

# Finsler metrics of Sectional Flag Curvature

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# acknowledgement

I would like to thank my doctoral supervisor professor Yibing Shen who brought me to the world of Finsler geometry and also professor Zhongmin Shen who led me to this problem and gave me the chance to stand here to talk about our recent work.

# Introduction

# Flag Curvature

# variations of arc length

Let  $F = F(x, y)$  be a Finsler metric on a manifold  $M^n$ .

There are various ways to deduce the flag curvature. The most natural way, I think, is the variations of arc length.

Let  $\sigma(t)$  be a smooth curve in  $M$ , then its **arc length** is the integral

$$L(\sigma) = \int_{\sigma} F(\sigma, \dot{\sigma}) dt.$$

**The First variation** of the arc length is

$$DL(\sigma)(V) = - \int_{\sigma} g_{\dot{\sigma}} \left( V, \nabla_{\dot{\sigma}} \frac{\dot{\sigma}}{F(\dot{\sigma})} \right) dt$$

which leads to the **spray coefficients**  $G^i$ , a **covariant derivative**  $\nabla$ , and the **geodesic equation**.

# variations of arc length

In order to understand the behavior of the geodesics, one shall derive the second variation which will define the **index form**

$$I(V, W) = \int \left[ g_{\dot{\sigma}}(\nabla_{\dot{\sigma}} V, \nabla_{\dot{\sigma}} W) - g_{\dot{\sigma}}(R_{\dot{\sigma}}(V), W) \right] dt.$$

It obviously introduces the **Riemann curvature tensor**

$$R_y = R^i_k(x, y) dx^k \otimes \frac{\partial}{\partial x^i}$$

which determines the index form.

The famous Berwald's formula says

$$R^i_k = 2(G^i)_{x^k} - (G^i)_{y^l} (G^l)_{y^k} - y^l (G^i)_{y^k x^l} + 2G^l (G^i)_{y^k y^l}.$$

# flag curvature

Let's consider the transformation  $R_y$ . Noting the symmetricity, it is enough to understand the normalized term on the diagonal

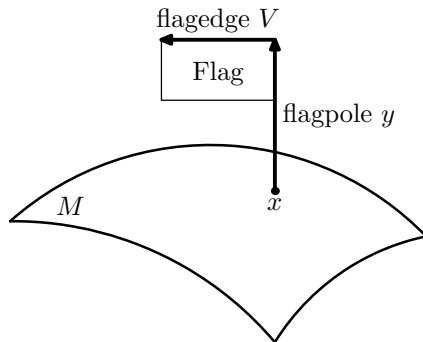
$$K(x, y, V) := \frac{g_y(R_y(V), V)}{g_y(y, y)g_y(V, V) - g_y(y, V)^2},$$

which is called the **flag curvature** of  $(y, P)$  with  $P = \text{span}\{y, V\}$ .

We call  $y$  the **flagpole**,  $V$  the **flagedge** and  $P$  the **section**.

Let's see a picture.

# flag curvature



# flag curvature

It is well known that the curvature of the flag  $(y, P)$  is independent of the choice of the flagedge  $V$ .  
Hence the flag curvature is usually denoted by

$$K(x, y, P) := K(x, y, V).$$

## ANOTHER EXPLANATION

Contrast to the Riemannian case, the section  $P$  can not completely determine the curvature in Finslerian realm. One should pick a flagpole  $y \in P$  to construct a flag  $(P, y)$ , and then deduce the curvature. We got this point of view from professor Zhongmin Shen.

# Sectional Flag Curvature

# special flag curvatures

An important problem in Finsler geometry is to study metrics with special flag curvatures, such as

- ✓ scalar flag curvature:  $K(x, y, P) = K(x, y)$ ;
- ✓ isotropic flag curvature:  $K(x, y, P) = K(x)$ ;
- ✓ constant flag curvature:  $K(x, y, P) = \text{constant}$ .

