The Complex Deicke and Brickell Theorem and the Monge-Ampere Equation

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1. THE COMPLEX HOMOGENEOUS MONGE-AMPÉRE EQUATION

Notations. We shall adopt the following notations throughout the article:

$$d = \partial + \bar{\partial}, \quad d^c = \sqrt{-1}(\bar{\partial} - \partial)/2$$

We have $dd^c = \sqrt{-1}\partial\bar{\partial}$, known as the Levi operator. We shall write $dz^{\bar{j}}$ for $\overline{dz^j}$ (in the literature many authors write this as $d\bar{z}^j$).

Definition 1.1. *A function $u : M \to \mathbb{R}$ of class $\mathbb{C}^\infty$ on a connected complex manifold M of complex dimension n is said to be a Monge-Ampère function if $(dd^c u)^n \equiv 0$ and that $dd^c u$ is of rank $n-1$ at every point. A Monge-Ampère function is said to be pseudoconvex if $dd^c u \geq 0$. It is said to be a Monge-Ampère exhaustion if every level set $\{u = c\}$ is compact.*

It is clear that the condition that a function be a Monge-Ampère function is invariant by change of coordinates. Define the annihilator of a Monge-Ampère function u by

$$\operatorname{ann}_z dd^c u = \{v \in T_z M \mid \iota_v dd^c u = 0\}$$

where ι_v is the interior product. The condition that $dd^c u$ is of rank $n-1$ means that, at each point $z \in M$ the annihilator $\operatorname{ann}_z dd^c u$ is one (complex) dimension.

Example. The usual Euclidean square norm $||z||^2$ on $\mathbb{C}^n$ is a strictly pseudoconvex (in fact strictly convex) exhaustion. The function $u = \log ||z||^2$ is a pseudoconvex (plurisubharmonic) (in fact convex) function of class $\mathcal{C}^\infty$ on $M = \mathbb{C}^n \setminus \{0\}$. It is well-known that

$$dd^c u = dd^c \log ||z||^2 = [\ ]^* \omega_{\mathrm{FS}}$$

where $[\ ] : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the usual Hopf fibration and ω_{FS} is the Fubini-Study metric on $\mathbb{P}^{n-1}$. We have obviously $\omega_{\mathrm{FS}}^n \equiv 0$ and that ω^{n-1} is a volume element on $\mathbb{P}^{n-1}$. This implies that $u = \log ||z||^2$ is a pseudoconvex Monge-Ampère exhaustion (the level sets are the spheres) on $\mathbb{C}^n \setminus \{0\}$. The differential $[\ ]_* : T(\mathbb{C}^n \setminus \{0\} \to T\mathbb{P}^{n-1}$ is a submersion and the kernel is spanned by the radial vector field $Z = z^i\partial/\partial z^i$. Observe that Z is also the gradient vector field (with respect to the Euclidean metric) of the function $||z||^2$. At each point $z \in \mathbb{C}^n \setminus \{0$ the annihilator is spanned by Z.

Suppose that $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a (complex) Finsler metric (the homogeneity is $\rho(\lambda z) = |\lambda|^2\rho(z), \lambda \in \mathbb{C}$). We assume that ρ is of class $\mathcal{C}^\infty$ on $\mathbb{C}^n \setminus \{0\}$. It is known $\rho = ||z||^2$ if it is smooth at the origin. In any case we have, for $z \neq 0$ and $\lambda \neq 0$

$$\frac{\rho(\lambda z)}{||\lambda z||^2} = \frac{\rho(z)}{||z||^2}$$

for all $\lambda \neq 0$. This means that

$$\rho(z) = \phi([z])||z||^2$$

for $z \neq 0$ and where ϕ is a smooth function on $\mathbb{P}^{n-1}$. Thus

$$\begin{aligned}
&(dd^c \log \rho)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (dd^c \log \phi \circ [\ ])^{n-k} \wedge (dd^c \log ||z||^2)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} [\ ]^* \big((dd^c \log \phi)^{n-k} \wedge \omega_{\mathrm{FS}}^k \big) \\
&= 0.
\end{aligned}$$

In other words, for any Finsler metric ρ *smooth on* $\mathbb{C}^n \setminus \{0\}$, *the function* $u_\rho = \log \rho$ *is a Monge-Ampère exhaustion.* Observe that the annihilator of $dd^c u$ is still spanned by the radial vector field. Thus if ρ is strictly pseudoconvex then $dd^c \log \rho$ is positive definite when restricted to the holomorphic tangent space of a level set $\rho = c$. This is equivalent to the condition that $\omega_{\mathrm{FS}} + dd^c \log \phi$ is positive definite on $\mathbb{CP}^{n-1}$.

