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The Curvature Properties in Finsler Geometry

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1 Definitions and Notations

A (positive definite) **Finsler metric** on a manifold M is a C^∞ scalar function $F = F(x, y)$ on $TM \backslash \{0\}$:

- $F(x, y) > 0, \quad y \neq 0.$
- $F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0.$
- $(g_{ij}(x, y))$ is positive definite ,

where $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$.

The Fundamental Form: $g_y : T_xM \times T_xM \to R :$

$$g_y(u, v) = g_{ij}(x, y)u^i u^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x, v = v^j \frac{\partial}{\partial x^j}|_x$.

By the homogeneity of F, $F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}$.

Remark. Special Finsler metrics

- Riemann metric: $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$ (g_{ij} are independent of y)
- Minkowski metric: $F(x, y) = \sqrt{g_{ij}(y)y^i y^j}$ (F is independent of x)
- Randers metric (G. Randers, 1941): $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha(x) := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$ for any $x \in M$.

Geodesic Equation:

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}.$$

(geodesic coefficients of F)

Riemann Curvature $\mathbf{R}_y : T_xM \to T_xM$,

$$\mathbf{R}_y(u) := R^i{}_k u^k \frac{\partial}{\partial x^i}|_x, \quad u = u^i \frac{\partial}{\partial x^i}|_x,$$

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Flag Curvature:

$$\mathbf{K}(P, y) := \frac{g_y(\mathbf{R}_y(u), u)}{g_y(y, y) g_y(u, u) - [g_y(u, y)]^2},$$

where $P := span\{y, u\} \subset T_xM$.

F is Riemannian: $\mathbf{K}(P, y) = \mathbf{K}(P)$ is independent of y
(the sectional curvature)

F is of scalar flag curvature: $\mathbf{K}(P, y) = \mathbf{K}(x, y)$
(independent of P)

F is of constant flag curvature: $\mathbf{K}(P, y) =$constant

Non-Riemannian Geometric Quantities:

Distortion

$$\tau(x,y) := \ln\left[\frac{\sqrt{det(g_{ij}(x,y))}}{\sigma_F(x)}\right],$$

where

$$\sigma_F(x) := \frac{Vol(B^n)}{Vol\left\{(y^i) \in R^n | F(x,y) < 1\right\}}$$

characterizes the Busemann-Hausdorff volume form.

Cartan torsion

$$\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbf{R}$$

$$\mathbf{C}_y(u,v,w) := C_{ijk}(x,y)u^iv^jw^k$$

where

$$C_{ijk}(x,y) := \frac{1}{4}[F^2]_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}(x,y)}{\partial y^k}.$$

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d] & \mathbf{I} : I_i = g^{jk} C_{ijk} = \tau_{y^i} \arrow[d] & \tau \arrow[l] \arrow[d] \\
\mathbf{L} : L_{ijk} := C_{ijk|m} y^m \arrow[r] & \mathbf{J} : J_i := g^{jk} L_{ijk} = I_{i|m} y^m & \mathbf{S} := \tau_{|m} y^m
\end{tikzcd}$$

where $(g^{ij}(x,y)) := (g_{ij}(x,y))^{-1}$.

L : **Landsberg curvature**
J : **mean Landsberg curvature**
S : **S-curvature** **(Z. Shen, 1997)**

Facts: F **is Riemannian** $\Longleftrightarrow \mathbf{C} = 0 \Longleftrightarrow \mathbf{I} = 0 \Longleftrightarrow \tau = 0$

Berwald curvature

$$\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$$

$$\mathbf{B}_y(u, v, w) := B^i_{jkl}(x, y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$$

where

$$B^i_{jkl}(x, y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

Berwald metric: $\mathbf{B} = 0$.

mean Berwald curvature

$$\mathbf{E}_y : T_xM \times T_xM \to \mathbf{R}$$

$$\mathbf{E}_y(u, v) := E_{jk}(x, y) u^j v^k$$

where

$$E_{jk}(x, y) := \frac{1}{2} B^m_{jkm}(x, y).$$

weak Berwald metric: $\mathbf{E} = 0$

- F is of isotropic S-curvature if there exists a scalar function $c(x)$ on M such that

  $$\mathbf{S}(x,y) = (n+1)c(x)F(x,y).$$

  If $c(x) = \mathit{constant}$, we say that F is of constant S-curvature.