All the three classes have been studied by many geometers, such as D.Bao, X. Y. Chen, X.H.Mo, C. Robles, Z. Shen and etc. The Schur's lemma tells us that the most interesting metrics in the second class are the Einstein surfaces.

# sectional flag curvature

The missing property is  $K(x, y, P) = K(x, P)$ . That is, the flag curvature is independent of the choice of the flagpole  $y$  in  $P$ .

Professor Zhongmin Shen suggests the following definition.

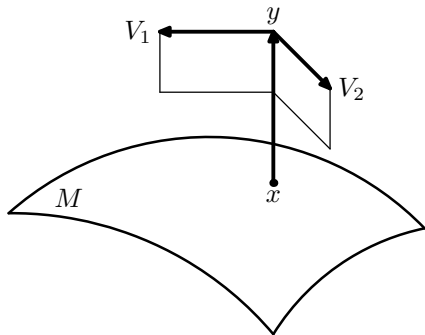
## Definition

A Finsler metric is of **sectional (flag) curvature** iff its flag curvature depends only on the section.

In other words, the flag curvature can be reduced to a function on the Grassmannian bundle of 2-planes, i.e.  $K : G_2(M) \rightarrow \mathbb{R}$ .

We will give some pictures to make this idea clear.

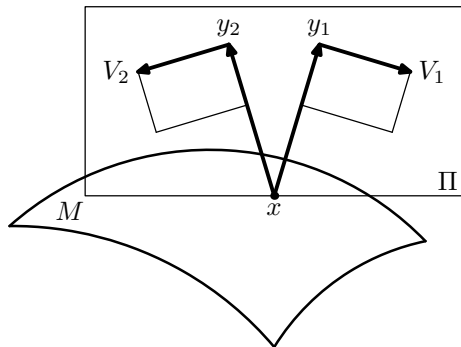
## scalar flag curvature



Scalar Flag Curvature:

$$K(x, y, V_1) = K(x, y, V_2) = K(x, y)$$

## sectional flag curvature



Sectional Flag Curvature:  
 $K(x, y_1, V_1) = K(x, y_2, V_2)$   
 $= K(x, \Pi)$

## some remarks

- ✓ Riemannian metrics are of sectional flag curvature, so the condition "of sectional flag curvature" is non-Riemannian;
- ✓ metrics with constant flag curvature are other trivial examples;
- ✓ sectional curvature + scalar curvature = isotropic curvature;
- ✓ Any Finsler surface is of scalar flag curvature. But the condition "of sectional flag curvature" is nontrivial even in dimension two. They are the Einstein surfaces mentioned above.

## questions

- ✓ Is the condition "of sectional flag curvature" rigid?

$$\text{sectional curvature} \stackrel{?}{\implies} \left\{ \begin{array}{l} \text{Riemannian} \\ \text{constant curvature} \end{array} \right.$$

- ✓ Look for some weak versions like

$$\text{sectional curvature} + ? \implies \left\{ \begin{array}{l} \text{Riemannian} \\ \text{constant curvature} \end{array} \right.$$

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# Rigidity Theorems

# Randers metrics of Sectional Curvature

# Randers metrics

Randers metrics are special members of Finsler metrics, which have the form

$$F = \alpha + \beta = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i.$$

It was introduced by G. Randers in his study of general relativity([Randers 1941]). G. Randers used this metric to describe the asymmetrical space-time. It seems to be an important model in physics.

For Finslerian geometers, Randers metrics maybe the most favorite objects in their studies. There are many great results until now.

# Randers metrics with constant flag curvature or Ricci curvature

## Theorem (Bao-Robles-Shen, 2003)

A Randers metric  $F = \alpha + \beta$  has constant flag curvature iff its Zermelo data  $(h, W)$  satisfy:

- $h$  is a Riemannian metric with constant sectional curvature ;
- $W$  is an infinitesimal homothety of  $h$ .