On the other hand, there are Monge-Ampère exhaustions which do not come from Finsler metrics. For instance we can take $\rho(z) = ||z||^{2\mu}$ for any positive real number μ. This is not a Finsler metric if $\mu \neq 1$ but $\log \rho = \mu \log ||z||^2$ is a Monge-Ampère exhaustion. We may take $\rho(z) = \phi([z])||z||^{2\mu}$ where ϕ is a smooth positive function on $\mathbb{CP}^{n-1}$. A computation shows that *the Ricci curvature of the Kähler metric* $dd^c\rho$ *vanishes if and only if* $\mu = 1$. As remarked earlier, the Monge-Ampère condition is independent of choices coordinates while the Finsler condition is defined in terms of the standard linear coordinates on $\mathbb{C}^n$. Thus we can take any Finsler metric ρ and any holomorphic automorphism Φ of $\mathbb{C}^n$ then the composite $\rho \circ \Phi$ is still a Monge-Ampère exhaustion but for general Φ the Finsler condition is destroyed.

2. UNIFORMIZATION VIA MONGE-AMPÈRE FUNCTIONS

In the theory of Monge-Ampere equation there is a famous result due to Stoll [10] (see also Burns [4] and Wong [11]) known as the Uniformization Theorem:

Theorem 2.1. *Let M be a connected complex manifold of complex dimension n. Assume that there is a smooth and strictly pseudoconvex exhaustion $\tau : M \to [0, R), R \in \mathbb{R}_{>0} \cup \{\infty\}$ and that $u : M_* = M \setminus \{\tau = 0\}$ is a pseudoconvex Monge-Ampere exhaustion. Then*
(i) $\{\tau = 0\}$ consists of one point $\{o\}$;
(ii) the exponential map

$$\exp_o|_{\mathbb{B}^n(R)} : \mathbb{B}^n(R) \to M$$

is a holomorphic isometry where $\mathbb{B}^n(R)$ is the ball of radius R in the tangent space T_oM equipped with the Euclidean metric, and M is equipped with the metric $dd^c\tau$;
(iii) the pull-back of τ via the exponential map is the standard square norm, i.e., $\tau \circ \exp_o(z) = ||z||^2$.

The assumption that the function τ is smooth on all of M (not merely on M_*) is crucial.
If $M = \mathbb{C}^n$ the preceding theorem asserts that, after a global biholomorphic change of coordinates, a Monge-Ampére exhaustion ρ

satisfying the condition of the theorem is the standard Euclidean square norm.

Question. *What can can be said about a continuous exhaustion (all level sets are compact) $\tau : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ satisfying the following conditions*
(a) $\{t = 0\} = \{0\}$,
(b) τ is smooth and strictly pseudoconvex on $\mathbb{C}^n \setminus \{0\}$,
(c) $u = \log \tau$ is a Monge-Ampére exhaustion on $\mathbb{C}^n \setminus \{0\}$?
In other words, τ satisfies all conditions in Theorem 2.1 on $\mathbb{C}^n$ except that it is only continuous at the origin.

