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On isometries of Finsler manifolds

László Kozma (University of Debrecen, Hungary)

- – Finsler metrics, examples
- – isometries of Finsler manifolds
- – the group of isometries
- – characterizations of isometries with area and angle
- – Finsler manifolds with many isometries
- – Weinstein theorem for Finsler manifolds

The notion of a Finsler metric

Approach I: $\forall p \in M \quad L_p : T_p M \longrightarrow \mathbb{R}^+$ norm

- $L_p(u) \geq 0 \quad = 0 \iff u = 0$
- $L_p(\lambda(u)) = \lambda L_p(u) \quad \lambda > 0$ positively homogeneous
- $L_p(u + v) \leq L_p(u) + L_p(v)$ convexity
- $L^2 : TM \setminus \{0\} \longrightarrow \mathbb{R}^+$ is of class C^2
- $L_p(-u) = L_p(u)$ symmetrical/ reversible

indicatrix: $\mathcal{I}_p = \{u \in T_p M \mid L_p(u) = 1\}$

Approach II: variational problem

$$\int_a^b L(x(t), \dot{x}(t)) dt \longrightarrow \text{Euler-Lagrange equations}$$

. \uparrow positively homogenous

Riemannian case: $L(x, \dot{x}) = \sqrt{g_{ij}(x) \dot{x}^i \dot{x}^j}$

Finslerian case: $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}$

$g(x, y)$: Riemannian metric in the Finsler vector bundle VTM

Approach III: $d : M \times M \longrightarrow R^+$ is a metric

$v \in T_p M; c : [0, 1] \longrightarrow M$ with $c(0) = p, \dot{c}(0) = v$

$$L_p(v) = \lim_{t \rightarrow 0} \frac{d(p, c(t))}{t}$$

Example 1: Funk metric

$\Omega \subset R^n$ strictly convex

$$d(p, q) = \ln \frac{|z - p|}{|z - q|}$$

$$p + \frac{y}{L(y)} \in \partial\Omega$$

$$B^n = \Omega; \quad L(y) = \frac{\sqrt{|y|^2 - (|p|^2|y|^2 - (p, y)^2)} + (p, y)}{1 - |p|^2}$$

— projectively flat

— constant negative curvature $-1/4$

— non-reversible

— Randers metric

Example 2: Hilbert metric

$$\tilde{d}(p, q) = \frac{1}{2}(d(p, q) + d(q, p)) = \frac{1}{2} \left| \ln \left(\frac{|z - p|}{|z - q|} : \frac{|v - p|}{|v - q|} \right) \right|$$

Example 3: Katok's example (1973), W. Ziller (1982)

S^2 ; standard Riemannian metric α

Φ_t : one parameter group of rotations

leaving the north & south poles invariant

X : Killing vector field

β : Killing form

$$L_\varepsilon(x, y) = \alpha(x, y) + \varepsilon\beta(x, y)$$

Theorem: For any irrational ε a curve c is a closed geodesic of L_ε if and only if c is a closed geodesic of α and invariant with respect to Φ_t .

Properties:

- the length of the two closed geodesics: $\frac{2\pi}{1+\varepsilon}$; $\frac{2\pi}{1-\varepsilon}$
- L_ε is a Finsler metric $\iff |\varepsilon| < 1$

Isometries of Finsler manifolds

(M, L) : Finsler manifold

d : the induced distance function, not necessarily reversible

The length of a curve in (M, L) is given as usual:

$$\ell(c) = \int_0^1 L(\dot{c}) dt.$$

The induced distance d between $x, y \in M$ can be defined by taking the infimum of the length of all curves joining x to y :

$$d(x, y) = \inf\{\ell(c) \mid c(0) = x, c(1) = y\}$$

1. an isometry: a diffeomorphism $\phi : M \longrightarrow M$ of M onto itself which preserves L :

$$L(d\phi(u)) = L(u) \quad \forall u \in TM$$

2. an isometry: a mapping $\phi : M \longrightarrow M$ of M onto itself which preserves the distance between each pair of points:

$$d(\phi(x), \phi(y)) = d(x, y) \quad \forall x, y \in M$$

[Deng, Shaoqiang and Hou, Zixin: The group of isometries of a Finsler space. Pacific J. Math. 207 (2002), no. 1, 149–155] generalizes the Myers-Steenrod theorem in Riemannian geometry:

the two definitions are equivalent.