## Theorem (Bao-Robles, 2004)

A Randers metric  $F = \alpha + \beta$  is Einstein iff its Zermelo data  $(h, W)$  satisfy:

- $h$  is a Riemann-Einstein metric;
- $W$  is an infinitesimal homothety of  $h$ .

# Randers metrics of scalar curvature

## Theorem (Mo-Shen,2003)

*In dimension  $\geq 3$ , any closed Finsler manifold with negative scalar curvature is of Randers type.*

## Theorem (Shen-Yildirim,2005)

*A Randers metric is of scalar curvature iff its  $(\alpha, \beta)$  satisfies certain equations.*

# Randers metrics of sectional curvature

All the results encourage us to study the Randers metrics first. It may be a good candidate and gives us nice results.

Theorem (with B. Chen, 2007)

*Let  $F$  be a non-Riemannian Randers metric. If it has sectional flag curvature, then it must have isotropic flag curvature. Hence*

- $\dim = 2$ , it is an Einstein surface which is determined by Bao-Robles's theorem;*
- $\dim \geq 3$ , it has constant flag curvature which has been classified by Bao-Robles-Shen.*

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# Landsberg Spaces of Sectional Flag Curvature

# Landsberg spaces

A Finler metric is Landsbergian iff its Landsberg tensor vanishes.

The geometrical explanation: All the embeddings

$$i_x : S_x M \longrightarrow SM$$

are totally geodesic w.r.t the Sasaki metric on  $SM$ .

If the embeddings are minimal, then it is a weakly Landsberg metric.

# Properties of Landsberg spaces

Landsberg spaces have many nice properties, such as Gauss-Bonnet-Chern theorem. There are many rigidity theorems for Landsberg spaces.

Theorem (Nimmo, 1975)

*In dimension  $\geq 3$ , a Landsberg space with nonzero scalar flag curvature is Riemannian.*

Theorem (Wu, 2007)

*A closed Landsberg space with negative flag curvature is Riemannian.*

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# Landsberg spaces of sectional curvature

What is the behavior of Landsberg spaces of sectional curvature?  
It is natural to have such rigidity theorems.  
Indeed, we do have certain results. But they are not complete.

Theorem (with B. Chen, 2003)

Let  $F$  be a Landsberg metric of sectional flag curvature.

- $\dim \leq 7, K \neq 0 \implies$  Riemannian;
- $\dim \geq 8, -\frac{n-1}{n-7} \leq K \leq -\frac{1}{4}$  or  $\frac{1}{4} \leq K \leq \frac{n-1}{n-7} \implies$  Riemannian.

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# sphere theorem

Being aware of

$$\frac{n-1}{n-7} \searrow 1,$$

we can deduce a universal pinching.

Corollary

*Let  $F$  be a Landsberg metric of sectional flag curvature. If  $\frac{1}{4} \leq K \leq 1$ , then  $F$  is Riemannian.*

Theorem

*Let  $M^n$  be a complete Landsberg space of sectional flag curvature. If  $\frac{1}{4} < K \leq 1$ , then the universal cover of  $M$  is diffeomorphic to the sphere  $S^n$ .*

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## remarks

- ✓ In the sense of dimension, it behaves as the Bernstein theorem;
- ✓ In the sense of pinching, it likes the  $1/4$ -sphere theorem;
- ✓ In our opinion, it should be rigid in every dimension. But it needs new ideas.

# Akbar-Zadeh Type Theorems

# The philosophy

The philosophy of Akbar-Zadeh type theorems:  
if  $\lambda < 0$ , then the solution of

$$f''(t) + \lambda f(t) = 0$$

has exponential growth order when  $t \rightarrow \infty$ , which is impossible on a compact manifold unless  $f$  is identically zero.

## Example A

The most famous theorem is the Akbar-Zadeh's rigidity theorem for Finsler space forms.

