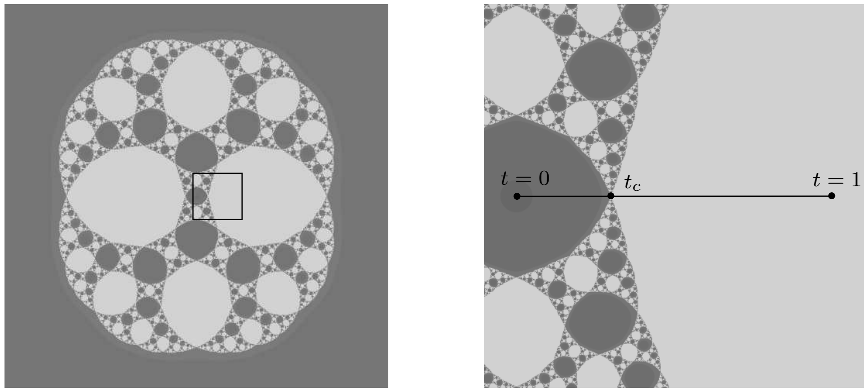

FIGURE 1.3. On the left is the Julia set for $\mathcal{R}|\,\mathcal{L}_{\mathrm{inv}}$. On the right is a zoomed-in view of a boxed region around the critical point t_c. The invariant interval $[0,1]$ corresponds to the states with real temperatures $T \in [0,\infty]$ and vanishing field $h=0$.

In this paper, we will use the Migdal-Kadanoff RG equations to study the global limiting distribution of Lee-Yang-Fisher zeros for the DHL. Their extension to $\mathbb{CP}^2$, which we will also denote by $\mathcal{S}_n^c$, is a curve of degree $2\cdot 4^n$. Our main result is:

Theorem (Global Lee-Yang-Fisher Current). *The currents* $\frac{1}{2\cdot 4^n}[\mathcal{S}_n^c]$ *converge distributionally to some* $(1,1)$*-current* $\mathcal{S}^c$ *on* $\mathbb{CP}^2$ *whose pluripotential coincides with the (lifted) free energy* $\hat{F}^{\#}$ *of the system.*

An important subtlety arises because the degrees of $\mathcal{R}$ do not behave properly under iteration:

$$4^n < \deg(\mathcal{R}^n) < (\deg(\mathcal{R}))^n = 6^n.$$

This *algebraic instability*[1] of $\mathcal{R}$ has the consequence that

$$\mathcal{S}_n^c \neq (\mathcal{R}^n)^*\mathcal{S}_0^c.$$

The issue is resolved by working with another rational mapping R coming directly from the Migdal-Kadanoff RG Equations, without passing to the "physical" (z,t)-coordinates. This map is semi-conjugate to $\mathcal{R}$ by a degree two rational map $\Psi : \mathbb{CP}^2 \to \mathbb{CP}^2$. Moreover, R is algebraically stable, satisfying $\deg(R^n) = (\deg(R))^n = 4^n$. For each $n \geq 0$, we have:

$$\mathcal{S}_n^c = \Psi^{-1}(R^{-n}S_0^c),$$

where S_0^c is an appropriate projective line.

Associated to any (dominant, algebraically stable) rational mapping $f : \mathbb{CP}^k \to \mathbb{CP}^k$ is a canonically defined invariant current S, called the Green current of f. It satisfies $f^*S = d\cdot S$, where $d = \deg f$. If f satisfies an additional (minor) technical hypothesis, then $\operatorname{supp} S$ equals the Julia set of f. (In our case, R does not satisfy this additional hypothesis, but we still have that $\operatorname{supp} S$ equals the Julia set for R; see Prop. 4.4).

[1]For the definition, see Appendix B.1.

Such invariant currents are a powerful tool of higher-dimensional holomorphic dynamics: see Bedford-Smillie [BS], Fornaess-Sibony [FS1], and others (see [Si] for a survey of this subject).

We will show that the normalized sequence of currents $\frac{1}{4^n}(R^n)^*[S_0^c]$ converges distributionally to the Green current S of R. Pulling everything back under Ψ, this will imply the Global Lee-Yang-Fisher Theorem. In this way, the classical Lee-Yang-Fisher theory gets linked to the contemporary dynamical pluripotential theory.

Remark 1.1. This dynamical approach to studying the limiting distribution of Lee-Yang-Fisher zeros for hierarchical lattices has independently been considered in [DeSMa] and studied numerically in [DeS].

Asymptotic distribution for pullbacks of algebraic curves under rational maps has been a focus of intense research in multidimensional holomorphic dynamics since the early 1990's; see [BS, FS1, RuSh, FaJ, Gu1, Gu2, DS2, DDG] for a sample of papers on the subject. Our result above is very close in spirit to this work. However, the above theorem does not seem to be a consequence of any available results.

1.3. **Structure of the paper.** We begin in §2 by recalling the definitions of free energy and the classical notion of thermodynamic limit for the Ising model. We then discuss the notion of global thermodynamic limit, which is sufficient in order to guarantee that some lattice have a $(1,1)$-current $\mathcal{S}^c$ describing its limiting distribution of LYF zeros in $\mathbb{C}^2$. Following this, we extend the whole description from $\mathbb{C}^2$ to $\mathbb{CP}^2$ by lifting the partition function and free energy to $\mathbb{C}^3$. We also give an alternative interpretation of the partition function as a section of (an appropriate tensor power of) the hyperplane bundle over $\mathbb{CP}^2$ that will be central to the proof of the Global LYF Current Theorem. We conclude §2 by summarizing material on the Migdal-Kadanoff RG equations.

In §3 we summarize the global features of the mappings $\mathcal{R}$ and R on the complex projective space $\mathbb{CP}^2$ that were studied in [BLR1], including their critical and indeterminacy loci, superattracting fixed points and their separatrices.

In the next section, §4, we define the Fatou and Julia sets for $\mathcal{R}$ and show that the Julia set coincides with the closure of preimages of the invariant complex line $\{z = 1\}$ (corresponding to the vanishing external field). It is based on M. Green's criteria for Kobayashi hyperbolicity of the complements of several algebraic curves in $\mathbb{CP}^2$ [G1, G2] that generalize the classical Montel Theorem. We then use this result to prove that points in the interior of the solid cylinder $\mathbb{D} \times I$ are attracted to a superattracting fixed point $\eta = (0, 1)$ of $\mathcal{R}$.

We prove the Global LYF Current Theorem in §5, relying on estimates of how volume is transformed under a single iterate of R that are completed in §6.

We finish with several Appendices. In Appendix A we collect needed background in complex geometry (normality, Kobayashi hyperbolicity, currents and their pluripotentials, line bundles over $\mathbb{CP}^2$, etc). In Appendix B we provide background on the complex dynamics in higher dimensions, including the notion of algebraic stability and information on the Green current. In Appendix C we collect several open problems.

1.4. **Basic notation and terminology.** $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $\mathbb{T} = \{|z| = 1\}$, $\mathbb{D}_r = \{|z| < r\}$, $\mathbb{D} \equiv \mathbb{D}_1$, and $\mathbb{N} = \{0, 1, 2 \dots\}$. Given two variables x and y, $x \asymp y$ means that $c \leq |x/y| \leq C$ for some constants $C > c > 0$.

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2. Description of the model

2.1. **Free Energy and Thermodynamic Limit.** The partition function of the Ising model on a graph Γ is a symmetric Laurent polynomial in (z, t) of the form

$$\mathsf{Z}_\Gamma = \sum_{n=0}^{d} a_n(t)(z^n + z^{-n}) \tag{2.1}$$

of degree d equal to the the number of edges in Γ.

The *free energy* of the system is defined as

$$F_\Gamma = -T \log |\mathsf{Z}_\Gamma|, \tag{2.2}$$

where T is the temperature (related to the temperature-like variable by $t = e^{-J/T}$, where J is the coupling constant of the model).

It will be more convenient to consider the following variant of the free energy:

$$F_\Gamma^\#(z,t) := -\frac{1}{T} F_\Gamma(z,t) + d(\log|z| + \frac{1}{2}\log|t|) = \log|\check{\mathsf{Z}}(z,t)|.$$

Here, $\check{\mathsf{Z}}(z,t) := z^d t^{d/2} \mathsf{Z}(z,t)$ is obtained by clearing the denominators of Z. The advantage of using $F_\Gamma^\#$, instead of F_Γ, is that it extends as a plurisubharmonic function on all of $\mathbb{C}^2$. We will also refer to $F_\Gamma^\#$ as the "free energy".

Assume that we have a lattice given by a hierarchy of graphs Γ_n with $d_n \to \infty$ edges. Let us consider its partition functions Z_n and free energies $F_n^\#$. To pass to the thermodynamic limit we normalize the free energy *per bond*. One says that the hierarchy of graphs has a (pointwise) *thermodynamic limit* if

$$\frac{1}{2d_n} F_n^\#(z,t) \to F^\#(z,t) \quad \text{for any } z \in \mathbb{R}_+,\ t \in (0,1). \tag{2.3}$$

In this case, the function $F^\#$ is called the (modified) *free energy* of the lattice. For many[2] lattices (e.g. $\mathbb{Z}^d$), existence of the thermodynamic limit can be justified by van Hove's Theorem [vH, R2]. If the classical thermodynamic limit exists, then one can justify existence of the limiting distribution of Lee-Yang zeros and relate it to the limiting free energy; see [BLR1, Prop. 2.2].

In order to prove existence of a limiting distribution for the Fisher zeros, one needs to prove existence of the thermodynamic limit in the $L^1_{\text{loc}}(\mathbb{C})$-sense:

$$\frac{1}{2d_n} F_n^\#(1,t) \to F^\#(1,t) \quad \text{in } L^1_{\text{loc}}(\mathbb{C}). \tag{2.4}$$

For the $\mathbb{Z}^2$ lattice this is achieved by the Onsager solution, which provides an explicit formula the limiting free energy; see, for example, [Ba]. Similar techniques apply to the triangular, hexagonal, and various homopolygonal lattices (see [MaSh1, MaSh2]

[2]Note that the DHL is not in this class—instead, dynamical techniques are used to justify its classical thermodynamic limit.

for suitable references and an investigation of the distribution of Fisher zeros for these lattices). For various hierarchical lattices, (2.4) can be proved by dynamical means. It seems to be an open question for other lattices, including $\mathbb{Z}^d$, when $d \geq 3$.

The situation is similar for the Lee-Yang-Fisher zeros:

Proposition 2.1. *Let* (Γ_n) *be a lattice for which the thermodynamic limit exists in the* $L^1_{\mathrm{loc}}(\mathbb{C}^2)$*-sense:*

$$\frac{1}{2d_n} F_n^{\#}(z,t) \to F^{\#}(z,t) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}^2). \tag{2.5}$$

Then, there is a closed positive $(1,1)$*-current* $\mathcal{S}^c$ *on* $\mathbb{C}^2$ *describing the limiting distribution of Lee-Yang-Fisher zeros. Its pluripotential coincides with the free energy* $F^{\#}(z,t)$.

For the DHL, we will prove existence of the limit (2.5) in the Global LYF Current Theorem. It is an open question whether such a limit exists for other lattices, including the classical $\mathbb{Z}^d$ lattices for $d \geq 2$; see Problem C.1.

Proof. The locus of Lee-Yang-Fisher zeros $\mathcal{S}_n^c$ are the zero set (counted with multiplicities) of the degree $2d_n$ polynomial $\check{\mathsf{Z}}_n(z,t)$. The Poincaré-Lelong Formula describes its current of integration:

$$[\mathcal{S}_n^c] = \frac{i}{\pi}\partial\bar{\partial}\log|\check{\mathsf{Z}}_n(z,t)| = \frac{i}{\pi}\partial\bar{\partial}F_n^{\#}(z,t).$$

Hypothesis (2.5) implies

$$\frac{1}{2d_n}[\mathcal{S}_n^c] = \frac{i}{\pi}\partial\bar{\partial}\frac{1}{2d_n}F_n^{\#}(z,t) \to \frac{i}{\pi}\partial\bar{\partial}F^{\#}(z,t) =: \mathcal{S}^c.$$

□

2.2. **Migdal-Kadanoff renormalization for the DHL.** The renormalized field-like and temperature-like variables z_n and t_n that appear in the Migdal- Kadanoff RG equations (1.1) are defined through certain "conditional partition functions of level n" in the following way:

$$z_n^2 = W_n/U_n, \quad t_n^2 = \frac{V_n^2}{U_n W_n}. \tag{2.6}$$

In the (U,V,W)-coordinates the Migdal-Kadanoff RG equation assumes the homogeneous form

$$U_{n+1} = (U_n^2 + V_n^2)^2, \quad V_{n+1} = V_n^2(U_n + W_n)^2, \quad W_{n+1} = (V_n^2 + W_n^2)^2,$$

and the total partition function becomes a linear form

$$\mathsf{Z}_n \equiv \mathsf{Z}_{\Gamma_n} = U_n + 2V_n + W_n.$$

(See Part I [BLR1] for the derivation of these equations.) This leads us to a homogeneous degree 4 polynomial map

$$\hat{R} : (U,V,W) \mapsto \left((U^2+V^2)^2, V^2(U+W)^2, (W^2+V^2)^2\right), \tag{2.7}$$

called the *Migdal Kadanoff Renormalization*, such that $(U_n, V_n, W_n) = \hat{R}^n(U_0, V_0, W_0)$. (The corresponding map $R : \mathbb{CP}^2 \to \mathbb{CP}^2$ will be referred to in the same way.) Moreover, letting $Y_0 := U + 2V + W$, we obtain:

$$\mathsf{Z}_n = Y_0 \circ \hat{R}^n, \tag{2.8}$$

so the partition functions Z_n are obtained by pulling the linear form $Y_0 \equiv \mathsf{Z}_0$ back by $\hat{R}^n$.