Theorem. Let $x \in M$ and $B_x(r)$ be a tangent ball of $T_x(M)$ such that \exp_x is a C^1 diffeomorphism from $B_x(r)$ onto $\mathcal{B}_x^+(r)$. For $A, B \in B_x(r)$, $A \neq B$, let $a = \exp_x A$, $b = \exp_x B$. Then

$$\frac{L(x, A - B)}{d(a, b)} \longrightarrow 1$$

as $(A, B) \longrightarrow (0, 0)$.

Theorem. Let $\|\cdot\|_1, \|\cdot\|_2$ be two Minkowski norms on R^n . Let ϕ be a mapping of R^n into itself such that $\|\phi(A) - \phi(B)\|_2 = \|A - B\|_1, \forall A, B \in R^n$. Then ϕ is a diffeomorphism.

Corollary. Let (M, L) be a Finsler space and ϕ be a distance-preserving mapping of M onto itself. Then ϕ is a diffeomorphism.

Theorem. [Deng, Hou, 2002] The group of isometries $I(M)$ is a Lie transformation group. The isotropy subgroup $I_x(M)$ is compact.

Area in Minkowski spaces

(\mathbb{R}^n, L) : Minkowski space

$\mathcal{B} = \{v \in \mathbb{R}^n : L(v) < 1\}$: Minkowski ball

Minkowski measure of $D \subset \mathbb{R}^n$:

$$\|D\|_M = \frac{\pi \|D\|_E}{\|\mathcal{B}\|_E}$$

independent of $\|\cdot\|_E$

Angles in Finsler geometry

Finsler angle of Finsler vectors; $U, V \in V_u TM$:

$$\sphericalangle_F(U, V) = \arccos \frac{g_u(U, V)}{\sqrt{g_u(U, U)} \sqrt{g_u(V, V)}}$$

Minkowski angle of tangent vectors, rays in the tangent spaces

u, v : non-parallel vectors in $T_x M$;

Σ : generated linear space by u, v ;

$\mathcal{B}^2 = \Sigma \cap \mathcal{B}$; $D = \text{conv}(u, v) \cap \mathcal{B}^2$

$$\sphericalangle_M(u, v) = \epsilon 2 \|D\|_M, \quad \epsilon = \pm 1$$

Properties: additive, symmetric; the measure of straight angle is π iff L is absolutely homogeneous (reversible).

Observation. $\phi : (M, L_1) \longrightarrow (\bar{M}, L_2)$ is an isometry if and only for indicatrices

$$d\phi(\mathcal{I}_p) = \bar{\mathcal{I}}_{\phi(p)} \quad \forall p \in M.$$

$$L_2(d\phi(u)) = L_2(L_1(u)d\phi\left(\frac{u}{L_1(u)}\right)) = L_1(u)L_2\left(d\phi\left(\frac{u}{L_1(u)}\right)\right) = L_1(u).$$

Theorem. [Tamássy, 2007)]

A diffeomorphism $\phi : (M, L_1) \longrightarrow (\bar{M}, L_2)$ is an isometry if and only if $d\phi$ preserves the 2-dimensional area and the Minkowski angle.

