On isometries of Finsler manifolds

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- Finsler metrics, examples
- isometries of Finsler manifolds
- the group of isometries
- characterizations of isometries with area and angle
- Finsler manifolds with many isometries
- Weinstein theorem for Finsler manifolds
The notion of a Finsler metric

Approach I: \( \forall p \in M \) \( L_p : T_p M \to R^+ \) norm

- \( L_p(u) \geq 0 \) \( = 0 \iff u = 0 \)
- \( L_p(\lambda(u)) = \lambda L_p(u) \) \( \lambda > 0 \) positively homogeneous
- \( L_p(u + v) \leq L_p(u) + L_p(v) \) convexity
- \( L^2 : TM \setminus \{0\} \to R^+ \) is of class \( C^2 \)
- \( L_p(-u) = L_p(u) \) symmetrical/ reversible

indicatrix: \( \mathcal{I}_p = \{ u \in T_p M \mid L_p(u) = 1 \} \)
Approach II: variational problem

\[ \int_{a}^{b} L(x(t), \dot{x}(t))dt \rightarrow \text{Euler-Lagrange equations} \]

\[ \uparrow \text{positively homogenous} \]

Riemannian case: \( L(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^{i}\dot{x}^{j}} \)

Finslerian case: \( g_{ij}(x, y) = \frac{1}{2} \frac{\partial^{2}L^{2}}{\partial y^{i}\partial y^{j}} \)

\( g(x, y) \): Riemannian metric in the Finsler vector bundle \( VTM \)

Approach III: \( d : M \times M \rightarrow R^{+} \) is a metric

\( v \in T_{p}M; c : [0, 1] \rightarrow M \) with \( c(0) = p, \dot{c}(0) = v \)

\[ L_{p}(v) = \lim_{t \rightarrow 0} \frac{d(p, c(t))}{t} \]
Example 1: Funk metric

\[ \Omega \subset \mathbb{R}^n \text{ strictly convex} \]

\[ d(p, q) = \ln \frac{|z - p|}{|z - q|} \]

\[ p + \frac{y}{L(y)} \in \partial \Omega \]

\[ B^n = \Omega; \quad L(y) = \frac{\sqrt{|y|^2 - (|p|^2|y|^2 - (p, y)^2)} + (p, y)}{1 - |p|^2} \]

— projectively flat
— constant negative curvature \(-1/4\)
— non–reversible
— Randers metric

Example 2: Hilbert metric

\[ \check{d}(p, q) = \frac{1}{2} (d(p, q) + d(q, p)) = \frac{1}{2} \ln \left( \frac{|z - p|}{|z - q|} : \frac{|v - p|}{|v - q|} \right) \]
Example 3: Katok’s example (1973), W. Ziller (1982)

$S^2$: standard Riemannian metric $\alpha$

$\Phi_t$: one parameter group of rotations leaving the north & south poles invariant

$X$: Killing vector field

$\beta$: Killing form

$$L_\varepsilon(x, y) = \alpha(x, y) + \varepsilon \beta(x, y)$$

**Theorem:** For any irrational $\varepsilon$ a curve $c$ is a closed geodesic of $L_\varepsilon$ if and only if $c$ is a closed geodesic of $\alpha$ and invariant with respect to $\Phi_t$.

Properties:

– the length of the two closed geodesics: $\frac{2\pi}{1 + \varepsilon}$; $\frac{2\pi}{1 - \varepsilon}$

– $L_\varepsilon$ is a Finsler metric $\iff |\varepsilon| < 1$
Isometries of Finsler manifolds

\((M, L)\) : Finsler manifold

d : the induced distance function, not necessarily reversible

The length of a curve in \((M, L)\) is given as usual:

\[ \ell(c) = \int_0^1 L(\dot{c})\,dt. \]

The induced distance \(d\) between \(x, y \in M\) can be defined by taking the infimum of the length of all curves joining \(x\) to \(y\):

\[ d(x, y) = \inf \{ \ell(c) \mid c(0) = x, c(1) = y \} \]
1. an isometry: a diffeomorphism \( \phi : M \rightarrow M \) of \( M \) onto itself which preserves \( L \):

\[
L(d\phi(u)) = L(u) \quad \forall u \in TM
\]

2. an isometry: a mapping \( \phi : M \rightarrow M \) of \( M \) onto itself which preserves the distance between each pair of points:

\[
d(\phi(x), \phi(x)) = d(x, y) \quad \forall x, y \in M
\]


*the two definitions are equivalent.*
Theorem. Let $x \in M$ and $B_x(r)$ be a tangent ball of $T_x(M)$ such that $\exp_x$ is a $C^1$ diffeomorphism from $B_x(r)$ onto $B_x^+(r)$. For $A, B \in B_x(r)$, $A \neq B$, let $a = \exp_x A, b = \exp_x B$. Then

$$\frac{L(x, A - B)}{d(a, b)} \to 1$$

as $(A, B) \to (0, 0)$.

Theorem. Let $\| \cdot \|_1, \| \cdot \|_2$ be two Minkowski norms on $R^n$. Let $\phi$ be a mapping of $R^n$ into itself such that $\|\phi(A) - \phi(B)\|_2 = \|A - B\|_1, \forall A, B \in R^n$. Then $\phi$ is a diffeomorphism.