- F is of isotropic mean Berwald curvature if there exists a scalar function $c(x)$ on M such that

  $$\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}\mathbf{h},$$

  h: angular tensor of F

- F is of relatively isotropic Landsberg curvature if there exists a scalar function $c(x)$ on M such that

  $$\mathbf{L} + cF\mathbf{C} = 0.$$

Example 1.1 **FunK metric Θ** ***on a strangly convex domain*** $\Omega \subset \mathrm{R}^n$:

$$x + \frac{y}{\Theta(x,y)} \in \partial\Omega.$$

- ***positively complete***
- $\mathbf{K} = -\frac{1}{4}$
- $\mathbf{S} = \frac{n+1}{2}\Theta$**,** $\quad c = \frac{1}{2}$
- $\mathbf{E} = \frac{n+1}{4}F^{-1}\mathbf{h}$
- $\mathbf{L} + \frac{1}{2}F\mathbf{C} = 0$
- ***the geodesics are straight lines***

Example 1.2 **(Cheng-Shen)** ***Let $F = \alpha + \beta$ be a Randers meric on an n-dimensional manifold M, the following are equivalent***

(a) $\mathbf{L} + c(x)F\mathbf{C} = 0$

(b) $\mathbf{E} = \frac{n+1}{2}c(x)F^{-1}\mathbf{h}$ ***and β is closed***

(c) $\mathbf{S} = (n+1)c(x)F$ ***and β is closed***

(d) $e_{ij} = 2c(x)(a_{ij} - b_i b_j)$ ***and β is closed***

where $e_{ij} := r_{ij} + b_i s_j + b_j s_i$ ***and***

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^i s_{ij}$$

"|": ***horizontal covariant derivative with respect to α***

Projectively Flat Finsler Metrics

Hilbert's Fourth Problem: to characterize the distance functions on an open subset in R^n such that straight lines are the shortest paths.

Projectively flat Finsler metrics: the smooth solutions of Hilbert's Fourth Problem in the regular case

- Finsler metric F is projectively flat if and only if the geodesic coefficients

$$G^i = P(x, y)y^i,$$

$$(P(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0)$$

- (G. Hamel, 1903) A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset R^n$ is locally projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

Facts:

- **projectively flat Finsler metrics must be of scalar flag curvature:**

$$\mathbf{K} = \frac{P^2 - P_{x^m} y^m}{F^2}.$$

- **Beltrami Theorem: a Riemannian metric is projectively flat if and only if it is of constant sectional curvature.**

Beltrami Theorem is no longer true in Finsler geometry

Example 1.3 Shen's fish tank

Put

$$\Omega := \{(x, y, z)|x^2 + y^2 < 1\}, \qquad p = (x, y, z) \in \Omega$$

$$\mathbf{y} = \{u, v, w\} \in T_p\Omega$$

Randers metric

$$F = \alpha + \beta$$

$$\alpha := \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2}$$

$$\beta := -\frac{-yu + xv}{1 - x^2 - y^2}$$

- $\mathbf{K} = 0$
- $\mathbf{S} = 0$
- ***F is not projectively flat (*** β ***is not closed)***

Example 1.4 (Z. Shen) ***Let***

$$\mathbf{x} = (x, y, z) \in \mathbf{B}^3(1) \subset \mathbf{R}^3, \qquad \mathbf{y} = (u, v, w) \in T_{\mathbf{x}}\mathbf{B}^3(1)$$