3. THE THEOREMS OF DEICKE AND BRICKELL

Let F be a (real) Finsler metric on $\mathbb{R}^n$ (we take the homogeneity condition to be $F(tx) = tx$ for $t > 0$ so it is the square of the definition of Finsler metric of many other authors). We assume that it is smooth and strictly convex on $\mathbb{R}^n \setminus \{0\}$. Thus

$$g = g_F = g_{ij}(x)\, dx^i \otimes dx^j \tag{3.1}$$

where

$$g_{ij}(x) := \frac{1}{2}\frac{\partial^2 F^2}{\partial x^i \partial x^j}(x)$$

is a well-defined positive definite Riemannian tensor (henceforth referred to as the *fundamental tensor* of F) on $\mathbb{R}^n \setminus \{0\}$. We shall referred to the pair $(\mathbb{R}^n, F)$ is referred to as a *Minkowski space*. It is said to be a *reversible Minkowski space* if F satisfies, in addition $F(-tx) = F(tx), t \geq 0$. The *Cartan tensor* C of a Minkowski space is defined by

$$C := C_{ijk}(x)\, dx^i \otimes dx^j \otimes dx^k,$$

where

$$C_{ijk} := \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}(x) = \frac{1}{4}\frac{\partial^3 F^2}{\partial x^i \partial x^j \partial x^k}(x)$$

and the *mean Cartan tensor* I by

$$\tilde{C} := \tilde{C}_i(x)\; dx^i, \tilde{C}_i(x) := g^{jk}(x) C_{ijk}(x)$$

where $(g^{jk}) := (g_{jk})^{-1}$. It is clear that F is the Euclidean square norm if and only if the

Cartan tensor C vanishes, namely, $C_{ijk} = 0$ for all i, j and k.

The following theorem is due to Diecke [5] (see also Brickell [3]):

Theorem 3.1. *A Minkowski space* $(\mathbb{R}^n, F^2)$ *is the flat Euclidean space* $(\mathbb{R}^n, ||x||^2)$ *if and only if the mean Cartan tensor* $\tilde{C}$ *vanishes.*

The following result is due to Brickell [3]:

Theorem 3.2. *Let* $(\mathbb{R}^n, F)$ *be an* $n(\geq 3)$ *dimensional reversible Minkowski space. If the curvature of* g_F*, as defined in* (3.1)*, on* $\mathbb{R}^n \backslash \{0\}$ *vanishes if and only if* F *is an Euclidean norm.*

We now consider the complex analogue. A complex Minkowski space is a pair $(\mathbb{C}^n, F)$ where F is a complex Finsler metric which is smooth and *strictly pseudoconvex* on $\mathbb{C}^n \setminus \{0\}$. Thus

$$g = g_F = g_{i\bar{j}}(z)\, dz^i \otimes dz^{\bar{j}}, g_{i\bar{j}}(z) := \frac{\partial^2 F^2}{\partial z^i \partial z^{\bar{j}}}(z)$$

is a well-defined positive definite Kählerian tensor (henceforth referred to as the *fundamental tensor* of F) on $\mathbb{C}^n \setminus \{0\}$. We shall referred to the pair $(\mathbb{C}^n, F)$ is referred to as a *complex Minkowski space*. The *complex Cartan tensor* C is defined to be

$$C(z) := C_{i\bar{j}k}(z)\, dz^i \otimes dz^{\bar{j}} \otimes dz^k,$$

where

$$C_{i\bar{j}k}(z) := \frac{\partial g_{i\bar{j}}}{\partial z^k}(z) = \frac{\partial^3 F^2}{\partial z^i \partial z^{\bar{j}} \partial z^k}(z)$$

and the mean Cartan tensor is defined by

$$\tilde{C}(z) := \tilde{C}_k(z) dz^k, \tilde{C}_k(z) := g^{\bar{j}i}(z) C_{i\bar{j}k}(z),$$

where $(g^{\bar{j}i}) = (g_{i\bar{j}})^{-1}$.

The indicatrix I_F of a complex Minkowski space $(\mathbb{C}^n, F)$ is by definition the unit sphere, relative to the fundamental Kähler tensor g_F, in the (holomorphic) tangent space $T_0\mathbb{C}^n$ at the origin:

$$I_F = \{v \in T_0\mathbb{C}^n \mid ||v||_{g_F} = 1\}.$$

Denote by g^I the Riemannian metric on I_F induced by $g = g_F$. We shall examine the curvature of g^I. It is clear that if $F = ||z||^2$

then g_F is the standard Euclidean metric on $T_0\mathbb{C}^n = \mathbb{C}^n = \mathbb{C}^n, I_F$ is the standard unit sphere with g^I the standard metric on the unit sphere. In this case the scalar curvature $r_{g_I} = (2n-1)(2n-2)$.