We will often write R in the system of local coordinates $u = U/V$ and $w = W/V$, in which it has the form

$$R : (u, w) \mapsto \left(\frac{u^2 + 1}{u + w}, \ \frac{w^2 + 1}{u + w} \right)^2. \tag{2.9}$$

Notice that the form Y_0 is *not a function* on $\mathbb{CP}^2$ but rather a *section* σ_{Y_0} of the hyperplane line bundle over $\mathbb{CP}^2$, see Appendix §A.4. Respectively, the partition functions Z_n are sections of the tensor powers of this line bundle. Accordingly, the Lee-Yang-Fisher loci S_n^c are the *zero divisors* of these sections.

The free energy is also no longer a function on $\mathbb{CP}^2$, rather it is lifted to become a function on $\mathbb{C}^3$, given by

$$\hat{F}_n^{\#} := \log |\hat{\mathsf{Z}}_n|. \tag{2.10}$$

The above formuli express the partition functions and free energies in terms of the U, V, W coordinates. To re-express them in terms of the physical coordinates, we pull each of them back by

$$\Psi : \mathbb{CP}^2 \to \mathbb{CP}^2, \quad (U : V : W) = \Psi(z, t) = (z^{-1} t^{-1/2} : t^{1/2} : z t^{-1/2}). \tag{2.11}$$

This change of variables also semi-conjugates the map

$$\mathcal{R} : (z, t) \mapsto \left(\frac{z^2 + t^2}{z^{-2} + t^2}, \ \frac{z^2 + z^{-2} + 2}{z^2 + z^{-2} + t^2 + t^{-2}} \right), \tag{2.12}$$

corresponding to RG equation (1.1), to R.

3. Global properties of the RG transformation in $\mathbb{CP}^2$.

We will now summarize (typically without proofs) results from [BLR1] about the global properties of the RG mappings.

3.1. **Preliminaries.** The renormalization mappings $\mathcal{R}$ and R are semi-conjugate by the degree two rational mapping $\Psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ given by (2.6).

Both mappings have topological degree 8 (see Prop. 4.3 from Part I). However, as noted in the introduction, their algebraic degrees behave differently: R is algebraically stable, while $\mathcal{R}$ is not. Since $\deg(R^n) = 4^n$, for any algebraic curve D of degree d, the pullback $(R^n)^* D$ is a divisor of degree $d \cdot 4^n$. (For background on divisors, see Appendix A.) For this reason, we will focus most of our attention on the dynamics of R.

The semiconjugacy Ψ sends Lee-Yang cylinder $\mathcal{C}$ to a Mobius band C that is invariant under R. It is obtained as the closure in $\mathbb{CP}^2$ of the topological annulus

$$C_0 = \{(u, w) \in \mathbb{C}^2 : \ w = \bar{u}, \ |u| \geq 1\}. \tag{3.1}$$

Let $\mathrm{T} = \{(u, \bar{u}) : \ |u| = 1\}$ be the "top" circle of C, while B be the slice of C at infinity. In fact, $\Psi : \mathcal{C} \to C$ is a conjugacy, except that it maps the bottom $\mathcal{B}$ of $\mathcal{C}$ by a 2-to-1 mapping to B (see Prop. 3.1 from Part I).

3.2. **Indeterminacy points for R.** In homogeneous coordinates on $\mathbb{CP}^2$, the map R has the form:

$$R : [U : V : W] \mapsto [(U^2+V^2)^2 : \; V^2(U+W)^2 : \; (V^2+W^2)^2)]. \tag{3.2}$$

One can see that R has precisely two points of indeterminacy $a_+ := [i : 1 : -i]$ and $a_- := [-i : 1 : i]$. Resolving all of the indeterminacies of R by blowing-up the two points $a_\pm$ (see Appendix A.3), one obtains a holomorphic mapping $\tilde{R} : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$.

In coordinates $\xi = u - i$ and $\chi = (w+i)/(u-i)$ near $a_+ = (i, -i)$, we obtain the following expression for the map $\tilde{R} : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$ near $L_{\rm exc}(a_+)$:

$$u = \left(\frac{\xi + 2i}{1+\chi}\right)^2, \quad w = \left(\frac{\chi^2\xi - 2i\chi}{1+\chi}\right)^2. \tag{3.3}$$

(Similar formulas hold near $a_- = (-i, i)$.) The exceptional divisor $L_{\rm exc}(a_+)$ is mapped by $\tilde{R}$ to the conic

$$G := \{(u-w)^2 + 8(u+w) + 16 = 0\}.$$

3.3. **Superattracting fixed points and their separatrices.** We will often refer to $L_0 := \{V = 0\} \subset \mathbb{CP}^2$ as the *line at infinity*. It contains two symmetric superattracting fixed points, $e = (1 : 0 : 0)$ and $e' = (0 : 0 : 1)$. Let $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$ stand for the attracting basins of these points. It will be useful to consider local coordinates ($\xi = W/U$, $\eta = V/U$) near e.

The line at infinity $L_0 = \{\eta = 0\}$ is R-invariant, and the restriction $R|L_0$ is the power map $\xi \mapsto \xi^4$. Thus, points in the disk $\{|\xi| < 1\}$ in L_0 are attracted to e, points in the disk $\{|\xi| > 1\}$ are attracted to e', and these two basins are separated by the unit circle B. We will also call L_0 the *fast separatrix* of e and e'.

Let us also consider the conic

$$L_1 = \{\xi = \eta^2\} = \{V^2 = UW\} \tag{3.4}$$

passing through points e and e'. It is an embedded copy of $\mathbb{CP}^1$ that is invariant under R, with $R|\,L_1(w) = w^2$, where $w = W/V = \xi/\eta$. Thus, points in the disk $\{|w| < 1\}$ in L_1 are attracted to e, points in the disk $\{|w| > 1\}$ are attracted to e', and these two basins are separated by the unit circle T (see §3.1 from Part I). We will call L_0 the *slow separatrix* of e and e'.

If a point x near e (resp. e') does not belong to the fast separatrix L_0, then its orbit is "pulled" towards the slow separatrix L_1 at rate ρ^{4^n}, with some $\rho < 1$, and converges to e (resp. e') along L_1 at rate r^{2^n}, with some $r < 1$.

The strong separatrix L_0 is transversally superattracting: all nearby points are pulled towards L_0 uniformly at rate r^{2^n} It follows that these points either converge to one of the fixed points, e or e', or converge to the circle B.

Given a neighborhood Ω of B, let

$$\mathcal{W}^s_{\mathbb{C},\rm loc}(\mathrm{B}) = \{x \in \mathbb{CP}^2 : \; R^n x \in \Omega \; (n \in \mathbb{N}) \text{ and } \mathbb{R}^n x \to \mathrm{B} \text{ as } n \to \infty\} \tag{3.5}$$

(where Ω is implicit in the notation, and an assertion involving $\mathcal{W}^s_{\mathbb{C},\rm loc}(\mathrm{B})$ means that it holds for arbitrary small suitable neighborhoods of B). It is shown in Part I (§9.2) that $\mathcal{W}^s_{\mathbb{C},\rm loc}(\mathrm{B})$ has the topology of a 3-manifold that is laminated by the union of holomorphic stable manifolds $W^s_{\mathbb{C},\rm loc}(x)$ of points $x \in \mathrm{B}$.

We conclude:

Lemma 3.1. *$\mathcal{W}^s(e) \cup \mathcal{W}^s(e') \cup \mathcal{W}^s_{\mathbb{C},\rm loc}(\mathrm{B})$ fills in some neighborhood of L_0.*

3.4. **Regularity of $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$.** For a diffeomorphism the existence and regularity of the local stable manifold for a hyperbolic invariant manifold N has been studied extensively in [HPS]. In order to guarantee a C^1 local stable manifold $\mathcal{W}^s_{\mathrm{loc}}(N)$, a strong form of hyperbolicity known as *normal hyperbolicity* is assumed. Essentially, N is normally hyperbolic for f if the expansion of Df in the unstable direction dominates the maximal expansion of Df tangent to N and the contraction of Df in the stable direction dominates the maximal contraction of Df tangent to N. See [HPS, Theorem 1.1]. If, furthermore, the expansion in the unstable direction dominates the r-th power of the maximal expansion tangent to N and the contraction in the stable direction dominates the r-th power of the maximal contraction tangent to N, this guarantees that the stable manifold is of class C^r. The corresponding theory for endomorphisms is less developed, although note that other aspects of [HPS], related to persistence of normally hyperbolic invariant laminations, have been generalized to endomorphisms in [Be].

In our situation, B is not normally hyperbolic because it lies within the invariant line L_0 and R is holomorphic. This forces the expansion rates tangent to B and transverse to B (within this line) to coincide. Therefore, the following result does not seem to be part of the standard hyperbolic theory:

Lemma 3.2. *$\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$ is a C^∞ manifold and the stable foliation is a C^∞ foliation by complex analytic discs.*

Proof. In Proposition 9.11 from Part I, we showed that within the cylinder $\mathcal{C}$ the stable foliation of $\mathcal{B}$ has C^∞ regularity and that the stable curve of each point is real analytic. Mapping forward under Ψ, we obtain the same properties for the stable foliation of B within C.

Let us work in the local coordinates $\xi = W/U$ and $\eta = V/U$. In these coordinates, $\mathrm{B} = \{\eta = 0, |\xi| = 1\}$. The stable curve of some $\xi_0 \in \mathrm{B}$ can be given by expressing ξ as the graph of a holomorphic function of η:

$$\xi = h(\eta, \xi_0) = \sum_{i=0}^{\infty} a_i(\xi_0)\eta^i. \tag{3.6}$$

The right hand side is a convergent power series with coefficients depending on ξ_0, having a uniform radius of convergence over every $\xi_0 \in \mathcal{B}$. The series is uniquely determined by its values on the real slice C, in which the leaves depend with C^∞ regularity on ξ_0. Therefore, each of the coefficients $a_i(\xi_0)$ is C^∞ in ξ_0. This gives that $W^s_{\mathbb{C},\mathrm{loc}}(x)$ is a C^∞ manifold.

□

Remark 3.1. The technique from the proof of Lemma 3.2 applies to a more general situation: Suppose that M is a real analytic manifold and $f : M \to M$ is a real analytic map. Let $N \subset M$ be a compact real analytic invariant submanifold for f, with $f|N$ expanding and with N transversally attracting under f. Then, N will have a stable foliation $\mathcal{W}^s_{\mathrm{loc}}(N)$ of regularity C^r, for some $r > 0$ (see the beginning of this subsection), with the stable manifold of each point being real-analytic. The stable manifold $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(N)$ for the extension of f to the complexification $M_{\mathbb{C}}$ of M will then also have C^r regularity.

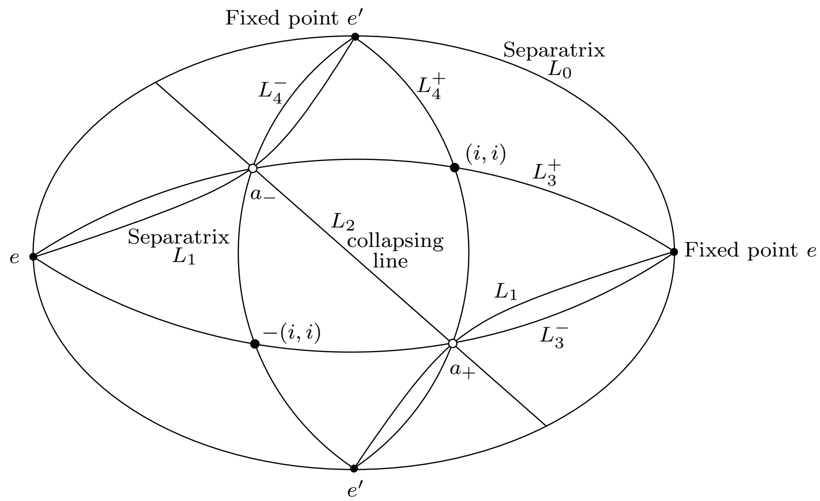

FIGURE 3.1. Critical locus for R shown with the separatrix L_0 at infinity.