Corollary. Let $(M, L)$ be a Finsler space and $\phi$ be a distance-preserving mapping of $M$ onto itself. Then $\phi$ is a diffeomorphism.

Theorem. [Deng, Hou, 2002] The group of isometries $I(M)$ is a Lie transformation group. The isotropy subgroup $I_x(M)$ is compact.
Area in Minkowski spaces

$(\mathbb{R}^n, L)$: Minkowski space
$\mathcal{B} = \{ v \in \mathbb{R}^n : L(v) < 1 \}$: Minkowski ball

Minkowski measure of $D \subset \mathbb{R}^n$:

$$\|D\|_M = \frac{\pi \|D\|_E}{\|\mathcal{B}\|_E}$$

independent of $\| \cdot \|_E$
Angles in Finsler geometry

**Finsler angle** of Finsler vectors; \( U, V \in V_uTM \):

\[
\angle_F(U, V) = \arccos \frac{g_u(U, V)}{\sqrt{g_u(U, U)} \sqrt{g_u(V, V)}}
\]

**Minkowski angle** of tangent vectors, rays in the tangent spaces \( u, v \): non-parallel vectors in \( T_xM \);
\( \Sigma \): generated linear space by \( u, v \);
\( \mathcal{B}^2 = \Sigma \cap \mathcal{B} \); \( D = \text{conv}(u,v) \cap \mathcal{B}^2 \)

\[
\angle_M(u, v) = \epsilon 2\| D \|_M, \quad \epsilon = \pm 1
\]

Properties: additive, symmetric; the measure of straight angle is \( \pi \) iff \( L \) is absolutely homogeneous (reversible).
Observation. \( \phi : (M, L_1) \longrightarrow (\bar{M}, L_2) \) is an isometry if and only for indicatrices

\[ d\phi(I_p) = \bar{I}_{\phi(p)} \quad \forall p \in M. \]

\[ L_2(d\phi(u)) = L_2(L_1(u)d\phi(\frac{u}{L_1(u)})) = L_1(u)L_2(d\phi(\frac{u}{L_1(u)})) = L_1(u). \]

Theorem. [Tamássy, 2007)]
A diffeomorphism \( \phi : (M, L_1) \longrightarrow (\bar{M}, L_2) \) is an isometry if and only if \( d\phi \) preserves the 2-dimensional area and the Minkowski angle.

Proof. Necessity: \( d\phi \) is linear \( \Rightarrow \) preserves the ratio of areas :

\[ \|d\phi(D)\|_{\bar{M}} = \frac{\pi\|d\phi(D)\|_E}{\|\bar{I}^2\|_E} = \frac{\pi\|D\|_E}{\|I^2\|_E} = \|D\|_M. \]
**Sufficiency.** Suppose: \( \phi : (M, L_1) \longrightarrow (\bar{M}, L_2) \) diffeomorphism; preserves area and angle. Let \( \hat{B}_p = (d\phi)^{-1}(B_{\phi(p)}) \).

If \( \hat{I}_p \neq I_p \), then there are two nearby rays \( u, v \) such that

\[
\text{conv} (u, v) \cap B_p \subset \text{conv} (u, v) \cap \hat{B}_p,
\]

however

\[
\|\text{conv} (u, v) \cap B_p\|_M^{\text{angle}} = \|\text{conv} (d\phi(u), d\phi(v)) \cap B_{\phi(p)}\|_{\bar{M}}^{\text{area}} = \|\text{conv} (u, v) \cap \hat{B}_p\|_M
\]

Remark: In this case the Finsler angle is preserved, too.

\[ n \neq 4, \quad \dim I^F(M) > \frac{1}{2}n(n - 1) + 1 \implies (M, L) \text{ is Riemannian} \]


There exist non-Riemannian Finsler spaces with

\[ \dim I^F(M) = \frac{1}{2}n(n - 1) + 1. \]
Study of all the non-Riemannian Finsler spaces having a group of motions of the largest order.

Theorem 1. If \((M, L)\) is a non-Riemannian Finsler space of dimension \(n > 4\) and its group of motions \(I(M)\) is of order \(n(n - 1)/2 + 1\), it must be of one of the following types:

1. \((M, L)\) is a symmetric Berwald space which is the non-Riemannian Cartesian product of Riemannian spaces \(U \) [resp. \(V\)], where \(U = R, S^{1}\) and \(V = R^{n-1}, S^{n-1}, H^{n-1}, P^{n-1}(R)\),

2. \((M, L)\) is a \(BLF^n\)-space.

Theorem 2. Every \(BLF^n\) space \((n \geq 2)\) is a non-Berwaldian Wagner space which is conformal to a Minkowski space.