The fundamental Kähler metric $g = g_F$ also induces a Kähler metric $\tilde{g}_F$ on $\mathbb{C}^{n-1}$ via the Hopf fibration. This is defined as follows. Recall that $F(z) = \phi([z])||z||^2$ where ϕ is defined on $\mathbb{P}^{n-1}$, hence

$$\begin{aligned} dd^c \log F &= dd^c \log ||z||^2 + dd^c \log \phi([z]) \\ &= [\ ]^*(\omega_{FS} + dd^c \log \phi) \end{aligned}$$

and that $\tilde{g}_F = \omega_{FS} + dd^c \log \phi$ is a Kähler metric on $\mathbb{CP}^{n-1}$. Note that in the case $F = ||z||^2$ we have $\gamma = 1$ and so the induced metric is just the Fubini-Study metric.

We have the following analogue of Deicke and Brickell type theorem:

Theorem 3.3. *Let* $(\mathbb{C}^n, F)$ *be a complex Minkowski space. The following statements are equivalent:*
(1) *The complex Minkowski space* $(\mathbb{C}^n, F)$ *is a complex Euclidean space* $(\mathbb{C}^n, ||\zeta||)$.
(2) *The Cartan tensor vanishes identically, namely,* $C_{i\bar{j}k} = 0, \forall i, j, k$.
(3) *The mean Cartan tensor vanishes identically, namely,* $C_i = 0, \forall i$.
(4) *The Ricci curvature Ric g of the Kähler metric* $g = G_{i\bar{j}} d\zeta^i d\zeta^{\bar{j}}, G = F^2$, *on* $\mathbb{C}^n \setminus \{0\}$ *is zero.*
(5) *The scalar curvature* r_g *of the Kähler metric g on* $\mathbb{C}^n \setminus \{0\}$ *is either non-negative* $(r_g \geq 0)$ *(resp. non-positive* $(r_g \leq 0)$*).*
(6) *The scalar curvature* r_{g^I} *of the metric* g^I, *induced by g, on the indicatrix* $S = \{G = 1\}$ *satisfies* $r_{g^I} \geq (2n-1)(2n-2)$ *(resp.* $r_{g^I} \leq (2n-1)(2n-2)$*).*
(7) *The scalar curvature* $r_{\tilde{g}}$ *of the metric* $\tilde{g}$ *on* $\mathbb{CP}^{n-1} \setminus \{0\}$, *induced by the Kähler metric g satisfies* $r_{\tilde{g}} \geq 4n(n-1)$ *(resp.* $r_{\tilde{g}} \leq 4n(n-1)$*).*

Some special cases of the preceding result is known. Yan [] showed that (1), (2) and (3) are equivalent. Y. B. Shen showed that (1) is equivalen to the condition that the Ricci curvature of g_F vanishes. If we replace the strictly pseudoconvex Finsler metric by a strictly pseudoconvex exhaustion such that $u = \log\tau$ is a pseudoconvex Monge-Ampère exhaustion on $\mathbb{C}^n \setminus \{0\}$ then

Theorem 3.4. *Let τ be a continuous exhaustion of $\mathbb{C}^n$. Assume that τ is strictly pseudoconvex on $\mathbb{C}^n \setminus \{0\}$ and that $u = \log\tau$ is a Monge-Ampère function. Then the following statements are equivalent:*
(1) *The space $(\mathbb{C}^n, \tau)$ is, up to global holomorphic change of coordinates, the complex Euclidean space $(\mathbb{C}^n, ||z||^2)$.*
(2) *The Ricci curvature of the metric $dd^c\tau$ vanishes.*

4. PROOF OF THEOREM 3.3

Proof of Theorem 3.3. The proof are separated into three parts.

Part I. The implications (1)$\Longleftrightarrow$(2), (2)$\Longrightarrow$(3) and (4) $\Longrightarrow$(5) are trivial.
Part II. The implication (5) $\Longrightarrow$(2).
Part III. the statements (5), (6) and (7) are equivalent.