Let

$$\begin{aligned} A &:= (x^2 + y^2 + z^2)u - 2x(xu + yv + zw) \\ &= (y^2 + z^2 - x^2)u - 2xyv - 2xzw, \\ B &:= 1 - (x^2 + y^2 + z^2)^2, \\ C &:= u^2 + v^2 + w^2. \end{aligned}$$

Define Randers metric $F = \alpha + \beta$ ***on*** $\mathbf{B}^3(1)$ ***by***

$$F := \frac{\sqrt{A^2 + BC}}{B} + \frac{A}{B}$$

(a) F ***is*** **not** ***projectively flat***

(b) $\mathbf{E} = -2xF^{-1}\mathbf{h}$ ***and*** $\mathbf{S} = -4xF \qquad (c(x) = -x)$

(c) F ***is of scalar curvature with flag curvature***

$$\mathbf{K} = -\frac{3u}{F} + x^2 - 2y^2 - 2z^2$$

2 Matsumoto Tensor and Randers Metrics

Matsumoto Tensor $\mathbf{M}_y : T_xM \times T_xM \times T_xM \longrightarrow \mathbf{R}$

$$M_{ijk}(x,y) := C_{ijk} - \frac{1}{n+1}\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\},$$

where $h_{ij} := FF_{y^i y^j} = g_{ij} - F^{-2} g_{ip} y^p g_{jq} y^q$**: angular tensor h**

♣ **(Matsumoto-Hōjō, 1972-1978)**

Finsler manifold $(M,F), \dim M \geq 3$. **Then**

$$\mathbf{M} = 0 \Longleftrightarrow \textbf{F is a Randers metric}$$

♣ **(Z. Shen et al. 2005)**

Finsler manifold $(M, F), \dim M \geq 3$

- **closed**
- $\mathbf{K} = \mathbf{K}(x, y)$
- $\mathbf{K} < 0$

Then F **is a Randers metric**

cf. (Akbar-Zadeh, 1988) Finsler manifold (M, F)

- **closed**
- $\mathbf{K} = constant$
- $\mathbf{K} < 0$

Then F **is a Riemannian**

3 Flag Curvature and Relatively Isotropic Landsberg Curvature

Theorem 3.1 (Cheng-Mo-Shen)

Finsler manifold (M, F) $(\dim M = n)$

- ***scalar flag curvature*** $\mathbf{K} = \mathbf{K}(x, y)$
- ***F is of relatively isotropic mean Landsberg curvature***

$$\mathbf{J} + cF\mathbf{I} = 0, \qquad c = c(x).$$

Then ***the flag curvature*** $\mathbf{K}$ ***and the distorsion*** τ ***satisfy***

$$\frac{n+1}{3}\mathbf{K}_{.l} + \left(\mathbf{K} + c^2 - \frac{c_{x^m} y^m}{F}\right)\tau_{.l} = 0.$$

Further,

(a) If $c = constant$***, then***

$$\mathbf{K} = -c^2 + \sigma(x) e^{-\frac{3\tau}{n+1}}$$

(b) If $\mathbf{K} = \mathbf{K}(x)$ ***is a scalar function on*** M***, then either*** F ***is Riemannian or*** $\mathbf{K}(x) = -c^2 = constant \leq 0.$

Theorem 3.2 (Cheng-Shen)

Randers space $(M, F = \alpha + \beta) \quad (\dim M = n)$

- $\mathbf{K} = \mathbf{K}(x)$
- ***F is of relatively isotropic mean Landsberg curvature***

$$\mathbf{J} + cF\mathbf{I} = 0, \qquad c = c(x).$$

Then $\mathbf{K} = -c^2 = constant \leq 0$ ***and***

- $\mathbf{K} = -c^2 = 0:$ ***F is locally Minkowskian.***
- $\mathbf{K} = -c^2 = -1/4:$ ***after a scaling, F can be expressed in the following form***

$$F = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - <\mathbf{x},\mathbf{y}>^2)}}{1 - |\mathbf{x}|^2} \pm \frac{<\mathbf{x},\mathbf{y}>}{1 - |\mathbf{x}|^2} \pm \frac{<\mathbf{a},\mathbf{y}>}{1 + <\mathbf{a},\mathbf{x}>},$$

$$\mathbf{y} \in T_{\mathbf{x}}\mathbf{R}^n,$$

where $\mathbf{a} \in \mathbf{R}^n$ ***is a constant vector with*** $|\mathbf{a}| < 1.$