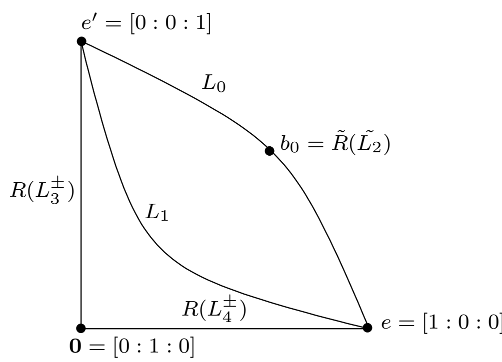

FIGURE 3.2. Critical values locus of R.

3.5. **Critical locus.** We showed in Part I [BLR1] that the critical locus of R consists of 6 complex lines and one conic:

$$\begin{aligned} L_0 &:= \{V=0\} = \text{line at infinity}, \\ L_1 &:= \{UW = V^2\} = \text{conic } \{uw=1\}, \\ L_2 &:= \{U=-W\} = \{u=-w\} = \text{ the collapsing line}, \\ L_3^{\pm} &:= \{U = \pm iV\} = \{u = \pm i\}, \\ L_4^{\pm} &:= \{W = \pm iV\} = \{w = \pm i\}. \end{aligned}$$

(Here the curves are written in the homogeneous coordinates $(U:V:W)$ and in the affine ones, $(u = U/V, w = W/V)$.) The critical locus is schematically depicted on Figure 3.1, while its image, the critical value locus, is depicted on Figure 3.2.

It will be helpful to also consider the critical locus for the lift $\tilde{R} : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$. Each of the critical curves L_i lifts by proper transform (see Appendix A.3) to a critical curve $\tilde{L}_i \subset \tilde{\mathbb{CP}}^2$ for $\tilde{R}$. Moreover, any critical point for $\tilde{R}$ is either one of these proper transforms or lies within the exceptional divisors of $L_{\mathrm{exc}}(a_\pm)$.

By symmetry, it is enough to consider the blow-up of a_+. We saw in Part I that there are four critical points on the exceptional divisor $L_{\mathrm{exc}}(a_+)$ occurring at $\chi = -1, 1, \infty$, and 0, where $\chi = (w+i)/(u-i)$. They correspond to intersections of $L_{\mathrm{exc}}(a_+)$ with the collapsing line $\tilde{L}_2$, the $\tilde{L}_1$, and the critical lines $\tilde{L}_3^+$ and $\tilde{L}_4^-$, respectively.

Whitney Folds are a normal form for the simplest type of critical points of a mapping (see §6). We have:

Lemma 3.3. *All critical points of $\tilde{R}$ except the fixed points e, e', the collapsing line $\tilde{L_2}$, and two points $\{\pm(i,i)\} = \widetilde{L_3^\pm} \cap \widetilde{L_4^\pm}$, are Whitney folds.*

The only critical values obtained as images of non-Whitney folds are: e, e', $b_0 = \tilde{R}(\tilde{L_2}) \in L_0$ and $\mathbf{0} := (0,0) = \tilde{R}(\pm(i,i))$.

4. Fatou and Julia sets and the measure of maximal entropy

4.1. **Julia set.** For a rational map $R : \mathbb{CP}^n \to \mathbb{CP}^n$, the *Fatou set* F_R is defined to be the maximal open set on which the iterates $\{R^m\}$ are well-defined and form a normal family. The complement of the Fatou set is the *Julia set* J_R.

If R is dominant and has no collapsing varieties, Lemma A.1 gives that R is locally surjective (except at indeterminate points), so that the Fatou set is forward invariant and consequently, the Julia set is backward invariant.

If R has indeterminate points, then, according to this definition they are in J_R. In this case, F_R and J_R are not typically totally invariant. One can see this by blowing up an indeterminate point and observing that the image of the resulting exceptional divisor typically intersects F_R. Moreover, if R is not algebraically stable, then any curve C that is mapped by some iterate to the indeterminacy points is in J_R.

The Migdal-Kadinoff renormalization R is not locally surjective at any $x \in L_2 \setminus L_0$. More specifically, if N is a small neighborhood of x that avoids L_0, then

$$R(N) \cap L_0 = b_0 = R(x),$$

since any point of $L_0 \setminus \{b_0\}$ only has preimages in L_0. However, we still have the desired invariance:

Lemma 4.1. *The Migdal-Kadinoff renormalization R has forward invariant Fatou set and, consequently, backward invariant Julia set.*

Proof. It suffices to show that $L_2 \subset J_R$, since R is locally surjective at any other point, by Lemma A.1. By definition, $\{a_\pm\} \subset J_R$, so we consider $x \in L_2 \setminus \{a_\pm\}$. Let N be any small neighborhood of x. Note that $R(x) = b_0$ is a fixed point of saddle-type, with one-dimensional stable and unstable manifolds. Therefore, in order for the iterates to form a normal family on N, we must have $R(N) \subset \mathcal{W}^s(b_0)$. However, this is impossible, since there are plenty of regular points for R in N. □

Lemmas 3.1 and 3.2 give a clear picture of J_R in a neighborhood of the line at infinity L_0.

Proposition 4.2. *Within some neighborhood N of L_0 we have that $J_R \cap N = \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathrm{B}) \cap N$. Within this neighborhood, J_R is a C^∞ 3-dimensional manifold.*

Let us consider the locus $\{h = 0\}$ of vanishing magnetic field in $\mathbb{CP}^2$ for the DHL. In the affine coordinates, it an R-invariant line $L_{\rm inv} = \{u = w\}$; in the physical coordinates, it is an $\mathcal{R}$-invariant line $\mathcal{L}_{\rm inv} = \{z = 1\}$. The two maps are conjugate by the Möbius transformation $\mathcal{L}_{\rm inv} \to L_{\rm inv}$, $u = 1/t$. Dynamics of $\mathcal{R} : t \mapsto \left(\frac{2t}{t^2+1}\right)^2$ on $\mathcal{L}_{\rm inv}$ was studied in [BL]. In particular, it is shown in that paper that the Fatou set for $\mathcal{R}|\,\mathcal{L}_{\rm inv}$ consists entirely of the basins of attraction of the fixed points β_0 and β_1 which are superattracting within this line: see Figure 1.3.

Proposition 4.3. $J_R = \overline{\bigcup_n R^{-n}(L_{\rm inv})}$.

Proof. Since $R|\,L_{\rm inv}$ is conformally conjugate to $\mathcal{R}|\,\mathcal{L}_{\rm inv}$, every point in the Fatou set of $R|L_{\rm inv}$ is in the basin of attraction of either b_0 or b_1. Since these points are of saddle-type in $\mathbb{CP}^2$, the family of iterates R^n cannot be normal at any point on $L_{\rm inv}$. Thus $L_{\rm inv} \subset J_R$. It follows that $\overline{\bigcup_n R^{-n}(L_{\rm inv})} \subset J_R$ since J_R is closed and backward invariant.

We will now show that $\bigcup_n R^{-n}(L_{\rm inv})$ is dense in J_R. Consider a configuration of five algebraic curves

$$
\begin{aligned}
X_0 &:= \{V = 0\} = \text{the separatrix } L_0,\\
X_1 &:= \{U = W\} = \text{the invariant line } L_{\rm inv},\\
X_2 &:= \{U = -W\} = \text{the collapsed line } L_2 \subset R^{-1}(L_{\rm inv}),\\
X_3 &:= \{U^2 + 2V^2 + W^2 = 0\} = \text{a component of } R^{-1}(L_{\rm inv}),\\
X_4 &:= \{U^4 + 2U^2V^2 + 2V^4 + 2W^2V^2 + W^4 = 0\} = \text{a component of } R^{-1}(X_3).
\end{aligned}
$$

We will use the results of M. Green to check that the complement of these curves, $M := \mathbb{CP}^2 \smallsetminus \bigcup_i X_i$, is a complete Kobayashi hyperbolic manifold hyperbolically embedded in $\mathbb{CP}^2$ (see Appendix A.6). We will first check that M is Brody hyperbolic, i.e., there are no non-constant holomorphic maps $f : \mathbb{C} \to M$. To this end we will apply Green's Theorem A.5. It implies that the image of f must lie in a line L that is tangent to the conic X_3 at an intersection point with X_i, for one of the lines X_i, $i = 0, 1, 2$, and contains the intersection point $X_j \cap X_l$ of the other two lines. It is a highly degenerate situation which does not occur for a generic configuration. However, this is exactly what happens in our case, as the lines X_0, X_1, X_2 form a self-dual triangle with respect to the conic X_3 (see §A.7). However, one can check by direct calculation that the last curve, X_4, must intersect each of these tangent lines L in at least one point away from $X_0, \dots, X_3$. Since any holomorphic map from $\mathbb{C}$ to $L \smallsetminus \bigcup_i X_i$ must then omit 3 points in L, it must be constant.

So, M is Brody hyperbolic. Moreover, for each $i = 0, \dots, 4$ the remaining curves $\bigcup_{j \neq i} X_j$ intersect X_i in at least three points so that there is no non-constant holomorphic map from $\mathbb{C}$ to $X_i \smallsetminus \bigcup_{j \neq i} X_j$. Therefore, another of Green's results (Theorem A.4) applies showing that M is complete hyperbolic and hyperbolically embedded. It then follows from Proposition A.3 that the family $\{R^n\}$ is normal on any open set $N \subset \mathbb{CP}^2$ for which $R^n : N \to M$ for all n.

Given any $\zeta \in J_R$ and any neighborhood N of ζ, we'll show that some preimage $R^{-n}(L_{\rm inv})$ intersects N. Since $\zeta \in J_R$, the family of iterates R^n are not normal on N, hence $R^n(N)$ must intersect $\bigcup_i X_i$ for some n. If the intersection is with X_i for $i > 0$ then $R^{n+2}(N)$ intersects $L_{\rm inv}$.

So, some iterate $R^n(N)$ must intersect $X_0 = L_0$. Suppose first that $\zeta \in L_0$. Then, by Lemma 3.1, $\zeta \subset \mathcal{W}^s(e) \cup \mathcal{W}^s(e') \cup \mathbb{T}$. Since the first two basins are contained in the Fatou set, $\zeta \in \mathbb{T}$, where preimages of the fixed point $\beta_0 \in L_{\rm inv}$ are dense.

Finally, assume $\zeta \notin L_0$. By shrinking N if needed, we can make it disjoint from L_0. Hence there is $n > 0$ such that $R^n(N)$ intersects L_0, while $R^{n-1}(L_0)$ is disjoint from L_0. But since $R^{-1}(L_0) = L_0 \cup L_2$, we conclude that $R^{n-1}(N)$ must intersect L_2. But L_2 collapses under R to the fixed point $\beta_0 \in L_{\rm inv}$. Hence $R^n(N)$ intersects $L_{\rm inv}$. □

We will now relate J_R to the Green current S. (See Appendix B for the definition and basic properties of S.)

Proposition 4.4. $J_R = \operatorname{supp} S$.

Proof. The inclusion $\operatorname{supp} S \subset J_R$ follows immediately from Theorem B.5. We will use Proposition 4.3 to show that $J_R \subset \operatorname{supp} S$. Since $\operatorname{supp} S$ is a backward invariant closed set, it is sufficient for us to show that $L_{\rm inv} \subset \operatorname{supp} S$.

Note that $L_{\rm inv} = \mathcal{W}^s(b_0) \cup \mathcal{W}^s(b_1) \cup J_{R|L_{\rm inv}}$. The basin $\mathcal{W}^s(L_0)$ is open and contained within the nice set for R (see Appendix B for the definition of nice set), since every point of $\mathcal{W}^s(L_0)$ has a neighborhood whose forward iterates remain bounded away from $\{a_\pm\}$. Therefore, $\mathcal{W}^s(b_0) \subset J_R \cap N \subset \operatorname{supp} S$, by Theorem B.5. Since $\operatorname{supp} S$ is closed, we also have that $J_{R|L_{\rm inv}} \subset \operatorname{supp} S$.