### Theorem (Akbar-Zadeh, 1988)

*Let  $(M, F)$  be a closed Finsler space of constant flag curvature  $K$ , then*

- If  $K < 0$ , then  $F$  is Riemannian;*
- If  $K = 0$ , then  $F$  is locally Mincowski.*

The key is

$$\ddot{A}_{ijk} + KA_{ijk} = 0$$

where  $A_{ijk} dx^i \otimes dx^j \otimes dx^k$  is the Cartan tensor.

## Example B

A similar form is about scalar curvature.

Theorem (Mo-Shen, 2003)

*In dimension  $\geq 3$ , any closed Finsler manifold with negative scalar curvature is of Randers type.*

The key is

$$\ddot{M}_{ijk} + KM_{ijk} = 0$$

where  $M_{ijk} dx^i \otimes dx^j \otimes dx^k$  is the Matsumoto tensor.

## Example C

Some trace forms say

Theorem (Z.Shen, 2005)

*Any closed Finsler space with negative flag curvature and constant S-curvature is Riemannian.*

Theorem (B.Y.Wu, 2007)

*Any closed weakly Landsberg space with negative flag curvature is Riemannian.*

The key is

$$\ddot{l}_k + R^i_k l_i = 0$$

where  $l_k dx^k$  is the Cartan form.

# A version for sectional flag curvature

In our study of sectional flag curvature, we find an Akbar-Zadeh type theorem.

## Theorem (with B.Chen, 2008)

*Let  $(M^n, F)$  be a complete Finsler space of sectional flag curvature. If  $n \leq 7$ ,  $\|I\|_{C^0} < \infty$  and  $K < 0$ , then  $F$  is Riemannian.*

## Corollary

*For  $\dim \leq 7$ , any closed Finsler manifold with negative sectional flag curvature is Riemannian.*

## remark

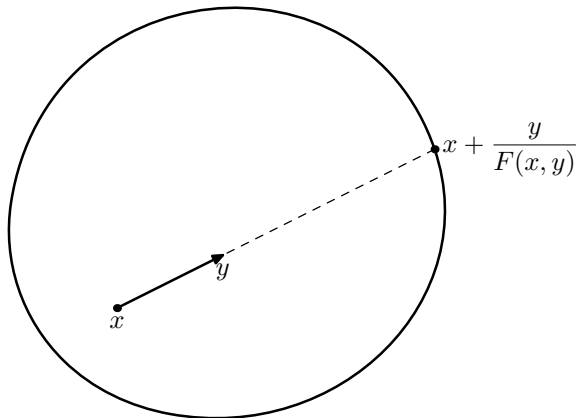
We should remark here that the completeness in the above theorem can not be replaced by positive completeness.

The Funk metric is a positive complete Finsler metric with constant flag curvature  $-1/4$ .

On the other hand, the Cartan form of the Funk metric (Randers metric) has the norm

$$\|I\| < \frac{n+1}{\sqrt{2}}.$$

# funk metric



# Proofs and Problems

# Proof for Randers metrics

# Proofs of Randers metrics

## Lemma

*A Finsler metric  $F$  is of sectional flag curvature, iff its flag curvature has the symmetric property  $K(x, u, v) = K(x, v, u)$  for any linear independent vectors  $u$  and  $v$ .*

## Proof.

The reason is  $K(x, u, P) = K(x, u, v)$  and  $K(x, v, P) = K(x, v, u)$  where  $P = \text{span}\{u, v\}$ . □

# Proofs of Randers metrics

That is

$$\frac{R_{jk}(y)V^jV^k}{F^2(y)h_{jk}(y)V^jV^k} = K(y, V) = K(V, y) = \frac{R_{ab}(V)y^ay^b}{F^2(V)h_{ab}(V)y^ay^b}$$

for linear independent vectors  $y$  and  $V$ .