Theorem 3.3 (Chern-Shen)

Finsler manifold (M, F) $\quad (\dim M = n)$

- ***scalar flag curvature*** $\mathbf{K} = \mathbf{K}(x, y)$
- ***F is of relatively isotropic Landsberg curvature***

$$\mathbf{L} + cF\mathbf{C} = 0, \qquad c = c(x).$$

Then

(a) If $c = constant$***, then***

$$\mathbf{K} = -c^2 + \sigma(x)e^{-\frac{3\tau}{n+1}}$$

(b) If $n \geq 3$ ***and*** $\mathbf{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$ ***, then*** F ***is a Randers metric***

4 Flag Curvature and Isotropic S-Curvature

Theorem 4.1 (Cheng-Mo-Shen)

Finsler manifold (M, F) $(\dim M = n)$

- ***scalar flag curvature*** $\mathbf{K} = \mathbf{K}(x, y)$
- $\mathbf{S} = (n+1)c(x)F(x, y)$

Then ***there is a scalar function*** $\sigma(x)$ ***on*** M ***such that***

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F(x,y)} + \sigma(x)$$

Further, $c = constant \Longleftrightarrow \mathbf{K} = \mathbf{K}(x)$ ***is a scalar function on*** M***.***

♣ Projectively Flat Randers Metrics with Isotropic S-Curvature

Theorem 4.2 **(Zhongmin Shen)** ***Randers metric*** $F = \alpha + \beta$***,*** $\beta \not\equiv 0$***,*** $\dim M = n:$

- ***F is locally projectively flat***
- ***F is of constant Ricci curvature*** **Ric** $= (n-1)\lambda F^2$

Then

$$\lambda \leq 0$$

- $\lambda = 0$***: F is locally Minkowskian***
- $\lambda = -1/4$***: F can be expressed in the following form***

$$F = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - <\mathbf{x},\mathbf{y}>^2)}}{1-|\mathbf{x}|^2} \pm \frac{<\mathbf{x},\mathbf{y}>}{1-|\mathbf{x}|^2} \pm \frac{<\mathbf{a},\mathbf{y}>}{1+<\mathbf{a},\mathbf{x}>},$$

$$\mathbf{y} \in T_{\mathbf{x}}\mathbf{R}^n,$$

where $\mathbf{a} \in \mathbf{R}^n$ ***is a constant vector with*** $|\mathbf{a}| < 1.$

(a) $\mathbf{K} = -1/4$

(b) $\mathbf{S} = \pm\frac{1}{2}(n+1)F$

Remark:(D. Bao and C. Robles, 2004) If Randers metric F is Einstein with Ric $= (n-1)\mathbf{K}(x)F^2$, then F is of constant S-curvature.

Hence, it is natural to consider projectively flat Randers metrics with isotropic S-curvature.

Theorem 4.3 **(Cheng-Mo-Shen)** ***Randers metric*** $F = \alpha + \beta$, $\dim M = n:$

- ***locally projectively flat (*** α ***is of constant sectional curvature*** μ ***and*** β ***is closed)***
- ***the*** S***-curvature is isotropic,*** $\mathbf{S} = (n+1)c(x)F$

Then F ***can be classified as follows***

(A) ***If*** $\mu + 4c(x)^2 \equiv 0$, ***then*** $c(x) = constant$ ***and*** $K = -c^2 \leq 0$.