None of the points of $\mathcal{W}^s(b_1)$ are nice because

$$b_1 \in \overline{\bigcup_{n\geq 0} R^{-n}\{a_\pm\}}.$$

Therefore, we cannot directly use Theorem B.5 to conclude that $\mathcal{W}^s(b_1) \subset \operatorname{supp} S$.

Notice that the points of $L_2 \setminus \{a_\pm\}$ are nice, since they are in $\mathcal{W}^s(L_0)$. Theorem B.5 gives that $L_2 \setminus \{a_\pm\} \subset \operatorname{supp} S$, since $L_2 \subset J_R$. Because $\operatorname{supp} S$ is closed, $L_2 \subset \operatorname{supp} S$. Let $D_2 \subset L_2$ be a small disc centered around a_+. Preimages of D_2 under appropriate branches of R^n will give discs intersecting L_1 transversally at a sequence of points converging to b_1. By the Dynamical λ-Lemma (see [PM, pp. 80-84]), this sequence of discs will converge to $\mathcal{W}^s_0(b_1) \subset L_{\rm inv}$, where $\mathcal{W}^s_0(b_1)$ is the immediate basin of b_1. Since each of the discs is in $\operatorname{supp} S$, and the latter is closed, we find that $\mathcal{W}^s_0(b_1) \subset \operatorname{supp} S$. Further preimages show that all of $\mathcal{W}^s(b_1) \subset \operatorname{supp} S$. □

4.2. **Fatou Set.** Because $J_R = \operatorname{supp} S$, we immediately have:

Corollary 4.5. *The Fatou set of R is pseudoconvex.*

For the definition of pseudoconvexity, see [Kra].

Proof. It is well-known that in the complement in $\mathbb{CP}^2$ of the support of a closed positive $(1,1)$-current is pseudoconvex. See [C, Theorem 6.2] or [U, Lemma 2.4]. □

Remark 4.1. We thank the referee for pointing out that Corollary 4.5 can also be obtained directly from Proposition 4.3. Suppose that $L_{\rm inv} = \{l_{\rm inv} = 0\}$ and $L_0 = \{l_0 = 0\}$. For any $n \geq 0$ one can define a holomorphic function which does not extend to $\cup_{n=0}^{N} R^{-n} L_{\text{inv}}$ by

$$z \mapsto \frac{\left(l_0(\hat{z})\right)^k}{\left(\prod_{n=0}^{N} l_{\text{inv}} \circ \hat{R}^n(\hat{z})\right)}, \text{ where } \pi(\hat{z}) = z \text{ and } k = \sum_{n=0}^{N} 4^n.$$

Therefore, the Fatou set of R is the interior of the intersection of domains of holomorphy, so it is also a domain of holomorphy. Hence, it is pseudoconvex.

Computer experiments indicate that the Fatou set of R may be precisely the union of the basins of attraction $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$ for the two superattracting fixed points e and e'. See Problem C.4.

Consider the solid cylinders

$$SC := \left\{[U:V:W] \;:\; \frac{V^2}{UW} \in [0,1] \text{ and } \left|\frac{W}{U}\right| < 1\right\} \text{ and}$$
$$SC' := \left\{[U:V:W] \;:\; \frac{V^2}{UW} \in [0,1] \text{ and } \left|\frac{W}{U}\right| > 1\right\}.$$

Then, we can prove the following more modest statement:

Theorem 4.6. *For the mapping R we have $SC \subset \mathcal{W}^s(e)$ and $SC' \subset \mathcal{W}^s(e')$.*

In the proof, we will need to use an important property of $R : C \to C$ that was proved in Part I. Recall from §3.1 that $C = \Psi(\mathcal{C})$ is the invariant real Möbius and that $C_0 = C \smallsetminus \mathrm{B}$ is the topological annulus obtained by removing the "core curve" B.

The key property is:

(P9′) Every proper path γ in C_0 lifts under R to at least 4 proper paths in C_0. If γ crosses G at a single point, then $R^{-1}\gamma = \cup\, \delta_i$.

Proof of Prop. 4.6: It suffices to prove the proposition for SC, since the statement for SC' follows from the symmetry ρ.

We will decompose SC as a union of complex discs and show that each disc is in $\mathcal{W}^s(e)$. Let

$$P_c := \left\{[U:V:W] \;:\; \frac{V^2}{UW} = c \in [0,1]\right\},$$

and

$$P_c^* := \left\{[U:V:W] \;:\; \frac{V^2}{UW} = c \in [0,1] \text{ and } \left|\frac{W}{U}\right| < 1\right\},$$

so that $SC = \bigcup_{c\in[0,1]} P_c^*$.

The discs P_0^* and P_1^* are in $\mathcal{W}^s(e)$ because they are each within the forward invariant critical curves L_0 and L_1, respectively, on which the dynamics is given by $(W/U) \to (W/U)^4$ and $(W/U) \to (W/U)^2$, respectively.

We now show that for any $c \in (0,1)$ we also have $P_c^* \subset \mathcal{W}^s(e)$. In fact $e \in P_c^*$, so it suffices to show that R^n forms a normal family on P_c^*. Consider any $x \in P_c^*$. If $x = e$, then $x \in \mathcal{W}^s(e)$ so that R^n is normal on some neighborhood of x in P_c^*.

Now consider any $x \in P_c^* \smallsetminus \{e\}$. There is a neighborhood of $N \subset P_c^*$ of x with $e \notin N$, on which we will show that R^n forms a normal family. Recall the family of curves $X_0, \ldots, X_4$ from the proof of Proposition 4.3, where we showed that $\mathbb{CP}^2 \smallsetminus \bigcup_i X_i$ is complete hyperbolic and hyperbolically embedded. We will show for every n that $R^n(N)$ is in $\mathbb{CP}^2 \smallsetminus \bigcup_i X_i$, so that R^n is normal on N.

Since $P_c^* \cap X_0 = \{e\}$, and $e \notin N$, we have that $N \cap X_0 = \emptyset$. Therefore, by reasoning identical to that in the proof of Proposition 4.3, if $R^n(N)$ intersects X_i for any $i = 0, \ldots, 4$ we must have that some iterate $R^m(N)$ intersects $X_1 = L_{\text{inv}}$.

We will check that forward iterates of $R^n(P_c^*)$ are disjoint from L_{inv}, which is sufficient since $N \subset P_c^*$. The line L_{inv} intersects the invariant annulus C_0 in two properly embedded radial curves, so Property (P9') gives that $(R^n)^* L_{\text{inv}}$ intersects C in at least $2 \cdot 4^n$ properly embedded radial curves.

One can check that P_c intersects the invariant annulus C in the horizontal curve

$$\left\{ [U : V : W] \ : \ \frac{V^2}{UW} = c \in [0,1] \text{ and } \left| \frac{W}{U} \right| = 1 \right\},$$

which corresponds to $|u| = 1/\sqrt{c} > 1$ in the u coordinate for C. Therefore, the $2 \cdot 4^n$ radial curves in C from $(R^n)^* L_{\text{inv}}$ intersect P_c in at least $2 \cdot 4^n$ distinct points within C.

We will now show that these are the only intersection points between $(R^n)^* L_{\text{inv}}$ and P_c in all of $\mathbb{CP}^2$. Since R is algebraically stable, Bezout's Theorem gives $\deg(P_c) \cdot \deg((R^n)^* L_{\text{inv}}) = 2 \cdot 4^n$ intersection points, counted with multiplicities, in all of $\mathbb{CP}^2$. Therefore $P_c \cap (R^n)^* L_{\text{inv}} \subset C$.

Since $P_c^* \subset P_c$ with $P_c^* \cap C = \emptyset$, we conclude that $P_c^* \cap (R^n)^* L_{\text{inv}} = \emptyset$ for ever n. In other words, $R^n(P_c^*) \cap L_{\text{inv}} = \emptyset$ for ever n. Thus, the same holds for $N \subset P_c^*$, implying that R^n is a normal family on N. □

Proposition 4.6 has an interesting consequence for $\mathcal{R}$. The fixed point e' for R has a single preimage $\eta' = \Psi^{-1}(e')$, which is a superattracting fixed point for $\mathcal{R}$. However, e has the entire collapsing line $Z = 0$ as preimage under Ψ. Within this line is another superattracting fixed point $\eta = [0 : 1 : 1]$ for $\mathcal{R}$ and every point in $\{Z = 0\} \smallsetminus \{\mathbf{0}, \gamma\}$ is collapsed by $\mathcal{R}$ to η.

We obtain:

Corollary 4.7. *For the mapping $\mathcal{R}$, the solid cylinder $\{(z,t) : |z| < 1, t \in (0,1]\}$ is in $\mathcal{W}^s(\eta)$ and, symmetrically, the solid cylinder $\{(z,t) : |z| > 1, t \in [0,1]\}$ is in $\mathcal{W}^s(\eta')$.*

Notice that we we had to omit the "bottom", $t = 0$, of the solid cylinder in $\mathcal{W}^s(\eta)$ because points on it are forward asymptotic to the indeterminate point $\mathbf{0}$.

4.3. **Measure of Maximal Entropy.** There is a conjecture specifying the expected ergodic properties of a dominant rational map of a projective manifold[3] in terms of the relationship between various "dynamical degrees" of the map; see [Gu4].

Since the Migdal-Kadanoff renormalization R is an algebraically stable map of $\mathbb{CP}^2$, there are only two relevant dynamical degrees, the topological degree $\deg_{top} R$ and the algebraic degree $\deg R$, which satisfy

$$\deg_{top} R = 8 > 4 = \deg R.$$

This case of *high topological degree* was studied by Guedj [Gu3], who made use of a bound on topological entropy obtained by Dinh and Sibony [DS1]. In our situation, his results give

[3] It is stated more generally in [Gu4], for meromorphic maps of compact Kähler manifolds.

Proposition 4.8. *R has a unique measure ν of maximal entropy $\log 8$ with the following properties*

(i) *ν is mixing;*
(ii) *The Lyapunov exponents of ν are bounded below by $\log\sqrt{2}$;*
(iii) *If θ is any probability measure that does not charge the postcritical set*[4] *of R, then $8^{-n}(R^n)^*\theta \to \nu$;*
(iv) *If P_n is the set of repelling periodic points of R of period n that are in $\operatorname{supp}\nu$, then $8^{-n}\sum_{a\in P_n}\delta_a \to \nu$.*

The measure ν satisfies the backwards invariance $R^*\nu = 8\nu$, hence its support is totally invariant. In our situation, $\operatorname{supp}\nu \subsetneq J_R$ because (for example) the points in $\mathcal{W}^s(\mathcal{B})$ are not in $\operatorname{supp}\nu$. It can be thought of as the "little Julia set" within J_R on which the "most chaotic" dynamics occurs.

Remark 4.2. The statement of (iv) is slightly different in [Gu3], where it is not emphasized that the repelling periodic points being considered are actually on $\operatorname{supp}\nu$. However, it follows from the proof in [Gu3] and the fact that $\operatorname{supp}\nu$ is totally invariant. See [DS3, Thm 1.4.13] for the analogous argument for holomorphic $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

Remark 4.3. We know very little about the support of ν. See Problem C.3.

5. Pluri-potential interpretation

We will now prove the Global Lee-Yang-Fisher Theorem, assuming some technical volume estimates that will be saved for §6. In fact, we will prove the stronger theorem:

Theorem 5.1. *Let A be an algebraic curve of degree a. Then, $\frac{1}{4^n a}(R^n)^*[A]$ converges distributionally to the Green current if and only if A contains neither of the superattracting fixed points $\{e, e'\}$.*

The principal LYF locus S_0^c is disjoint from $\{e, e'\}$, so we have:

Corollary 5.2. *$\frac{1}{4^n}(R^n)^*[S_0^c]$ converges distributionally to the Green current S.*

Remark 5.1. The Global Lee-Yang-Fisher Theorem follows from Corollary 5.2, after pulling everything back under Ψ.

Remark 5.2. Note that the principal Lee-Yang-Fisher locus S_0^c does not lie within the Julia set J_R. In fact, the only holomorphic curves inside the stable manifold $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathrm{B})$ are the stable manifolds $\mathcal{W}^s_{\mathrm{loc}}(\zeta)$ of various points $\zeta \in \mathrm{B}$. Indeed, the maximal complex subspace within the tangent space $E_x := T_x(\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathrm{B}))$ is the complex line $E_x^c := E_x \cap iE_x$, which thus must coincide with $T_x\mathcal{W}^s_{\mathrm{loc}}(\zeta)$. Consequently, any holomorphic curve inside $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathrm{B})$ must be tangent to the line field $T_x\mathcal{W}^s_{\mathrm{loc}}(\zeta)$.

For the same reason, none of the LY loci $S_n = (R^n)^*(S_0)$ lies inside J_R either. However, in the limit they are distributed within the Julia set $J_R = \operatorname{supp} S$.