In other words, it is true for any  $y$  and  $V$  that

$$F^2(y)h_{jk}(y)V^jV^kR_{ab}(V)y^ay^b = F^2(V)h_{ab}(V)y^ay^bR_{jk}(y)V^jV^k.$$

# Proofs of Randers metrics

The formula of the Riemann curvature of a Randers metric  $\alpha + \beta$  has been obtained by [Bao-Robles 2004] and Z.Shen independently,

$$\begin{aligned}
 R^i_k &= {}^a R^i_k + \frac{1}{4} c_j^i c_0^j y_k - \frac{1}{4} \alpha^2 c_j^i c_k^j + \frac{3}{4} c_0^i c_{k0} + \alpha c_{0|k}^i \\
 &\quad - \frac{1}{2} \alpha c_{k|0}^i - \frac{1}{2} (1/\alpha) c_{0|0}^i y_k - \frac{1}{4} (\alpha/F) y^i l_{j0} c_k^j - \frac{1}{2} (1/F) y^i l_{k0|0} \\
 &\quad + \delta \text{Coeff} \cdot \delta_k^i + y y \text{Coeff} \cdot y^i y_k + y b \text{Coeff} \cdot y^i b_k \\
 &\quad + \frac{1}{4} (\alpha^2/F) y^i \theta_j c_k^j - \frac{3}{4} (1/F) y^i \theta_0 c_{k0} + \frac{1}{2} (\alpha/F) y^i c_0^j l_{jk} \\
 &\quad + \frac{1}{2} (\alpha/F) y^i \theta_{k|0} - (\alpha/F) y^i \theta_{0|k} + \frac{1}{2} (1/F) y^i l_{00|k}.
 \end{aligned}$$

Here

$$l_{ij} := b_{i|j} + b_{j|i}, \quad c_{ij} := b_{i|j} - b_{j|i}, \quad c_j^i := a^{ik} c_{kj}, \quad \theta_j := b^i c_{ij}$$

where " $|$ " means covariant derivative with respect to the Riemannian metric  $\alpha$ , and  $l, c$  interpret "Lie derivative" and "curl" respectively. Unless being pointed out, the indices are always raised and lowered by  $a$ , e.g.  $y_i := a_{ij}y^j$ .

# Proofs of Randers metrics

## Lemma

The flag curvature of the flag  $(y, V)$  is

$$\begin{aligned}
 & R_{jk}(y)V^jV^k \\
 = & \frac{F}{\alpha^3} \left( {}^aR_{**}\alpha^2 + \frac{1}{2}\alpha^2c_{*j}c_0^jy_* + \frac{1}{4}\alpha^4c_{j*}c_*^j + \frac{3}{4}\alpha^2c_{*0}^2 + \frac{1}{4}c_{j0}c_0^jy_*^2 \right) \textcircled{A} \\
 & + \frac{F}{\alpha^2} \left( \alpha^2c_{*0|*} + c_{0*|0}y_* \right) \textcircled{B} \\
 & + \frac{1}{\alpha^3} \left( -\frac{1}{4}l_{00|0} - \frac{1}{2}\alpha^2c_0^j\theta_j \right) \textcircled{C} H + \frac{1}{\alpha^2} \left( \frac{1}{2}c_0^j l_{j0} + \frac{1}{2}\theta_{0|0} \right) \textcircled{X} H \\
 & + \frac{1}{F\alpha^3} \left( \frac{3}{16}l_{00}^2 + \frac{3}{4}\alpha^2\theta_0^2 \right) \textcircled{Y} H + \frac{1}{F\alpha^2} \left( -\frac{3}{4}l_{00}\theta_0 \right) \textcircled{Z} H.
 \end{aligned}$$