Theorem 3:. A \(BLF^n\)-space \((n \geq 2)\) is projectively flat if and only if it is an \(HBLF^n\)-space, and all these spaces are of non-constant scalar curvature.
$H^n$: hyperbolic space

$G = \{ \text{isometries of } H^n \text{ leaving } S \text{ and } S^* \text{ invariantly} \}$

$G^0_p$ isotropy group at $p \in H^n$

$r : (0, 2\pi) \rightarrow \mathbb{R}$

$(\varphi, r(\varphi))$ indicatrix of a Minkowski (non-Euclidean) norm

$g^*$ Riemannian metric tensor of $H^n$

$\|X\| = \sqrt{g^*(X, X)}$

$L(X) = r(\arctan \frac{g^*(N, X)}{\|X - g^*(N, X) N\|}) \|X\|$
Alan Weinstein (1968):

Let \( f \) be a an isometry of a compact oriented Riemannian manifold \( M \). Suppose that \( M \) has positive sectional curvature and that \( f \) preserves the orientation of \( M \) if the dimension is even, and reverses if it is odd. Then \( f \) has a fixed point: \( f(p) = p \).

Weinstein’s Theorem for Finsler manifolds: (Kozma & Peter, 2006)

Let \( f \) be an isometry of a compact oriented positively homogeneous Finsler manifold \( M \) of dimension \( n \). If \( M \) has positive flag curvature and \( f \) preserves the orientation of \( M \) for \( n \) even and reverses the orientation of \( M \) for \( n \) odd, then \( f \) has a fixed point.
flag curvature:

\[ K(y, V) = \frac{g_y(R(V, y)y, V)}{g_y(y, y)g_y(V, V) - g_y^2(y, V)} \]

second variation formula:

Consider now the variation of \( \sigma \) given by

\[ \Sigma: (-\epsilon, \epsilon) \times [0, \ell] \to M \]

\[
\frac{d^2 \ell}{ds^2} \Sigma(0) = \int_0^\ell \left\{ g_{\dot{\sigma}}(\nabla_{\dot{\sigma}} U, \nabla_{\dot{\sigma}} U) - g_{\dot{\sigma}}(R_{\dot{\sigma}}(U), U) \right\} dt \\
+ g_{\dot{\sigma}(\ell)}(\kappa_\ell(0), \dot{\sigma}(\ell)) - g_{\dot{\sigma}(0)}(\kappa_0(0), \dot{\sigma}(0)) \\
+ T_{\dot{\sigma}(0)}(U(0)) - T_{\dot{\sigma}(\ell)}(U(\ell))
\]

where \( T = \dot{\sigma} \) and \( U \) are the tangential and transversal vector fields, resp; of the variation \( \Sigma \).
Proof:

Step 1:
Suppose that the isometry $f$ has no fixed points: $f(x) \neq x$ for all $x \in M$.
Since the manifold $M$ is compact, the function $h : M \to \mathbb{R}$, given by $h(x) = d(x, f(x))$ attains its minimum at a point $x \in M$: $h(x) > 0$.
The completeness of the manifold $M$ implies that there exists a minimizing normalized geodesic $\sigma : [0, \ell]$ joining $x$ and $f(x)$.
Show that the curves formed by $\sigma$ and $f \circ \sigma$ form a geodesic.
Then $df_x(\sigma'(0))) = \sigma'(\ell))$. 
Step 2:

Find a unit parallel vector field $E(t)$ which is $g_{\dot{\sigma}(t)}$-orthogonal complement of $\dot{\sigma}(t)$.
Then $df_x(E(0)) = E(\ell))$.

Step 3:

Construct a variation $\Sigma$ of $\sigma$ given by

$$\Sigma : (-\epsilon, \epsilon) \times [0, \ell] \to M$$

$$\Sigma(s, t) = \exp_{\sigma(t)}(sE(t)), \ s \in (-\epsilon, \epsilon), \ t \in [0, \ell].$$

Then

$$U(t) = \frac{\partial}{\partial s} \exp_{\sigma(t)}(sE(t))|_{s=0} = E(t),$$

so the transversal vector of the variation $\Sigma$ is parallel transported along $\sigma$. 
Step 4:

The second variation formula reduces to:

$$\frac{d^2 \ell \Sigma}{ds^2}(0) = - \int_0^\ell g_{\sigma}(R(U, \dot{\sigma})\dot{\sigma}, U)dt < 0,$$

which contradicts the minimality of the curve $\sigma$, which joins $x$ and $f(x)$.
Therefore $d(x, f(x)) > 0$ is impossible.
**Killing vector field** $X \in \mathfrak{X}(M)$ of $(M, L)$: if any local one-parameter transformation group of $X$ consists of local isometries.

zeros of $X \iff$ fixed points of isometries

**Theorem** [S. Deng, 2007]

$(M, L)$: connected, forward complete

$V = \{ p \in M \mid X(p) = 0 \} = \bigcup V_i$; $V_i$ are connected components.

- each $V_i$ is a totally geodesic closed submanifold of $M$;
- codim $V_i$ is even;
- $\forall x \in V_i, y \in V_j, i \neq j$ there is a one-parameter family of geodesics connecting $x$ and $y$; $\Rightarrow x$ and $y$ are conjugate points.
- $M$ compact; then for the Euler number :

$$\chi(M) = \sum \chi(V_i)$$

Corollary: the flag curvature is non-positive $\iff V$ is empty or connected.