$(A1)$ ***if*** $c = 0$, ***then*** F ***is locally Minkowskian ;***

$(A2)$ ***if*** $c \neq 0$, ***then after a scaling,*** F ***is locally isometric to the following Randers metric on the unit ball*** $\mathbf{B}^n \subset \mathbf{R}^n$,

$$F = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - <\mathbf{x},\mathbf{y}>^2)}}{1-|\mathbf{x}|^2} \pm \frac{<\mathbf{x},\mathbf{y}>}{1-|\mathbf{x}|^2} \pm \frac{<\mathbf{a},\mathbf{y}>}{1+<\mathbf{a},\mathbf{x}>},$$

$$\mathbf{y} \in T_{\mathbf{x}}\mathbf{R}^n,$$

where $\mathbf{a} \in \mathbf{R}^n$ ***is a constant vector with*** $|\mathbf{a}| < 1$.

(B) ***If*** $\mu + 4c(x)^2 \neq 0,$ ***then*** F ***is given by***

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2}$$

and the flag curvature of F ***is given by***

$$\begin{aligned} \mathbf{K} &= 3\left\{\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\right\} + \mu \\ &= \frac{3}{4}\left\{\mu + 4c(x)^2\right\}\frac{F(x,-y)}{F(x,y)} + \frac{\mu}{4}. \end{aligned}$$

Further, we can completely determine c(x).

Theorem 4.4 **(Cheng-Mo-Shen)** ***Let*** $S^n = (M, \alpha)$ ***is the standard unit sphere and*** $F = \alpha + \beta$ ***be a Randers metric on*** S^n***.***

- ***F is projectively flat;***
- ***the*** S***-curvature is isotropic,*** $\mathbf{S} = (n+1)c(x)F$***,***

Then

$$c(x) = f(x)/2\sqrt{1 - f(x)^2},$$

and

$$F(x, y) = \alpha(x, y) - \frac{f_{x^k}(x)y^k}{\sqrt{1 - f(x)^2}},$$

where $f(x)$ ***is an eigenfunction of*** S^n ***corresponding to the first eigenvalue. Moreover,***

(a) $\delta := \sqrt{|\nabla f|^2_\alpha(x) + f(x)^2} < 1$ ***is a constant and***

$$\frac{2 - \delta}{2(1 + \delta)} \leq \mathbf{K} \leq \frac{2 + \delta}{2(1 - \delta)}.$$

(b) The geodesics of F ***are the great circles on*** S^n ***with*** F***-length*** 2π***.***

♣ Randers Metrics of Scalar Flag Curvature with Isotropic S-Curvature

Technique: **The Solution of Zermelo Navigation Problem**

Finsler metric F is Randers metric

$$F = \alpha + \beta, \quad \alpha = \sqrt{a_{ij}(x)y^i y^j}, \quad \beta = b_i(x)y^i$$

if and only if there exists a Riemannian manifold (M, h) with $h(x, y) = \sqrt{h_{ij}(x)y^i y^j}$ and a vector field on M

$$W = W^i(x)\frac{\partial}{\partial x^i}, \quad h(x, -W) < 1,$$

such that F is the unique solution to the following equation

$$h(x, \frac{y}{F} - W_x) = 1.$$

$$a_{ij} = \frac{(1 - \|W\|^2)h_{ij} + W_i W_j}{(1 - \|W\|^2)^2}, \quad b_i = -\frac{W_i}{1 - \|W\|^2}$$

$$\|W\|^2 := h_{ij}W^i W^j (= \|\beta\|_\alpha^2), \quad W_i := h_{ij}W^j.$$

Theorem 4.5 (Bao-Robles) ***Let*** $F = \alpha + \beta$ ***be a Randers metric which is the solution of Zermelo's problem of navigation on the Riemannian manifold*** (M, h) ***under the external influence*** W ***as follows***

$$h(x, \frac{y}{F} - W_x) = 1.$$

Then F ***is of constant flag curvature*** $\mathbf{K}$ ***if and only if there is a constant*** σ ***such that***