# Proofs of Randers metrics

## Lemma

A Randers metric  $F = \alpha + \beta$  is of sectional flag curvature iff there exists a polynomial  $T(y, V) = y^i V^j T_{ijkl}(x) V^k y^l$  with  $T(y, V) = T(V, y)$ , such that

$$\left(\alpha^4 + 6\alpha^2\beta^2 + \beta^4\right) T = \left((\alpha^2 + \beta^2)A + 2\alpha^2\beta B + \beta CH + \alpha^2 XH + YH\right),$$

$$\left(4\alpha^2\beta + 4\beta^3\right) T = \left(2\beta A + (\alpha^2 + \beta^2)B + CH + \beta XH + ZH\right)$$

where  $H = \|y \wedge V\|_{\alpha}^2$ . Then

$$K(x, y, V) = \frac{T(y, V)}{H(y, V)} = \frac{T(V, y)}{H(V, y)} = K(x, V, y).$$

# Proofs of Randers metrics

The key point is the basic equation

$$l_{ik} + b_i\theta_k + b_k\theta_i = \sigma(x)(a_{ik} - b_ib_k).$$

which leads the following lemma.

## Lemma

*A necessary and sufficient condition for a Randers metric to be of sectional flag curvature is*

$$\begin{aligned} l_{00} + 2\theta_0\beta &= \sigma(\alpha^2 - \beta^2), \\ 2\beta T &= B - \left( \frac{1}{4}\sigma_{|0} + \frac{1}{2}\sigma\theta_0 + \frac{1}{8}\sigma^2\beta + \frac{1}{2}c_0^j\theta_j \right) H, \\ (\alpha^2 + \beta^2)T &= A + \left( \frac{3}{16}\sigma^2\alpha^2 - \frac{1}{16}\sigma^2\beta^2 + \frac{1}{4}\sigma_{|0}\beta + \frac{1}{4}\theta_0^2 + \frac{1}{2}\theta_{0|0} \right) H. \end{aligned}$$

# Proofs of Randers metrics

From the above lemma, in dimension  $\geq 3$ , one will get

$$T(y, V) = \lambda(x)H(y, V).$$

It means that the flag curvature is

$$K(x, y, V) = \frac{T(y, V)}{H(y, V)} = \lambda(x).$$

Finally, we reach the rigidity theorem for Randers metrics.  $\square\square$

# Proof for Landsberg Spaces

# Proofs of Landsberg spaces

The idea is  $\frac{d}{dt}K(x, y + tV, V) = 0$ .

## Lemma

*If  $F$  is of sectional flag curvature, then  $\frac{d}{dt}K(x, y + tV, V)|_{t=0} = 0$  for any  $y, V$ . If  $F$  is reversible in addition, then the condition is also sufficient.*

## Proof.

We only prove the second part.

For any  $u = ay + bV (a \neq 0)$ , since  $F$  is reversible, we have

$$K(x, u, V) = K(x, ay + bV, V) = K(x, y + \frac{b}{a}V, V) = K(x, y, V).$$

Then  $K(x, u, P) = K(x, y, P)$  unless  $u = bV$ . But by continuity, one can see  $u = bV$  is also true.  $\square$

# Proofs of Landsberg spaces

Applying some Bianchi's identities, the  $t$ -derivative will tell us

## Lemma

*The condition "of sectional flag curvature" implies*

$$\ddot{A}(V, V, V) + 2KA(V, V, V) = -R(V, V, V, \ell) = A(V, V, R_y(V)).$$

For a Landsberg space we have

$$2R(\ell, V, V, \ell)A(V, V, V) = h(V, V)A(V, V, R_y(V))$$

# Proofs of Landsberg spaces

Consider it as a polynomial of  $V$ .

By assuming  $K \neq 0$ , we can see that  $R(\ell, V, V, \ell)$  is irreducible in  $V$ . On the other hand,  $h(V, V)$  is clearly irreducible.