[4] Here, the postcritical set is defined without taking the closure.

5.1. **Proof of Theorem 5.1.** First, suppose that A contains one of the (totally invariant) superattracting fixed points, say e. Since the restriction of R to L_0 is given by $(U/W) \mapsto (U/W)^4$, the Bezout intersection multiplicity at e between $(R^n)^*A$ and L_0 grows like 4^n. Therefore, $\frac{1}{4^n}(R^n)^*[A]$ has definite mass in any neighborhood of e, preventing it from converging to a multiple of S, whose support is J_R.

Now suppose that $A \cap \{e, e'\} = \emptyset$. Let

$$A = \{x \in \mathbb{CP}^2 \,:\, P(\hat{x}) = 0\},$$

with P a homogeneous polynomial of degree a, which, without loss of generality, we assume to be irreducible. We must show that the limit

$$G = \lim_{n\to\infty} \frac{1}{4^n a} \log |P \circ \hat{R}^n| \tag{5.1}$$

exists in $L^1_{\text{loc}}(\mathbb{C}^3)$ and is equal to the Green potential of R (see Appendix B.2).

The homogeneious polynomial P determines a section σ_P of the a-th tensor power of the hyperplane bundle. Let us endow this bundle with the standard Hermitian structure $\|\cdot\|$ (see (A.5) from §A.4).

Theorem 5.3. $\frac{1}{4^n a} \log \|\sigma_P \circ R^n\| \to 0 \quad \text{in } L^1_{\text{loc}}(\mathbb{CP}^2) \quad \text{as} \quad n \to \infty.$

We will first show that Theorem 5.3 implies the convergence $\frac{1}{4^n a}(R^n)^*[A] \to S$.

Proof. Let $X = (U, V, W) \in \mathbb{C}^3 \smallsetminus \{0\}$ and $x = (U : V : W) \in \mathbb{CP}^2$. By (A.5) we have:

$$\|\sigma_P(x)\| = \frac{|P(X)|}{\|X\|^a} \qquad \text{and} \qquad \|\sigma_P(R^n x)\| = \frac{|P(\hat{R}^n X)|}{\|\hat{R}^n X\|^a}.$$

Outside of the measure zero set $\hat{R}^{-n}\{P = 0\}$, we can take $4^{-n}a^{-1}\log$ and apply Theorem 5.3 to obtain

$$\frac{1}{4^n a} \log |P(\hat{R}^n X)| = \frac{1}{4^n} \log \|\hat{R}^n X\| + o(1)$$

in $L^1_{\text{loc}}(\mathbb{C}^3)$. But since R is algebraically stable the quantity in the right-hand side converges to the Green potential (Theorem B.3). □

Proof of Theorem 5.3: We will use the following general convergence criterion:

Lemma 5.4. *Let ϕ_n be a sequence of L^2 functions on a finite measure space (X, m) with bounded L^2-norms. If $\phi_n \to 0$ a.e. then $\phi_n \to 0$ in L^1.*

Proof. Take any $\epsilon > 0$ and $\delta > 0$. By Egorov's Lemma, there exists a set $X' \subset X$ with $m(X \smallsetminus X') < \epsilon$ such that $\phi_n \to 0$ uniformly on X'. So, eventually the sup-norms of the ϕ_n on X' are bounded by δ. Hence

$$\int |\phi_n|\, dm = \int_{X'} |\phi_n|\, dm + \int_{X \smallsetminus X'} |\phi_n|\, dm \le \delta \cdot m(X) + B\sqrt{\epsilon},$$

where the last estimate follows from the Cauchy-Schwarz Inequality (with B the L^2-bound on the ϕ_n). The conclusion follows. □

Next, we will estimate how badly volume can increase under iterated pullbacks by R. Any volume form on $\mathbb{CP}^2$ will suffice for our discussion. Let us normalize so that $\operatorname{vol} \mathbb{CP}^2 = 1$.

Lemma 5.5. *For any measurable set X of distance at least d from the fixed points e, e', there exists $C_1(d) > 0$ so that* $\operatorname{vol} R^{-n} X \le C_1(d)\,(\operatorname{vol} X)^{1/3^n}$.

Proof. Since the fixed points e, e' are (super)attracting, all inverse images $R^{-j}X$ remain bounded away from them. Therefore, we can apply Proposition 6.4 to each iterate. □

Remark 5.3. With additional care, one can replace $1/3^n$ with $\gamma/2^n$, for a suitable constant γ. However, for our proof, any exponent γ/λ^n with $\lambda < 4$ will suffice.

Let

$$\phi_n = \frac{1}{4^n} \log \|\sigma_P \circ R^n\|.$$

Let us estimate distribution of the tales of these "random variables".

Lemma 5.6. *Let $M = \sup \log \|\sigma_P\|$. Then, there exists $C > 0$ so that*

$$\operatorname{vol}\{|\phi_n| > r\} \le C \exp\left(-2r \left(\frac{4}{3}\right)^n\right) \quad \text{for any } r > M4^{-n}.$$

Proof. We have:

$$X_n(r) := \{|\phi_n| > r\} = \{\log \|\sigma_P \circ R^n\| > r4^n\} \cup \{\log \|\sigma_P \circ R^n\| < -r4^n\} =$$

$$= \{\log \|\sigma_P \circ R^n\| < -r4^n\} = R^{-n}\{\|\sigma_P\| < \exp(-r4^n)\}.$$

(We have used that $\log \|\sigma_P\| < r4^n$.) One can use a finite sequence of blow-ups to resolve any singular points of the algebraic curve A (and the fact that P is irreducible) to show that a tubular neighborhood $\{\|\sigma_P\| < k\}$ of A has volume $\le C_2 k^2$. Thus, Lemma 5.5 gives

$$\operatorname{vol} X_n(r) \le C \exp\left(-2r \left(\frac{4}{3}\right)^n\right).$$

□

We are ready to show that the functions ϕ_n satisfy the conditions of Lemma 5.4.

Lemma 5.7. *The sequence ϕ_n is L^2-bounded.*

Proof. We have:

$$\|\phi_n\|^2 \le \sum_{k=0}^{\infty} (k+1)^2 \operatorname{vol}\{|\phi_n| \ge k\}.$$

By Lemma 5.6, this sum is bounded by

$$\sum_{k=0}^{M+1} (k+1)^2 + C \sum_{k>M} (k+1)^2 \exp\left(-2k \left(\frac{4}{3}\right)^n\right)$$

$$\le C_0 + C \sum_{k=0}^{\infty} (k+1)^2 \exp(-k) < \infty.$$

□

Lemma 5.8. *The sequence ϕ_n exponentially converges to 0 almost everywhere.*

Proof. Fix any $\lambda \in (1, 4/3)$. For sufficiently large n, we have $\lambda^{-n} > M4^{-n}$, hence Lemma 5.6 gives

$$\mathrm{vol}\{|\phi_n| > \lambda^{-n}\} \le C \exp\left(-2\left(\frac{4}{3}\lambda^{-1}\right)^n\right).$$

Since the sum of these volumes converges, the Borel-Cantelli Lemma gives that for a.e. $x \in \mathbb{CP}^2$, we eventually have $|\phi_n(x)| \le \lambda^{-n}$. □

This completes the proof of Theorem 5.3. □

6. Complex Whitney folds and Volume Estimates

To simplify calculations near the critical points, it is convenient to bring R to a normal form. A complex Whitney fold is generic and the simplest one (see [AGV]). Let $R : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a germ of holomorphic map with a critical point at 0. The map R (and the corresponding critical set) is called a *complex Whitney fold* if

(W1) The critical set L is a non-singular curve near 0;
(W2) $DR(0)$ has rank 1 and $\mathrm{Ker}\, DR(0)$ is transverse to L;
(W3) The second differential $D^2R(0)$ is not vanishing in the direction of $\mathrm{Ker}\, DR(0)$.

The following is a standard result from singularity theory:

Lemma 6.1. *A Whitney fold can be locally brought to a normal form $(u, w) \mapsto (u, w^2)$ in holomorphic coordinates.*

We now consider how volume is transformed under a mapping near a Whitney fold. Let us begin with the 1D power map:

Lemma 6.2. *Let $Q : w \mapsto w^d$. For any measurable set $Y \subset \mathbb{C}$,*

$$\mathrm{area}(Q^{-1}X) \le (\mathrm{area}\, X)^{1/d}.$$

Proof. Of course, we can assume that $\mathrm{area}\, X > 0$. Let us take the radius $r > 0$ such that $\pi r^2 = \mathrm{area}\, X$. Let $X_- = X \cap \mathbb{D}_r$, $X_+ = X \setminus X_-$, $X_c = \mathbb{D}_r \setminus X_-$. Then

$$(\mathrm{area}\, X)^{1/d} = (\mathrm{area}\, \mathbb{D}_r)^{1/d} = \mathrm{area}(Q^{-1}\mathbb{D}_r) = \mathrm{area}(Q^{-1}X_-) + \mathrm{area}(Q^{-1}X_c) \ge$$
$$\mathrm{area}(Q^{-1}X_-) + d\, \mathrm{Jac}\, Q^{-1}(r)\, \mathrm{area}\, X_c \ge \mathrm{area}(Q^{-1}X_-) + \mathrm{area}(Q^{-1}X_+) = \mathrm{area}(Q^{-1}X).$$

□

Corollary 6.3. *Let L be a complex Whitney fold for a map R, and let $L' \Subset L$. Then for any measurable set X sufficiently close to $R(L')$, we have:*

$$\mathrm{vol}(R^{-1}X) \le C\sqrt{\mathrm{vol}\, X},$$

where R^{-1} is the one-to-two branch of the inverse map associated to the fold, and $C = C(R, L, L')$.

Proof. As the assertion is local, we can use the Whitney normal coordinates (u, w) near L' (Lemma 6.1). Let X^h be the projection of X (in this coordinates) onto the u-axis L, and let X^v_u be the slice of X be the vertical complex line through $(u, 0) \subset L$. Then

$$\mathrm{vol}(R^{-1}X) \le \int_{X^h} \mathrm{area}(Q^{-1}(X^v_u))\, d\,\mathrm{area}(u)$$
$$\le \int_{X^h} \sqrt{\mathrm{area}\, X^v_u}\, d\,\mathrm{area}(u) \le \sqrt{\mathrm{area}\, X^h \cdot \mathrm{vol}\, X},$$

where the estimates follow from Fubini, Lemma 6.2, and Cauchy-Schwarz respectively. □

We will now prove the following estimate on how volumes transform under a single iterate of R that was used in the proof from §5.

Proposition 6.4. *For any measurable set X of distance at least d from the fixed points e, e', we have:*

$$\operatorname{vol}(R^{-1}X) \le C(d)(\operatorname{vol} X)^{1/3}.$$

The proof will follow from a sequence of lemmas.

Lemma 6.5. *Let $X \subset \mathbb{CP}^2$ be a measurable set such that $d := \operatorname{dist}(X, \{e, e', b_0, \mathbf{0}\}) > 0$. Then $\operatorname{vol}(R^{-1}X) = O(\sqrt{\operatorname{vol} X})$, with the constant depending on d.*

Proof. By Lemma 3.3, $e, e', b_0, \mathbf{0}$ are the only critical values of $\tilde{R}$ that are images of critical points more complicated than Whitney folds. By Corollary 6.3, $\operatorname{vol}(\tilde{R}^{-1}X) = O(\sqrt{\operatorname{vol} X})$. Since $R^{-1}X = \pi(\tilde{R}^{-1}X)$, where the projection $\pi : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$ is regular, we are done. □

The only remaining critical values (other than e and e') are b_0 and $\mathbf{0}$, with corresponding critical points consisting of the collapsing line L_2 and the two points $\pm(i, i)$, respectively. We first estimate the Jacobian near each of these critical points. We will say that a system of local coordinates (λ, η) is *centered* at x if $\lambda(x) = \eta(x) = 0$.

Lemma 6.6. *Centered at any point of the collapsing line L_2, except for $L_2 \cap L_0$ and two indeterminacy points $a_\pm$, there is a system of coordinates (λ, η) such that near this point we have $\det DR \asymp \lambda^2$.*

Proof. Let one of the coordinates be

$$\lambda = \frac{u + w}{1 + u^2}$$

near the collapsing line $L_2 = \{u + w = 0\}$ and outside the indeterminacy points $a_\pm$.