(i) h ***is of constant sectional curvature*** $\mathbf{K} + \sigma^2$***;***

(ii) W ***is an infinitesimal homothety of*** h***, namely***

$$(\mathcal{L}_W h)_{ik} = W_{i;k} + W_{k;i} = -4\sigma h_{ik}.$$

$$(\iff \mathbf{S} = (n+1)\sigma F)$$

Remark: Bao-Robles-Shen (2004) **have given the complete classification of Randers metrics of constant flag curvature.**

Remark:

- $\{$Projectively flat Finsler metrics$\}$
  $\subset \{$Finsler metrics of scalar flag curvature$\}$
- $\{$Finsler metrics of constant flag curvature$\}$
  $\subset \{$Finsler metrics of scalar flag curvature$\}$
- Randers metrics of constant flag curvature must have constant S-curvature

Then, it is natural to classify Randers metrics of scalar flag curvature with isotropic S-curvature

Theorem 4.6 (Cheng-Shen) *Let* $F = \alpha + \beta$ *be a Randers metric on* n*-dimensional* $(n \geq 3)$ *manifold* M *defined by*

$$h(x, \frac{y}{F} - W_x) = 1.$$

Then

(a) F *is of* scalar flag curvature $\mathbf{K} = K(x, y)$ and

(b) F *is of* isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$

if and only if *at any point, there is a local coordinate system in which* h*,* c *and* W *are given by*

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2},$$

$$c = \frac{\delta + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}},$$

$$W = -2\{(\delta\sqrt{1 + \mu|x|^2} + \langle a, x\rangle)x - \frac{|x|^2 a}{\sqrt{1 + \mu|x|^2} + 1}\}$$
$$+ xQ + b + \mu\langle b, x\rangle x,$$

where δ, μ *are constants,* $Q = (q_j{}^i)$ *is an anti-symmetric matrix and* $a, b \in \mathrm{R}^n$ *are constant vectors.*

Example 4.1 (Cheng-Shen) ***Let*** $\mu = 0, \delta = 0, Q = 0$ ***and*** $b = 0$***. We get***