Then by some knowledge of polynomials, one will reach

## Lemma

*If  $K \neq 0$ , then it holds*

$$4\text{Ric} \cdot I_k = (n - 7)R_k^s I_s.$$

# Proofs of Landsberg spaces

If the Cartan form  $I = I_k dx^k \neq 0$  at some  $y$ , then

$$4Ric(x, y) = (n - 7)K(x, y, I).$$

Since  $K \neq 0$ , we see either  $K$  always positive or always negative. Hence, in dimension  $\leq 7$ , it will be never true unless  $I = 0$  which means  $F$  is Riemannian.

# Proofs of Landsberg spaces

For dimension  $\geq 8$ ,

$$4Ric(x, y) = (n - 7)K(x, y, l).$$

- $-\frac{n-1}{n-7} \leq K \leq -\frac{1}{4},$

$$-(n-1) \leq (n-7)K(x, y, l) = 4Ric \leq -4(n-1)\frac{1}{4}.$$

- $\frac{1}{4} \leq K \leq \frac{n-1}{n-7},$

$$(n-1) \leq 4Ric(x, y) = (n-7)K(x, y, l) \leq (n-1).$$

# Proof for Negatively Curved Closed Spaces

# Proofs of negatively curved closed spaces

Recall the key equation

$$\ddot{A}(V, V, V) + 2KA(V, V, V) = -R(V, V, V, \ell) = A(V, V, R_y(V)).$$

**Remark.** By using Berwald's Riemann curvature (4,0)-tensor, one will find that

$$\ddot{A}(V, V, V) + KA(V, V, V) = -\frac{1}{2}R^b(V, V, V, \ell).$$

If  $F$  has constant flag curvature, then

$${}^bR_{ijkl} = K(g_{jk}g_{il} - g_{jl}g_{ik})$$

then the equation is well known as  $\ddot{A}_{ijk} + KA_{ijk} = 0$ .

# Proofs of negatively curved closed spaces

Without the Landsbergian assumption, a similar analysis gives

## Lemma

If  $K \neq 0$ , then it holds

$$3(n+1)\ddot{I}_k + 4Ric \cdot I_k - (n-7)R_k^s I_s = 0.$$

The following steps are well known. Let  $\gamma(t)$  be a normal geodesic with initial data  $(x, y)$ . Putting  $f(t) = g(I, I)_{(\gamma(t), \dot{\gamma}(t))}$

# Proofs of negatively curved closed spaces

Then under some curvature assumption

$$\begin{aligned} f''(t) &= 2g(\ddot{l}, l) + 2g(\dot{l}, \dot{l}) \\ &= \frac{2}{3(n+1)}g(-4Ric \cdot l + (n-7)R_y(l), l) + 2g(\dot{l}, \dot{l}) \\ &= \frac{2}{3(n+1)}\left(-4Ric + (n-7)K(x, y, l)\right)g(l, l) + 2g(\dot{l}, \dot{l}) \\ &\geq 0. \end{aligned}$$

So if  $f'(t_0) \neq 0$  at some point  $t_0$ , then  $f(t) \rightarrow \infty$  when  $t \rightarrow +\infty$  or  $t \rightarrow -\infty$  which is contradict to  $\|l\|_\infty < \infty$ . Hence  $f'(t) = 0$  and then  $f''(t) = 0, l = 0, \dot{l} = 0$ .  $\square\square$

# Problems and further plan





- ✓ Prove or disprove the negative rigidity in high dimensions.
- ✓ To find nontrivial examples.
- ✓ Tell the difference between the two sets

$$\mathcal{KF}_M = \left\{ G_2(M) \xrightarrow{K_F} \mathbb{R} : F \text{ is a SFC Finsler metric on } M \right\},$$

$$\mathcal{KR}_M = \left\{ G_2(M) \xrightarrow{K_F} \mathbb{R} : F \text{ is a Riemann metric on } M \right\}.$$

- ✓ Verify the properties that Riemann metrics hold. For instance, prove the 1/4-sphere theorem without the Landsbergian assumption.

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# Thank You