Recall that $R(L_2) = b_0$. In local coordinates $(z = 1/u,\ \zeta = u/w)$ near b_0, the map R (2.9) assumes form

$$z = \lambda^2, \quad \zeta = \sigma^2, \quad \text{where } \sigma = \frac{1 + u^2}{1 + w^2}.$$

So, in local coordinates $R : (\lambda, u) \mapsto (z, \zeta)$ we have:

$$\det DR = 4\lambda\sigma \frac{\partial \sigma}{\partial u}. \tag{6.1}$$

Moreover, $w = (1 + u^2)\lambda - u$, $\partial w/\partial u = 2u\lambda - 1$, and an elementary calculation yields:

$$\frac{\partial \sigma}{\partial u} = 2\frac{1 + u^2}{(1 + w^2)^2}(1 - uw)\,\lambda \asymp \lambda,$$

as long as we stay near L_2 and away from $a_\pm$. The conclusion follows. □

A similar estimate holds in the blow-up coordinates near the indeterminacy points $a_\pm \in L_2$. Recall that $\tilde{R}$ is given by (3.3) in the blow-up coordinates (ξ, χ). Moreover, $\tilde{L_2} = \{\chi = -1\}$.

Lemma 6.7. *There is a system of coordinates* (λ, η) *centered at* $\tilde{L_2} \cap L_{\rm exc}(a_\pm)$ *so that near this point we have* $\det D\tilde{R} \asymp \lambda^2$.

Proof. It is sufficient to treat $\tilde{R}$ near $\tilde{L_2} \cap L_{\rm exc}(a_+)$. In the local coordinates

$$\lambda = \frac{1+\chi}{\xi + 2i}, \quad \xi = u - i$$

near $L_{\rm exc}(a)$ and local coordinates $(z = 1/u,\ \zeta = u/w)$ near b_1, the map $\tilde{R}$ assumes form

$$z = \lambda^2, \quad \zeta = \sigma^2, \quad \text{where } \sigma = \frac{\xi + 2i}{\chi^2 \xi - 2i\chi}.$$

So

$$\det DR = 4\lambda\sigma \frac{\partial \sigma}{\partial \xi}.$$

Moreover, $\dfrac{\partial \chi}{\partial \xi} = \lambda$, and an elementary calculation yields: $\dfrac{\partial \sigma}{\partial \xi} \asymp \lambda$ near $\tilde{L_2} \cap L_{\rm exc}(a_+)$. □

Lemma 6.8. *There is a system of coordinates* (λ, η) *centered at* $L_2 \cap L_0$ *so that near this point we have* $\det DR \asymp \lambda^2 \eta$.

Proof. Let $\xi = W/U$ and $\eta = V/U$. Near $\{x\} = L_2 \cap L_0$, we will use the local coordinates $(\lambda = \xi + 1, \eta)$ and near b_0 we use (ξ, η). We have

$$\xi' = \left(\frac{\eta^2 + (\lambda - 1)^2}{1 + \eta^2} \right)^2, \qquad \eta' = \frac{\lambda^2 \eta^2}{(1 + \eta^2)^2},$$

where $(\xi', \eta') = R(\lambda, \eta)$. The result follows. □

Lemma 6.9. *There is a system of coordinates* (λ, η) *centered at* $\pm(i, i)$ *so that near this point we have* $\det DR \asymp \lambda\eta$.

Proof. In the coordinates

$$\lambda = \frac{u^2 + 1}{u + w}, \qquad \eta = \frac{w^2 + 1}{u + w},$$

we have $(u', w') = (\lambda^2, \eta^2)$. □

Lemma 6.10. *Let* $f : \mathbb{C}^2 \to \mathbb{C}^2$ *be a holomorphic mapping such that near the origin* $\det Df \asymp \lambda^a \eta^b$. *For any* $\gamma > \max(a, b) + 1$ *there is a* $C > 0$ *and a neighborhood* D *of the origin such that for any measurable set* $X \subset \mathbb{CP}^2$ *we have*

$$\text{vol}(R^{-1}X \cap D) \leq C(\mathit{vol}X)^\gamma. \tag{6.2}$$

(If $a \neq b$*, then the same holds for* $\gamma = \max(a, b) + 1$*.)*

Proof. Let us take a bidisc neighborhood of the origin $D = \{|\lambda| < \epsilon\} \times \{|\eta| < \epsilon\}$. For any measurable set $Y \subset D$ we have:

$$\text{vol}(f(Y)) \asymp \int_{Y^v} |\eta|^{2b} d\,\text{area}(\eta) \int_{Y^h_\eta} |\lambda|^{2a}\, d\,\text{area}(\lambda),$$

where Y^v is the projection of Y onto the η-axis and Y^h_η are the slices of Y by horizontal lines. The inner integral above is exactly $\text{area}(Q_{a+1}(Y^h_\eta))/(a+1)^2$, where

$Q_{a+1}(\lambda) = \lambda^{a+1}$. By Lemma 6.2, it is bounded below by $(\mathrm{area}\, Y^h_\eta)^{a+1}/(a+1)^2$. Using the Hölder inequality, with $p = \gamma/(\gamma-1)$ and $q = \gamma$, we find

$$\left(\int_{Y_v} 1/|\eta|^{2b/(\gamma-1)}\, d\,\mathrm{area}(\eta)\right)^{\gamma-1} \quad \int_{Y^v} |\eta|^{2b}(\mathrm{area}\, Y^h_\eta)^{a+1}\, d\,\mathrm{area}(\eta)$$
$$\geq \left(\int_{Y^v} (\mathrm{area}\, Y^h_\eta)^{(a+1)/\gamma}\, d\,\mathrm{area}(\eta)\right)^{\gamma} \geq (\mathrm{vol}\, Y)^{\gamma}.$$

The conclusion follows since $b < \gamma - 1$ implies that $1/|\eta|^{2b/(\gamma-1)}$ is locally integrable. (If $b < a$, then $\gamma = a + 1$ is sufficient for local integrability.) □

Proof of Proposition 6.4: By Lemma 6.5, we need only consider X near b_0 and $\mathbf{0}$. For X near b_0, we combine Lemma 6.10 with the estimates of $\det D\tilde{R}$ obtained in Lemmas 6.6, 6.7, and 6.8, to find that $\mathrm{vol}(\tilde{R}^{-1}X) \leq C(\mathrm{vol}\, X)^{1/3}$. This is sufficient, since $\pi : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$ is regular.

Meanwhile, if X is near $\mathbf{0}$, the result follows by combining Lemma 6.10 with Lemma 6.9. □

Appendix A. Elements of complex geometry

We are primarily interested in rational maps between complex projective spaces in two dimensions. However, greater generality will be useful in many circumstances (for example, in order to study a rational mapping near its indeterminate points). Much of the below material can be found with greater detail in [Da, De, GH, Shaf, K, La].

A.1. **Projective varieties and rational maps.** Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{CP}^k$ denote the canonical projection. Given $z \in \mathbb{CP}^k$, any $\hat{z} \in \pi^{-1}(z)$ is called a *lift* of z. One calls $V \subset \mathbb{CP}^k$ a (projective) algebraic hypersurface if there is a homogeneous polynomial $P : \mathbb{C}^{k+1} \to \mathbb{C}$ so that

$$V = \{z \in \mathbb{CP}^k : \quad P(\hat{z}) = 0\}.$$

More generally, a (projective) algebraic variety is the locus satisfying a finite number homogeneous polynomial equations. Any algebraic variety V has the structure of a smooth manifold away from a proper subvariety $V_{\mathrm{sing}} \subset V$ and the dimension of $V \setminus V_{\mathrm{sing}}$ is called *the dimension of* V. One calls V a *projective manifold* if $V_{\mathrm{sing}} = \emptyset$.

A rational map $R : \mathbb{CP}^k \to \mathbb{CP}^l$ is given by a homogeneous polynomial map $\hat{R} : \mathbb{C}^{k+1} \to \mathbb{C}^{l+1}$ for which we will assume the components have no common factors. One defines $R(z) := \pi(\hat{R}(\hat{z}))$ if $\hat{R}(\hat{z}) \neq 0$, and otherwise we say that z is an *indeterminacy point for R.* Since $\hat{R}$ is homogeneous, the above notions are well-defined. Because the components of $\hat{R}$ have no common factors, the set of indeterminate points $I(R)$ is a projective variety of codimension greater than or equal to two.

Given two projective varieties, $V \subset \mathbb{CP}^k$ and $W \subset \mathbb{CP}^l$, a *rational map* $R : V \to W$ is the restriction of a rational map $R : \mathbb{CP}^k \to \mathbb{CP}^l$ such that $R(V \setminus I(R)) \subset W$. As above, $I(R) \subset V$ is a projective subvariety of codimension greater than or equal to two in V. If $I(R) = \emptyset$, we say that R is a (globally) *holomorphic* map.

A rational mapping $R : V \to W$ between projective manifolds is *dominant* if there is a point $z \in V \setminus I(R)$ such that $\mathrm{rank}\, DR(z) = \dim W$.

We will call a subvariety $U \subset V$ a *collapsing variety* [5] if $\dim(R(U)) < \dim(U)$.

Lemma A.1. *Let $R : V \to W$ be a dominant rational map between projective manifolds of the same dimension. If z is not an indeterminate point for R and not on any collapsing variety for R, then R is locally surjective at z.*

It is a consequence of the Weierstrass Preparation Theorem—see for example, [De, Ch. II, §4.2] or [GH, Ch. 0.1].

A.2. **Divisors.** Divisors are a generalization of algebraic hypersurfaces that behave naturally under dominant rational maps. We will present an adaptation of material from [Da, Ch. 3] and [Shaf] suitable for our purposes.

A *divisor* D on a projective manifold V is a collection of irreducible hypersurfaces $C_1, \ldots, C_r$ with assigned integer multiplicities $k_1, \ldots, k_r$. One writes D as a formal sum

$$D = k_1 C_1 + \cdots + k_r C_r. \tag{A.1}$$

Alternatively, D can be described by choosing an open cover $\{U_i\}$ of V and rational functions $g_i : U_i \to \mathbb{C}$ with the compatibility property that g_i/g_j is a non-vanishing holomorphic function on $U_i \cap U_j \neq \emptyset$. Taking zeros and poles of the g_i counted with multiplicities, we obtain representation (A.1).

If $f : V \to W$ is a dominant holomorphic map, and $D = \{U_i, g_i\}$ is a divisor on W, the *pullback* f^*D is the divisor on V given by $\{f^{-1}U_i, f^*g_i\} \equiv \{f^{-1}U_i, g_i \circ f\}$. If $R : V \to W$ is a dominant rational map (with $I(R) \neq \emptyset$), we define R^*D by first pull-backing D under $R : V \smallsetminus I(R) \to W$. Since $I(R)$ is a finite collection of points, the result (in terms of local defining functions) can be extended trivially to obtain a divisor R^*D on all of V. Since the trivial extension of a divisor is unique, the result is well-defined.

A.3. **Blow-ups.** Given a pointed projective surface (V, p), the *blow-up* of V at p is another projective surface $\tilde{V}$ with a holomorphic projection $\pi : \tilde{V} \to V$ such that

- $L_{\text{exc}}(p) := \pi^{-1}(p)$ is a complex line $\mathbb{CP}^1$ called the *exceptional divisor*;
- $\pi : \tilde{V} \smallsetminus L_{\text{exc}}(p) \to V \smallsetminus \{p\}$ is a biholomorphic map.

See [GH, Shaf].

The construction has a local nature near p, so it is sufficient to provide it for $(\mathbb{C}^2, 0)$. The space of lines $l \subset \mathbb{C}^2$ passing through the origin is $\mathbb{CP}^1$, by definition. Then $\tilde{\mathbb{C}^2}$ is realized as the surface X in $\mathbb{C}^2 \times \mathbb{CP}^1$ given by equation $\{(u, v) \in l\}$ with the natural projection $(u, v, l) \mapsto (u, v)$. In this model, points of the exceptional divisor $L_{\text{exc}} = \{(0, 0, l) : \ l \in \mathbb{CP}^1\}$ get interpreted as the *directions* l at which the origin is approached.

Any line $l \subset \mathbb{C}^2$ naturally lifts to the "line" $\tilde{l} = \{(u, v, l) : \ (u, v) \in l\}$ in $\tilde{\mathbb{C}^2}$ crossing the exceptional divisor at $(0, 0, l)$. This turns $\tilde{\mathbb{C}^2}$ into a line bundle over $\mathbb{CP}^1$ known as the *tautological* line bundle. (The tautological line bundle over $\mathbb{CP}^m$ is constructed analogously.) Moreover, $\tilde{\mathbb{C}^2} \smallsetminus \tilde{l}$ is isomorphic to $\mathbb{C}^2$. Indeed, let $\phi(u, v) = au + bv$ a linear functional that determines l. It is linearly independent from one of the coordinate functionals, say with v (so $a \neq 0$). Then

$$(u, v, l) \mapsto (\phi, \kappa := v/\phi)$$

[5] Algebraic geometers call such a variety "exceptional". However this term has a conflicting meaning in complex dynamics, where the "exceptional set of R" consists of the largest proper algebraic variety that is completely invariant under R.

is a local chart that provides a desired isomorphism. In particular, two charts corresponding to the coordinate axes in $\mathbb{C}^2$ provide us with local coordinates $(u, \kappa = v/u)$ and $(v, \kappa = u/v)$ which are usually used in calculations.