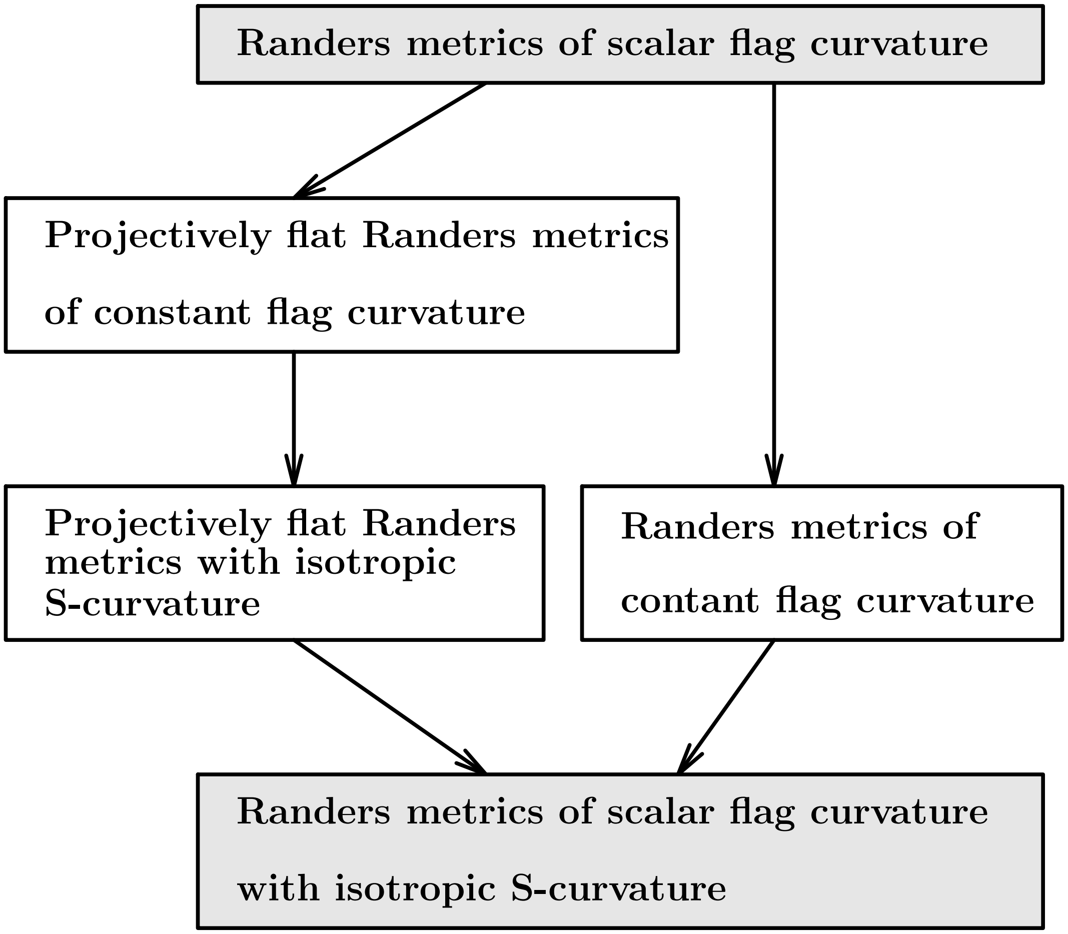
$$h = |y|, \qquad W = -2\langle a, x\rangle x + |x|^2 a, \qquad c = \langle a, x\rangle.$$

The Randers metric $F = \alpha + \beta$ ***is given by***

$$\begin{aligned} F \;&=\; \frac{\sqrt{(1 - |a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1 - |a|^2|x|^4} \\ &\quad - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1 - |a|^2|x|^4}. \end{aligned}$$

The above defined Randers metric F ***is of isotropic S-curvature and scalar flag curvature, i.e.,***

$$\mathbf{S} = (n+1)\langle a, x\rangle \; F, \qquad \mathbf{K} = \frac{3\langle a, y\rangle}{F} + 3\langle a, x\rangle^2 - 2|a|^2|x|^2.$$


Randers metrics of scalar flag curvature
Projectively flat Randers metrics of constant flag curvature
Projectively flat Randers metrics with isotropic S-curvature
Randers metrics of contant flag curvature
Randers metrics of scalar flag curvature with isotropic S-curvature

5 Projectively Flat Finsler Metrics with Isotropic S-Curvature

Theorem 5.1 (Cheng-Shen) *F is a Finsler metric on an open subset $\Omega \subset \mathbf{R}^n$:*

- *locally projectively flat*
- $\mathbf{S} = (n+1)c(x)F$

Then

(a) If $\mathrm{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$ on Ω, then F is a projectively flat Randers metric $F = \alpha + \beta$ with isotropic S-curvature $\mathbf{S} = (n+1)cF$

(b) If $\mathrm{K} \equiv -c^2 + \frac{c_{x^m}y^m}{F}$ on Ω, then $c = constant$ and F is either locally Minkowskian ($c = 0$) or, up to a scaling, locally isometric to the metric

$$\Theta_a := \Theta(x,y) + \frac{< a, y >}{1+ < a, x >} \quad (c = \frac{1}{2})$$

or its reverse

$$\bar{\Theta}_a := \Theta(x,-y) - \frac{< a, y >}{1+ < a, x >} \quad (c = -\frac{1}{2}),$$

where $a \in \mathbf{R}^n$ is a constant vector and $\Theta(x,y)$ is Funk metric on Ω.

Thank you very much for your attention!