Conic sections
If we slice a cone with a plane, we obtain a cross-section that is called a conic section. Hence conic sections are planar curves.

When viewed from the transverse end, the projection of the cone on a plane that contains the axis of the cone appears as two straight lines intersecting at the vertex of the cone, each making an acute angle \( \alpha \) with the axis of the cone. When viewed from the transverse end, the slicing plane appears as a straight line. Suppose this straight line, not passing through the vertex of the cone, makes an angle \( \beta \) with the axis of the cone.

\[
0 \leq \beta < \alpha \quad \Rightarrow \quad \text{Resulting cross-section cut by slicing plane, in two parts, is called a hyperbola}
\]

\[
\beta = \alpha \quad \Rightarrow \quad \text{Resulting cross-section cut by slicing plane is a parabola}
\]

\[
\alpha < \beta \leq \frac{\pi}{2} \quad \Rightarrow \quad \text{Resulting cross-section cut by slicing plane is an ellipse}
\]

The case of \( \beta = \frac{\pi}{2} \) actually yields a circle as a special limiting case of an ellipse.

The case of \( \beta = 0 \), where the slicing plane also contains the axis of the cone, yields two intersecting straight lines making the angle \( \alpha \) with the axis of the cone. Hence one can view the transverse-sectional outline of a cone as the limiting, degenerate case of a hyperbola.

The equations giving the algebraic description of the conic sections are, however, more easily derived from the classical definition of the conic sections.

The parabola is defined as the locus of points that are equidistant from a given point called the focus and a given straight line called the directrix. Based on this description, we derive the equation of the parabola as

\[
y^2 = 4px
\]

where we have placed the focus at \((p, 0)\) and the directrix at \(x = -p\). The graph of the parabola is symmetric about its axis, the x-axis, namely \(y = 0\), which passes through the focus and is perpendicular to the directrix. The point \((0, 0)\) is called the vertex of the parabola.

The ellipse is defined as the locus of points from which the sum of the distances to two given points called foci is constant. Let this constant be \(2a\), and let the two foci be placed at \((-c, 0)\) and \((c, 0)\). The equation of the ellipse can then be derived as

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

where \(c^2 = a^2 - b^2\).

The points \((-a, 0)\) and \((a, 0)\) are called vertices, the line segment connecting the vertices is called the focal or major axis of the ellipse. The segment connecting \((-b, 0)\) and \((b, 0)\) is called the minor axis. The point \((0, 0)\), halfway between the foci is called the center.

Note that in the case of \(a = b\), the equation simplifies to
\[ x^2 + y^2 = a^2, \] that of a circle centered at \((0, 0)\), radius \(a\).
Hence the **circle** is a special ellipse.

The **hyperbola** is defined as the locus of points from which the magnitude of the difference of the distances to two given points called **foci** is constant. Let this constant be \(2a\), and let the two **foci** be placed at \((-c, 0)\) and \((c, 0)\). The equation of the hyperbola can then be derived as

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

where \(c^2 = a^2 + b^2\).

The points \((-a, 0)\) and \((a, 0)\) are called **vertices**, the line segment connecting the vertices is called the **focal** or **transverse axis** of the hyperbola. The point \((0, 0)\) halfway between the foci is called the **center**. The hyperbola has **slant asymptotes** \(y = \pm \frac{b}{a} x\).

In the standard form of equation, the vertex of the parabola or the centers of the ellipse and hyperbola are placed at the origin \((0, 0)\). If these were placed at the point \((h, k)\) instead, while the axes retained their orientation, then replacing \(x\) and \(y\) in the standard-form equations by \(x - h\) and \(y - k\) respectively would give us the equations for the shifted conic sections. After all, this is merely the translation of graphs.

R.Tam