The value of this construction lies in the fact that it can be used to *resolve* the indeterminacy of a rational map; see [Shaf, Ch. IV, §3.3]. Moreover, any analytic curve C on V lifts to an analytic curve $\tilde{C} := \overline{\pi^{-1}(C \smallsetminus \{p\})}$ on $\tilde{V}$, known as the *proper transform* of C, which tends to have milder singularities than C; see [Shaf, Ch. IV, §4.1]. Taking multiplicities into consideration, the proper transform of a divisor D is defined similarly.

A.4. **Hyperplane bundle.** Associated to any homogeneous polynomial $P : \mathbb{C}^{m+1} \to \mathbb{C}$ is a divisor D_P given by $\{U_i, P \circ \sigma_i\}$, where the $\{U_i\}$ form an open covering of $\mathbb{CP}^m$ that admits local sections $\sigma_i : U_i \to \mathbb{C}^{m+1} \smallsetminus \{0\}$ of the canonical projection π. Furthermore, every divisor can be described as a difference $D = D_P - D_Q$ for appropriate P and Q. The following simple formula describes the pull-back:

$$R^* D_P = D_{\hat{R}^* P} \equiv D_{P \circ \hat{R}}, \tag{A.2}$$

where $\hat{R} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ is the lift of R.

The *degree* of a divisor $D = D_P - D_Q$ is $\deg D = \deg P - \deg Q$. On $\mathbb{CP}^2$, *Bezout's Theorem* asserts that two divisors D_1 and D_2 intersect $\deg D_1 \cdot \deg D_2$ times in $\mathbb{CP}^2$, counted with appropriate *intersection multiplicities*. Suppose that D_1 and D_2 are irreducible algebraic curves assigned multiplicity one. Then, an intersection point z is assigned multiplicity one if and only if both curves are non-singular at z, meeting transversally there. See [Shaf, Ch. IV].

The *algebraic degree* of a rational map $R : \mathbb{CP}^m \to \mathbb{CP}^m$ (denoted $\deg R$) is the common degree of the homogeneous equations in $\hat{R}$. Equation (A.2) implies

Lemma A.2. *Given a dominant rational map $R : \mathbb{CP}^m \to \mathbb{CP}^m$ and a divisor D in $\mathbb{CP}^m$, we have:*

$$\deg(R^* D) = \deg R \cdot \deg D.$$

We can also describe divisors on $\mathbb{CP}^m$ using sections of appropriate line bundles:

The fibers of $\pi : \mathbb{C}^{m+1} \smallsetminus \{0\} \to \mathbb{CP}^m$ are punctured complex lines $\mathbb{C}^*$. Compactifying each of these lines at infinity, we add to $\mathbb{C}^{m+1} \smallsetminus \{0\}$ the line at infinity $L_\infty \approx \mathbb{CP}^m$ obtaining the total space $(\mathbb{C}^{m+1})^* \cup L_\infty \approx (\mathbb{CP}^{m+1})^* := \mathbb{CP}^{m+1} \smallsetminus \{0\}$. The projection naturally extends to $\pi : (\mathbb{CP}^{m+1})^* \to \mathbb{CP}^m$, whose fibers are complex lines $\mathbb{C}$. The resulting line bundle is called the *hyperplane bundle* over $\mathbb{CP}^n$. It is naturally dual to the tautological bundle over $\mathbb{CP}^n$.

In homogeneous coordinates $(z_0 : \cdots : z_m : t)$ on $\mathbb{CP}^{m+1}$, this projection is just

$$\pi : (z_0 : \cdots : z_m : t) \mapsto (z_0 : \cdots : z_m), \tag{A.3}$$

with $L_\infty = \{t = 0\}$, $(\mathbb{C}^{m+1})^* = \{t = 1\}$, and the map $(z : t) \mapsto t/\|z\|$ parameterizing the fibers (here $\|z\|$ stands for the Euclidean norm of $z \in \mathbb{C}^{m+1} \smallsetminus \{0\}$). This line bundle is endowed with the natural *Hermitian structure*: $\|(z : t)\| = |t|/\|z\|$.

Any non-vanishing linear form Y on $\mathbb{C}^{n+1}$ determines a section of the hyperplane bundle:

$$\sigma_Y : z \mapsto (z : Y(z)), \quad z \in \mathbb{C}^{n+1}. \tag{A.4}$$

The divisor D_Y (a projective line counted with multiplicity 1) is precisely the zero divisor of σ_Y.

The dth tensor power of the hyperplane bundle can be described as follows. Its total space X^d is the quotient of $(\mathbb{C}^{m+2})^*$ by the $\mathbb{C}^*$-action

$$(z_0, \dots, z_m, t) \mapsto (\lambda z_0, \dots, \lambda z_m, \lambda^d t), \quad \lambda \in \mathbb{C}^*.$$

We denote the equivalence class of $(\hat{z}, t)$ using the "homogeneous" coordinates $(\hat{z} : t)$. The projection $X^d \to \mathbb{CP}^m$ is natural, as above (A.3). A non-vanishing homogeneous polynomial P on $\mathbb{C}^{m+1}$ of degree d defines a holomorphic section σ_P of this bundle given by $\sigma_P(z) = (\hat{z} : P(\hat{z}))$. This bundle is endowed with the Hermitian structure:

$$\|(z : t)\| = |t|/\|z\|^d. \tag{A.5}$$

More generally, any divisor $D = D_P - D_Q$ defines a section σ_D of the deg(D)-th tensor power of the hyperplane bundle, defined by $\sigma_D(z) = (\hat{z} : P(\hat{z})/Q(\hat{z}))$. One can recover D from σ_D by taking its zero divisor.

A.5. **Currents.** We will now give a brief background on currents; for more details see [dR, Le] and the appendix from [Si]. Currents are naturally defined on general complex (or even smooth) manifolds, however to continue our discussion of rational maps, divisors, etc, we restrict our attention to projective manifolds.

A $(1,1)$-*current* T on V is a continuous linear functional on $(m-1, m-1)$-forms with compact support. It can be also defined as a generalized differential $(1,1)$-form $\sum T_{ij} dz_i d\bar{z}_j$ with distributional coefficients.

A basic example is the current $[C]$ of integration over the regular points C_{reg} of an algebraic hypersurface C:

$$\omega \mapsto \int_{C_{\text{reg}}} \omega,$$

where ω is a test $(m-1, m-1)$-form. The current of integration over a divisor D is defined by extending linearly.

The space of currents is given the distributional topology: $T_n \to T$ if $T_n(\omega) \to T(\omega)$ for every test form ω.

A differential $(m-1, m-1)$-form ω is called positive if its integral over any complex subvariety is non-negative. A (1,1)-current T is called *positive* if $T(\omega) \geq 0$ for any positive (1,1)-form. A current T is called *closed* if $dT = 0$, where the differential d is understood in the distributional sense.

In this paper, we focus on closed, positive $(1,1)$-currents. They have a simple description in terms of local potentials, rather analogous to the definition of divisors.

Recall that ∂ and $\bar{\partial}$ stand for the holomorphic and anti-holomorphic parts of the external differential $d = \partial + \bar{\partial}$. Their composition $\frac{i}{\pi}\partial\bar{\partial}$ can be[6] thought of as a kind of "pluri-Laplacian" because, given a C^2-function h, the restriction of $\frac{i}{\pi}\partial\bar{\partial}h$ to any non-singular complex curve X is equal to the form $\frac{1}{2\pi}\Delta(h|X) dz \wedge d\bar{z}$, where z is a local coordinate on X and Δ is the usual Laplacian in this coordinate.

If U is an open subset of $\mathbb{C}^m$ and $h : U \to [-\infty, \infty)$ is a plurisubharmonic (PSH) function, then $\frac{i}{\pi}\partial\bar{\partial}h$ is a closed (1,1)-current on U. Conversely, the $\partial\bar{\partial}$-Poincaré Lemma asserts that every closed, positive $(1,1)$-current on U is obtained this way.

Therefore, any closed positive $(1,1)$-current T on a manifold V can be described using an open cover $\{U_i\}$ of V together with PSH functions $h_i : U_i \to [-\infty, \infty)$ that are chosen so that $T = \frac{i}{\pi}\partial\bar{\partial}h_i$ in each U_i. The functions h_i are called *local*

[6] Many authors introduce real operators $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ and write dd^c instead of $\frac{i}{\pi}\partial\bar{\partial}$. We use $\frac{i}{\pi}\partial\bar{\partial}$ to avoid confusion between the operator d and the algebraic degree of a map.

potentials for T and they are required to satisfy the compatibility condition that $h_i - h_j$ is pluriharmonic (PH) on any non-empty intersection $U_i \cap U_j \neq \emptyset$. The *support* of T is defined by:

$$\operatorname{supp} T \quad := \quad \{z \in V \,:\, \text{if } z \in U_j \text{ then } h_j \text{ is not pluriharmonic at } z\}.$$

The compatibility condition assures that that above set is well-defined.

The Poincaré-Lelong formula describes the current of integration over a divisor $D = \{U_i, g_i\}$ by the system of local potentials $h_i := \log |g_i|$. I.e., on each U_i we have $[D] = \frac{i}{\pi}\partial\bar{\partial}\ \log |g_i|$. The result is closed $(1,1)$-current, which is positive iff D is effective (i.e. the multiplicities $k_i, \ldots, k_r$ are non-negative).

Suppose $R : V \to W$ is a dominant rational map and T is a closed-positive $(1,1)$-current on W. The pullback R^*T is closed positive $(1,1)$-current on V, defined as follows. First, one obtains a closed positive $(1,1)$-current R^*T defined on $V \smallsetminus I(R)$ by pulling-back the system of local potentials defining T under $R : V \smallsetminus I(R) \to W$. One then extends R^*T trivially through $I(R)$, to obtain a closed, positive $(1,1)$-current defined on all of V. (By a result of Harvey and Polking [HaPo], this extension is closed.) See [Si, Appendix A.7] for further details. Pullback is continuous with respect to the distributional topology.

Similarly to divisors, there is a particularly convenient description of closed, positive $(1,1)$-currents on $\mathbb{CP}^m$. Associated to any PSH function $H : \mathbb{C}^{m+1} \to [-\infty, \infty)$, having the homogeneity (for some $c > 0$) that

$$H(\lambda\hat{z}) = c\log|\lambda| + H(\hat{z}), \tag{A.6}$$

is a closed, positive $(1,1)$-current, denoted by $\pi_*(\frac{i}{\pi}\partial\bar{\partial}H)$, given by the system of local potentials $\{U_i, H \circ \sigma_i\}$, where the $\{U_i\}$ form an open covering of $\mathbb{CP}^m$ that admits local sections $\sigma_i : U_i \to \mathbb{C}^{m+1} \smallsetminus \{0\}$ of the canonical projection π. (In each U_i, it is defined by $\pi_*(\frac{i}{\pi}\partial\bar{\partial}H) = \frac{i}{\pi}\partial\bar{\partial}\ H \circ \sigma_i$.) Moreover, every closed positive $(1,1)$-current on $\mathbb{CP}^m$ is described in this way; See [Si, Thm A.5.1]. The function H is called the *pluripotential* of $\pi_*(\frac{i}{\pi}\partial\bar{\partial}H)$.

If $R : \mathbb{CP}^m \to \mathbb{CP}^m$ is a rational map, the action of pull-back is described by

$$R^*\pi_*(\frac{i}{\pi}\partial\bar{\partial}H) = \pi_*(\frac{i}{\pi}\partial\bar{\partial}\ H \circ R). \tag{A.7}$$

A.6. **Kobayashi hyperbolicity and normal families.** In §4 we use the Kobayashi metric in order to prove that the iterates R^n form a normal family on certain subspaces of $\mathbb{CP}^2$. Here we recall the relevant definitions and some important results that we use. The reader can consult the books [K, La] and the original papers by M. Green [G1, G2] for more details. For more dynamical applications, see e.g. [Si].