Proof of Part III. We begin with a lemma in [11]:

Lemma 4.1. *Let (M, τ) be a strictly parabolic manifold. Denote by Z the complex gradient vector field of τ with respect to the Kähler metric $dd^c\tau$. If Z is holomorphic, then*

$$\tilde{\nabla}_W Z = W, \quad \tilde{\nabla}_{\overline{W}} Z = 0,$$

for all $W \in T^{1,0}M_*$ where ∇ is the Levi-Civita connection of the metric $g = dd^cG$. Consequently, for any constant c the level set $S_c = \{\tau = c\}$ is an umbilical CR hypersurface.

As was shown in section 1, for the case $M = \mathbb{C}^n$ and $\tau = G = F^2$ where F is a strictly pseudoconvex Finsler metric the gradient vector field Z is the radial vector field and is a holomorphic vector field. Thus the indicatrix $I = I_F = \{F = 1\}$ is an umbilical hypersurface. Denote by

• g the Kähler metric dd^cG on $\mathbb{C}^n \backslash \{0\}$ (with connection ∇, curvature R and Laplace-Beltrami operator Δ),

• g^I the Riemannian metric on I^c induced by g (with connection ∇^I, curvature R^I and Laplace-Beltrami operator Δ^I),

• $\tilde{g}$ the Kähler metric on $\mathbb{CP}^{n-1}$ induced by g (with connection $\tilde{\nabla}$, curvature $\tilde{R}$ and Laplace-Beltrami operator $\tilde{\Delta}$).

The almost complex structure on $\mathbb{C}^n$ is denoted by J.

Lemma 4.2. *Let* $(\mathbb{C}^n, F)$ *be a Minkowski space. Then the indicatrix* $I = I_F = \{F = 1\}$ *is an umbilical hypersurface, that is*

$$\nabla_X N = X$$

for all real vector field X *where* N *is the real part of the radial vector field. The vector field* N *is a unit normal to the indicatrix,* JN *is normal to the maximal complex subbundle* $\mathcal{H} = T_{\mathbb{R}}I \cap JT_{\mathbb{R}}I$ *of* $T_{\mathbb{R}}I$*. The integral curves of* JN *are precisely the circles* $L \cap I$ *where* L *is a complex line through the origin. These circles are geodesics in* I*. Furthermore, we have*
(1) $R(X, N)N = R(X, JN)JN = 0$ *for all real vector field* X *on* $\mathbb{C}^n \setminus \{0\}$,
(2) $\nabla^I_X Y = \nabla_X Y + g(X, Y)N$, *for all* $X, Y \in T_{\mathbb{R}}I$,
(3) (Gauss Equation) $g(R^I(X, Y)Y, X) = g(R(X, Y)Y, X) + g(X, X)g(Y, Y) - g(X, Y)^2$ *for all* $X, Y \in T_{\mathbb{R}}I$.

Let $e_2 = Je_1, e_3, ..., e_{2n} \in T_{\mathbb{R}}I$ be orthonormal vector fields. We may complete this to an orthonormal basis of vector fields on $\mathbb{C}^n$ by adding the vector field $e_1 = N$. Then the scalar curvature of the metric g is given by (using Lemma 3.6 (a))

$$r_g = \sum_{i=1}^{2n} \tilde{g}(\tilde{R}(e_i, e_1)e_1, e_i) + \sum_{i,j=2}^{2n} g(\tilde{R}(e_i, e_j)e_j, e_i)$$
$$= \sum_{i,j=2}^{2n} g(\tilde{R}(e_i, e_j)e_j, e_i).$$

From this and the Gauss equation (Corollary 3.1) we get

Corollary 4.1. *The scalar curvature of $\tilde{g}$ and g are related by the formula*

$$r_{g^I} = r_g + (2n-1)(2n-2).$$

Consequently, assertions (5) *and* (6) *of Theorem 3.3 are equivalent.*

Let $[\ ]|_I : I \longrightarrow \mathbb{CP}^{n-1}$ be the restriction of the Hopf fibration.