Proof. Necessity: $d\phi$ is linear \Rightarrow preserves the ratio of areas :

$$\|d\phi(D)\|_{\bar{M}} = \frac{\pi \|d\phi(D)\|_E}{\|\bar{\mathcal{I}}^2\|_E} = \frac{\pi \|D\|_E}{\|\mathcal{I}^2\|_E} = \|D\|_M.$$

Sufficiency. Suppose: $\phi : (M, L_1) \longrightarrow (\bar{M}, L_2)$ diffeomorphism; preserves area and angle. Let $\hat{\mathcal{B}}_p = (d\phi)^{-1}(\mathcal{B}_{\phi(p)})$.

If $\hat{\mathcal{I}}_p \neq \mathcal{I}_p$, then there are two nearby rays u, v such that

$$\text{conv}(u, v) \cap \mathcal{B}_p \subset \text{conv}(u, v) \cap \hat{\mathcal{B}}_p,$$

however

$$\|\text{conv}(u, v) \cap \mathcal{B}_p\|_M \stackrel{\text{angle}}{=} \|\text{conv}(d\phi(u), d\phi(v)) \cap \mathcal{B}_{\phi(p)}\|_{\bar{M}} \stackrel{\text{area}}{=} \|\text{conv}(u, v) \cap \hat{\mathcal{B}}_p\|_M$$

Remark: In this case the Finsler angle is preserved, too.

H. C. Wang, J. London Math. Soc. **22** (1947):

$n \neq 4, \quad \dim I^F(M) > \frac{1}{2}n(n-1) + 1 \implies (M, L) \text{ is Riemannian}$

Ku Chao-Hao, Sci. Records N.S. **1** (1957), 215–218.

A. I. Egorov, Gos. Ped. Inst. Ucen. Zap. (1974), 17–21.

There exist non-Riemannian Finsler spaces with

$$\dim I^F(M) = \frac{1}{2}n(n-1) + 1.$$

[Szabo, Z. I. Generalized spaces with many isometries. Geom.Dedicata 11 (1981), no. 3, 369-383.]:

Study of all the non-Riemannian Finsler spaces having a group of motions of the largest order.

Theorem 1. If (M, L) is a non-Riemannian Finsler space of dimension $n > 4$ and its group of motions $I(M)$ is of order $n(n-1)/2 + 1$, it must be of one of the following types:

(1) (M, L) is a symmetric Berwald space which is the non-Riemannian Cartesian product of Riemannian spaces U [resp. V], where $U = R, S^1$ and $V = R^{n-1}, S^{n-1}, H^{n-1}, P^{n-1}(R)$,

(2) (M, L) is a BLF^n -space.

Theorem 2. Every BLF^n space ($n \geq 2$) is a non-Berwaldian Wagner space which is conformal to a Minkowski space.

Theorem 3:. A BLF^n -space ($n \geq 2$) is projectively flat if and only if it is an $HBLF^n$ -space, and all these spaces are of non-constant scalar curvature.

H^n : hyperbolic space

$G = \{\text{isometries of } H^n \text{ leaving } S \text{ and } S^* \text{ invariantly}\}$

G_p^0 isotropy group at $p \in H^n$

$r : (0, 2\pi] \longrightarrow \mathbb{R}$

$(\varphi, r(\varphi))$ indicatrix of a Minkowski (non-Euclidean) norm

g^* Riemannian metric tensor of H^n

$$\|X\| = \sqrt{g^*(X, X)}$$

$$L(X) = r\left(\arctan \frac{g^*(N, X)}{\|X - g^*(N, X)N\|}\right) \|X\|$$

Alan Weinstein (1968):

Let f be an isometry of a compact oriented Riemannian manifold M . Suppose that M has positive sectional curvature and that f preserves the orientation of M if the dimension is even, and reverses if it is odd. Then f has a fixed point: $f(p) = p$.