The Kobayashi pseudometric is a natural generalization of the Poincaré metric on Riemann surfaces. Let $\|\cdot\|$ stand for the Poincaré metric on the unit disk $\mathbb{D}$. Let M by a complex manifold. Pick a a tangent vector $\xi \in TM$, and let $\mathcal{H}(\xi)$ be a family of holomorphic curves $\gamma : \mathbb{D} \to M$ tangent to the line $\mathbb{C} \cdot \xi$ at $\gamma(0)$. Then $Df(v) = \xi$ for some $v \equiv v_\gamma \in T_0\mathbb{D}$, and the Kobayashi pseudometric is defined to be:

$$ds_M(\xi) = \inf_{\gamma \in \mathcal{H}(\xi)} \|v_\gamma\|. \tag{A.8}$$

The Kobayashi pseudometric is designed so that holomorphic maps are distance decreasing: if $f : U \to M$ is holomorphic then $ds_M(Df(\xi)) \leq ds_U(\xi)$.

The reason for "pseudo-" is that for certain complex manifolds M, $ds(\xi)$ can vanish for some non-vanishing tangent vectors $\xi \neq 0$. For example, ds identically vanishes on $\mathbb{C}^n$ or $\mathbb{CP}^n$. A complex manifold M is called *Kobayashi hyperbolic* if ds is non-degenerate: $ds(\xi) > 0$ for any non-vanishing $\xi \in TM$. Then it induces a (Finsler) metric on M.

Let N be a compact complex manifold. Endow it with some Hermitian metric $|\cdot|_N$. A complex submanifold $M \subset N$ is called *hyperbolically embedded in* N if the Kobayashi pseudometric on M dominates the Hermitian metric on N, i.e., there exists $c > 0$ such that $ds_M(\xi) \geq c|\xi|_N$ for all $\xi \in TM$. Obviously, M is Kobayashi hyperbolic in this case.

A complex manifold M is called *Brody hyperbolic* if there are no non-constant holomorphic mappings $f : \mathbb{C} \to M$. If M is Kobayashi hyperbolic, it is also Brody hyperbolic, but the converse is generally not true unless M is compact.

An open subset of $\mathbb{CP}^2$ that is Kobayashi hyperbolic, but not hyperbolically embedded in $\mathbb{CP}^2$ is described in [K, Example 3.3.11] and an open subset of $\mathbb{C}^2$ that is Brody hyperbolic but not Kobayashi hyperbolic is described in [K, Example 3.6.6].

A family $\mathcal{F}$ of holomorphic mappings from a complex manifold U to a complex manifold M is called *normal* if every sequence in $\mathcal{F}$ either has a subsequence converging locally uniformly or a subsequence that diverges locally uniformly to infinity in M. In the case that M is embedded into some compact manifold Z, a stronger condition is that $\mathcal{F}$ is *precompact* in $\mathrm{Hol}(U, Z)$ (where $\mathrm{Hol}(U, Z)$ is the space of holomorphic mappings $U \to Z$ endowed with topology of uniform convergence on compact subsets of U).

Proposition A.3. *Let M be a hyperbolically embedded complex submanifold of a compact complex manifold N. Then for any complex manifold U, the family $\mathrm{Hol}(U, M)$ is precompact in $\mathrm{Hol}(U, N)$.*

See Theorem 5.1.11 from [K].

The classical Montel's Theorem asserts that the family of holomorphic maps $\mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$ is normal (as $\mathbb{C} \setminus \{0, 1\}$ is a hyperbolic Riemann surface). It is a foundation for the whole Fatou-Julia iteration theory. Several higher dimensional versions of Montel's Theorem, due to M. Green [G1, G2], are now available. Though their role in dynamics is not yet so prominent, they have found a number of interesting applications. Below we will formulate two particular results used in this paper (see §4). The following is Theorem 2 from [G1]:

Theorem A.4. *Let X be a union of (possibly singular) hypersurfaces $X_1, \dots, X_m$ in a compact complex manifold N. Assume $N \setminus X$ is Brody hyperbolic and*

$$X_{i_1} \cap \cdots \cap X_{i_k} \setminus (X_{j_1} \cup \cdots X_{j_l}) \text{ is Brody hyperbolic}$$

for any choice of distinct multi-indices $\{i_1, \dots, i_k, j_1, \dots, j_l\} = \{1, \dots, m\}$. Then $N \setminus X$ is a complete hyperbolically embedded submanifold of N.

In the last section of [G1], the following result is proved:

Theorem A.5. *Let $M = \mathbb{CP}^2 \setminus (Q \cup X_1 \cup X_2 \cup X_3)$, where Q is a non-singular conic and X_1, X_2, X_3 are lines. Then any non-constant holomorphic curve $f : \mathbb{C} \to M$ must lie in a line L that is tangent to Q at an intersection point with one of the lines, X_i, and that contains the intersection point $X_j \cap X_l$ of the other two lines.*

The configurations that appear in this theorem are related to amusing projective triangles:

A.7. **Self-dual triangles.** Let $Q(z) = \sum q_{ij} z_i z_j$ be a non-degenerate quadratic form in $E \approx \mathbb{C}^3$, and $X = \{Q = 0\}$ be the corresponding conic in $\mathbb{CP}^2$. The form Q makes the space E Euclidean, inducing duality between points and lines in $\mathbb{CP}^2$. Namely, to a point $z = (z_0 : z_1 : z_2)$ corresponds the line $L_z = \{\zeta : \; Q(z,\zeta) = 0\}$ called the *polar* of z with respect to X (here we use the same notation for the quadratic form and the corresponding inner product). Geometrically, this duality looks as follows. Given a point $z \in \mathbb{CP}^2$, there are two tangent lines from z to X. Then L_z is the line passing through the corresponding tangency points. (In case $z \in X$, the polar is tangent to X at z).

Three points z_i in $\mathbb{CP}^2$ in general position are called a "triangle" Δ with vertices z_i. Equivalently, a triangle can be given by three lines L_i in general position, its "sides". Let us say that Δ is *self-dual* (with respect to the conic X) if its vertices are dual to the opposite sides.

Lemma A.6. *A triangle Δ with vertices z_i is self-dual if and only if the corresponding vectors $\hat{z}_i \in E$ form an orthogonal basis with respect to the inner product Q.*

All three sides of a self-dual triangle satisfy the condition of Theorem A.5, so they can give us exceptional holomorphic curves $\mathbb{C} \to \mathbb{CP}^2 \smallsetminus (Q \cup X_1 \cup X_2 \cup X_3)$.

Remark A.1. In a similar way one can define self-dual tetrahedra in higher dimensions.

Appendix B. Elements of complex dynamics in several variables

We now provide a brief background on the dynamics of rational maps in several variables. We refer the reader to the survey [Si] for more details.

B.1. **Algebraic Stability.** The following statement appears in [Si, Prop. 1.4.3]:

Lemma B.1. *Let R and S be two rational maps $\mathbb{CP}^m \to \mathbb{CP}^m$. Then,* $\deg(S \circ R) = \deg(S) \cdot \deg(R)$ *if and only if there is no algebraic hypersurface $V \subset \mathbb{CP}^m$ that is collapsed by R to an indeterminate point of S.*

A rational mapping $R : \mathbb{CP}^m \to \mathbb{CP}^m$ is called *algebraically stable* if there is no integer n and no collapsing hypersurface $V \subset \mathbb{CP}^m$ so that $R^n(V)$ is contained within the indeterminacy set of R, [Si, p. 109]. A consequence of Lemma B.1 is that R is algebraically stable if and only if $\deg R^n = (\deg R)^n$.

A direct consequence of Lemma A.2 is:

Lemma B.2. *If R is an algebraically stable map, then for any divisor D we have:*

$$\deg((R^n)^* D) = (\deg R)^n \cdot \deg D \tag{B.1}$$

B.2. **Green potential.**

Theorem B.3 (see [Si],Thm 1.6.1)**.** *Let $R : \mathbb{CP}^m \to \mathbb{CP}^m$ be an algebraically stable rational map of degree d. Then the limit*

$$G = \lim_{n\to\infty} \frac{1}{d^n} \log \|\hat{R}^n\|$$

exists in $L^1_{\text{loc}}(\mathbb{C}^3)$ and determines a plurisubharmonic function. This function satisfies the following equivariance properties:

$$G(\lambda z) = G(z) + \log|\lambda|, \quad \lambda \in \mathbb{C}^*, \tag{B.2}$$

$$G \circ \hat{R} = dG.$$

It is called the *Green potential* of R.

B.3. **Green current.** Applying $\frac{i}{\pi}\partial\bar{\partial}$ to the Green potential, we obtain:

Theorem B.4 (see [Si],Thm 1.6.1)**.** *Let $R : \mathbb{CP}^m \to \mathbb{CP}^m$ be an algebraically stable rational map of degree d. Then $S = \pi_*(\frac{i}{\pi}\partial\bar{\partial}G)$ is a closed positive (1,1)-current on $\mathbb{CP}^2$ satisfying the equivariance relation: $R^*S = d \cdot S$.*

The current S is called the *Green current* of R.

The set of *nice* [7] points for an algebraically stable rational map $R : \mathbb{CP}^m \to \mathbb{CP}^m$ is:

$$N := \left\{ \begin{array}{cc} x \in \mathbb{CP}^m : & \text{there exits neighborhoods } U \text{ of } x \text{ and } V \text{ of } I(R) \\ & \text{so that } f^n(U) \cap V = \emptyset \text{ for every } n \in \mathbb{N}. \end{array} \right\}$$

The nice points form an open subset of $\mathbb{CP}^m$.

One primary interest in the Green current S is the following connection between its support $\operatorname{supp} S$ and the Julia set J_R. (See §4 for the definitions of the Fatou and Julia sets.) Note that $\operatorname{supp} S$ is closed and backwards invariant, $R^{-1} \operatorname{supp} S \subset \operatorname{supp} S$, since $R^*S = d \cdot S$.

Theorem B.5 (see [Si],Thm 1.6.5)**.** *Let $f : \mathbb{CP}^m \to \mathbb{CP}^m$ be an algebraically stable rational map. Then:*

$$J_R \cap N \subset \operatorname{supp} S \subset J_R.$$

An algebraically stable rational map $R : \mathbb{CP}^2 \to \mathbb{CP}^2$ for which $\operatorname{supp} S \subsetneq J_R$ is given in [FS1, Example 2.1].

Appendix C. Open Problems

Problem C.1 (**Existence of Fisher and Lee-Yang-Fisher distributions**). Consider a sequence of graphs Γ_n for which van Hove's Theorem [vH, R2] justifies existence of the limiting free energy (2.3). Under what circumstances does the limit (2.5) exist? As explained in Proposition 2.1, this would justify consideration of the limiting distributions of Lee-Yang-Fisher zeros for more general lattices.

For $\mathbb{Z}^d$, existence of the limit (2.5) seems to be open if $d \geq 2$.

Problem C.2 (**Geometric properties of the Lee-Yang-Fisher current**). The theory of geometric currents has become increasingly useful in complex dynamics, see [RS, BLS, Du, Di, dT, DDG] as a sample.

The Green current S is strongly laminar in a neighborhood of B. The structure is given by the stable lamination of B (see §3.3) together with transverse measure obtained under holomomy from the Lebesgue measure on B. However, S is not strongly laminar in a neighborhood of the topless Lee-Yang "cylinder" C_1.

One can see this also follows: a disc within the invariant line L_{inv} centered at $L_{\text{inv}} \cap \text{B}$ is within the stable lamination of B. Therefore, an open neighborhood within L_{inv} of $L_{\text{inv}} \cap C_1$ would have to be a leaf of the lamination. However, S restricts to L_{inv} in a highly non-trivial way, coinciding with the measure of maximal entropy for $R|L_{\text{inv}}$. (It is supported on the Julia set shown in Figure 1.3).

[7] They are called *normal* points in [Si], but we prefer *"nice"* to avoid confusion with the notion of normal families.

Does S have a weaker geometric structure? For example, is it non-uniformly laminar [BLS, Du] or woven [Di, dT, DDG]?

Problem C.3 (**Support for the measure of maximal entropy**). What can be said about the support of the measure of maximal entropy ν that was discussed in §4.3? Is the critical fixed point $b_c \in L_{\rm inv}$ within $\operatorname{supp}\nu$? A positive answer to this question is actually equivalent to $C \cap \operatorname{supp}\nu \neq \emptyset$ and also to $\operatorname{supp}\nu \cap L_{\rm inv} = J_{R|L_{\rm inv}}$.

Problem C.4 (**Fatou Set**). In Theorem 4.7 we showed that certain "solid cylinders" are in $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$. Computer experiments suggest a much stronger result:

Conjecture. $\mathcal{W}^s(e) \cup \mathcal{W}^s(e')$ *is the entire Fatou set for* R.

Problem C.5 (**Julia Set**). Proposition 4.2 gives that in a neighborhood of B, J_R is a C^∞ 3-manifold. What can be said about the global topology of J_R?

Remark C.1. Note that each of the above problems C.2 – C.5 has a natural counterpart for $\mathcal{R}$.

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