Lemma 4.3. *The map* $[\ ]|_I : I \to \mathbb{CP}^{n-1}$ *is a totally geodesic Riemannian submersion where the indicatrix is equipped with the Riemannian metric* g^I *and* $\mathbb{CP}^{n-1}$ *is equipped with the Kähler metric* $\tilde{g}$*. The fibers of* $[\ ]|_I$ *are the geodesic circles* $I \cap L$ *where* L *are the complex lines through the origin. The kernel of* $[\ ]_*|_{T_{\mathbb{R}}I}$ *is spanned by* JN *and* $[\ ]_*|_{\mathcal{H}} : \mathcal{H} \to T\mathbb{CP}^{n-1}$ *is an isomorphism.*

A Riemannian submersion is said to totally geodesic if every fiber is totally geodesic. We use O'Neill's formula for submersions (see [8] and also [3], p.241) to relate the curvature of the metric g^I on I and the curvature of the metric $\tilde{g}$ on $\mathbb{CP}^{n-1}$. In the theory of Riemannian submersions there are two operators T and A defined as follows:

$$\begin{cases} T_{E_1}E_2 = (\nabla_{E_1^{\text{ver}}}E_2^{\text{ver}})^{\text{hor}} + (\nabla_{E_1^{\text{ver}}}E_2^{\text{hor}})^{\text{ver}}, \\ A_{E_1}E_2 = (\nabla_{E_1^{\text{hor}}}E_2^{\text{ver}})^{\text{hor}} + (\nabla_{E_1^{\text{hor}}}E_2^{\text{hor}})^{\text{ver}} \end{cases} \tag{4.1}$$

where $E_1, E_2 \in T_{\mathbb{R}}I$. The superscript 'ver' and 'hor' indicate the vertical (the direction of JN) and horizontal component (in the subbundle $\mathcal{H}$) respectively. It is known (see [1] 9.26, p.240) that T vanishes identically if and only if the submersion is totally geodesic. This implies (keeping in mind that $g = g^I$ on $T_{\mathbb{R}}I$) the next two identities (see [1] p. 241 formulas (2.29b) and (2.29c))

$$g(R^I(X, JN)JN, X) = |A_X JN|^2,$$

$$\begin{aligned} &g(R^I(X, Y)Y, X) \\ &= \tilde{g}(\tilde{R}([\,]_*X, [\,]_*Y)[\,]_*Y, [\,]_*X) - 3|A_X Y|_g^2 \end{aligned}$$

for all $X, Y \in \mathcal{H}$. Since $(JN)^{\text{hor}} = 0$ and $X^{\text{ver}} = 0$ for $X \in \mathcal{H}$, the formula (4.1) yields:

$$\begin{aligned} A_X JN &= (\nabla_X JN)^{\text{hor}} \\ &= (J\nabla_X N)^{\text{hor}} = JX = 0, \end{aligned}$$

and

$$\begin{aligned} A_X Y &= (\nabla_X Y)^{\text{ver}} \\ &= g(\nabla_X Y, JN)JN = g(X, JY)JN, \end{aligned}$$

for all $X, Y \in \mathcal{H}$. Thus, the identities reduce to

$$g(R^I(X, JN)JN, X) = 0$$

and

$$g(R^I(X,Y)Y,X) = \tilde{g}(\tilde{R}([\,]_*X,[\,]_*Y)[\,]_*Y,[\,]_*X) - 3|g(X,JY)JN|_g^2$$

for all $X, Y \in \mathcal{H}$. Choosing an orthogonal J-basis $e_2, Je_2, \cdots, e_n, Je_n$ for $\mathcal{H}$ and $e_1 = N$ (so $Je_1 = JN$ is tangent to the fibers of the submersion $[\,]|_I$). A direct computation using the formulas established above yields

Corollary 4.2. *The scalar curvature r_{g^I} of the metric g^I on the indicatrix I and the scalar curvature $r_{\tilde{g}}$ of $\tilde{g}$ on $\mathbb{CP}^{n-1}$ are related by the formula*

$$r_{\tilde{g}} = r_g + 4n(n-1).$$

Consequently, the assertions (6) *and* (7) *of Theorem 3.3 are equivalent.*

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