Weinstein's Theorem for Finsler manifolds: (Kozma & Peter, 2006)

Let f be an isometry of a compact oriented positively homogeneous Finsler manifold M of dimension n . If M has positive flag curvature and f preserves the orientation of M for n even and reverses the orientation of M for n odd, then f has a fixed point.

flag curvature:

$$K(y, V) = \frac{g_y(R(V, y)y, V)}{g_y(y, y)g_y(V, V) - g_y^2(y, V)}$$

second variation formula:

Consider now the variation of σ given by

$$\Sigma : (-\epsilon, \epsilon) \times [0, \ell] \rightarrow M$$

$$\begin{aligned} \frac{d^2 \ell_\Sigma}{ds^2}(0) &= \int_0^\ell \{g_{\dot{\sigma}}(\nabla_{\dot{\sigma}} U, \nabla_{\dot{\sigma}} U) - g_{\dot{\sigma}}(R_{\dot{\sigma}}(U), U)\} dt \\ &\quad + g_{\dot{\sigma}(\ell)}(\kappa_\ell(0), \dot{\sigma}(\ell)) - g_{\dot{\sigma}(0)}(\kappa_0(0), \dot{\sigma}(0)) \\ &\quad + \mathbf{T}_{\dot{\sigma}(0)}(U(0)) - \mathbf{T}_{\dot{\sigma}(\ell)}(U(\ell)) \end{aligned}$$

where $T = \dot{\sigma}$ and U are the tangential and transversal vector fields, resp; of the variation Σ .

Proof:

Step 1:

Suppose that the isometry f has no fixed points:

$f(x) \neq x$ for all $x \in M$.

Since the manifold M is compact, the function $h : M \rightarrow \mathbb{R}$, given by $h(x) = d(x, f(x))$ attains its minimum at a point $x \in M$:
 $h(x) > 0$.

The completeness of the manifold M implies that there exists a minimizing normalized geodesic $\sigma : [0, \ell]$ joining x and $f(x)$.

Show that the curves formed by σ and $f \circ \sigma$ form a geodesic.

Then $df_x(\sigma'(0)) = \sigma'(\ell)$.

Step 2:

Find a unit parallel vector field $E(t)$ which is $g_{\dot{\sigma}(t)}$ -orthogonal complement of $\dot{\sigma}(t)$.

Then $df_x(E(0)) = E(\ell)$.

Step 3:

Construct a variation Σ of σ given by

$$\begin{aligned}\Sigma &: (-\epsilon, \epsilon) \times [0, \ell] \rightarrow M \\ \Sigma(s, t) &= \exp_{\sigma(t)}(sE(t)), \quad s \in (-\epsilon, \epsilon), \quad t \in [0, \ell].\end{aligned}$$

Then

$$U(t) = \frac{\partial}{\partial s} \exp_{\sigma(t)}(sE(t))|_{s=0} = E(t),$$

so the transversal vector of the variation Σ is parallel transported along σ .

Step 4:

The second variation formula reduces to:

$$\frac{d^2 \ell_{\Sigma}}{ds^2}(0) = - \int_0^{\ell} g_{\dot{\sigma}}(R(U, \dot{\sigma})\dot{\sigma}, U) dt < 0,$$

which contradicts the minimality of the curve σ , which joins x and $f(x)$.

Therefore $d(x, f(x)) > 0$ is impossible.

Killing vector field $X \in \mathfrak{X}(M)$ of (M, L) : if any local one-parameter transformation group of X consists of local isometries.

zeros of $X \iff$ fixed points of isometries

Theorem [S. Deng, 2007]

(M, L) : connected, forward complete

$V = \{p \in M \mid X(p) = 0\} = \cup V_i$; V_i are connected components.

- each V_i is a totally geodesic closed submanifold of M ;

codim V_i is even;

- $\forall x \in V_i, y \in V_j, i \neq j$ there is a one-parameter family of geodesics connecting x and y ; $\Rightarrow x$ and y are conjugate points.

- M compact; then for the Euler number :

$$\chi(M) = \sum \chi(V_i)$$

Corollary: the flag curvature is non-positive $\implies V$ is empty or connected.