Synge type theorems for positively curved Finsler manifolds

Chang-Wan Kim
Korea Institute for Advanced Study

February 16, 2008 (Saturday) 9:30AM-10:20AM

Finsler manifolds

Let M be a differentiable manifold and $\gamma : [a, b] \to M$ is a curve. The length of γ is

$$\int_a^b ||\dot{\gamma}(t))||dt.$$

Finsler geometry is actually the geometry of a simple integral and is as old as the calculus of variations.

- 1854 Riemann, 1869 Christoffel
- Hilbert: Paris address of 1900 devoted Problem 23 to the variational calculus of an invariant integral and its geometrical.
- 1918 Finaler, 1928 Funk, Cartan
- Berwald, Busemann, Akbar-Zadeh, Dazord, S. S. Chern

Finsler structure

Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties:

(i) F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;

(ii) $F(ty) = |t|F(y), t \in \mathbf{R},\ \ y \in T_xM$, reversible;

(iii) F^2 is strongly convex, i.e., $g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$ is positive definite for all $(x, y) \in \widetilde{TM}$.

- $g_{ij}(x) = g_{ij}(x, y)$ Riemannian metric
- $g_{ij}(y) = g_{ij}(x, y)$ Minkowski metric

(i) F^2 is C^2 on all TM iff F is the Riemannian metric.

(ii)
- Homogeneous of degree one in y
- Hilbert form $\omega = F_{y_i} dx^i$ is essentially Hilbert's invariant integral in the calculus of variations
- The length of curve is $\int_a^b w$.
- Symmetric condition for dist_F

(iii) Regularity Hypothesis:
- The Hilbert form ω is a global one form on SM and define a contact structure $\omega \wedge d\omega \neq 0$ on SM [Chern '48]
- The critical point theory for the closed geodesics problem [Mercuri '77]; Klingenberg
- Triangle inequality for dist_F

Flag curvature

For a fixed $v \in T_xM$ let γ_v be the geodesic from $\gamma_v(0) = x$ with $\dot\gamma_v(0) = v$. Along γ_v, we have the *osculating Riemannian metrics*

$$g^{\dot\gamma_v(t)} := g\big(\gamma_v(t), \dot\gamma_v(t)\big)$$

in $T_{\gamma_v(t)}M$.

Define the *flag curvature* $R^{\dot\gamma_v(t)}(u(t)) : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$ by

$$R^{\dot\gamma_v(t)}(u(t)) := R\big(U(t), V(t)\big)V(t),$$

where $U(t) = \big(\hat\gamma_v(t); u(t)\big), V(t) = \big(\hat\gamma_v(t); \gamma_v(t)\big) \in \pi^* TM$.

- The flag curvature is independent of connections, that is, this term appears in the second variation formula of arc length, thus is of particular interest to us.
- If F is Riemannian, then the flag curvature coincides with the sectional curvature.

Relationships between topology and curvature

[Auslander '55] Cartan-Hadamard, Bonnet-Myers, Synge theorems

[Bao-Chern '93] The first comparison theorem of Rauch

[Bao-Chern '96, Shen] Gauss-Bonnet-Chern theorem

[Egloff '97] Ansosov theorem, growth of the fundamental group in negative flag curvature

[Shen '97] Bishop-Gromov volume comparison theorem

[Dazord '68, Kern '71, Rademacher '02] Homotopy sphere theorem

[Wu-Xin '07] McKean type theorems and the first eigenvalue in terms of Ricci curvature

[Kozma-Peter '07] Connectedness principle in Finsler manifolds

[Józefowicz '08] The Finslerian foliations on compact manifolds is Riemannian

Space Forms

All of simply connected complete Riemannian manifolds with constant sectional curvature K are isometric to Eucldean $K = 0$, sphere $K > 0$, or hyperbolic spaces $K < 0$; Cartan, Hopf.

Zadeh Theorem '88

Let (M, F) be a compact Finsler manifold with constant flag curvature R.

(1) If $R < 0$, then F is Riemannian.

(2) If $R = 0$, then F is locally Minkowskian.

- The non-Riemannian $R = 1$ examples constructed by Bryant '97 and Bao-Shen '02; Foulon '04, Bryant '06, Kim '07
- Negatively curved manifolds; Foulon '97, Boland-Newberger '01, Colbois-Verovic '02, Colbois-Newberger-Verovic '07

Randers metrics

For a Riemannian metric

$$a := a_{ij}(x)dx^i \otimes dx^j,$$

a linear term

$$b := b_i(x)dx^i,$$

Randers metrics

$$F(x,y) := \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i, y = y^i\frac{\partial}{\partial x^i} \in T_xM$$

with

$$a_{ij}(x)dx^i \otimes dx^j(b_i(x)dx^i, b_i(x)dx^i) < 1.$$

Non-reversible metrics with positive constant curvature

[Bryant '97] Finsler metrics on S^2

[Bejancu-Farran '00] Unit horizontal Reeb vector field is Killing on the unit tangent bundle.

[Bao-Shen '02] Nonprojectively flat Randers metric on S^3

[Bejancu-Farran '02] Nonprojectively flat Randers metric on S^{2n+1}

[Bao-Robles-Shen '04] Necessary and sufficient conditions for a Randers spaces to have constant flag curvature; Zermelo navigation problems

[Robles '06] Classify the geodesics on Randers spaces; either all the geodesics close or only finitely many of geodesics close.

Reversible metrics with $R = 1$ is Riemannian, Kim '07

[Bryant '06] On dim 2, by using the fundamental and deep result of LeBrun and Mason.

(*sketch of proof*)

1. The universal covering $\overline{M}$ of M is an n-dimensional Finsler manifold with flag curvature $R = 1$, then every geodesic is closed with same length 2π and $\overline{M}$ is a diffeomorphic sphere. By Weinstein, Yang and Àlvarez Paiva's result, the symplectic volume of $\overline{M}$, $V(S\overline{M})$ is equal to $V(SS^n)$.

2. Define a map $\psi_x : (0, \infty) \times S_x\overline{M} \to \overline{M}$ by $\psi_x(t, v) = \gamma_v(t)$.

[Shen '95] The osculating Riemannian metric $g^{\dot{\gamma}_v(t)}$ has the form $(\psi_x)^* g^{\dot{\gamma}_v(t)} = dt^2 \oplus \sin^2 t\, \dot{g}_x$. By the Blaschke-Santalo's inequality, $vol_{g^{\dot{\gamma}_v(t)}}(M) \leq vol(S_1^n)$, and equality holds if and only if (T_xM, F_x) is Euclidean; **Need reversible condition.**

3. For all $x \in \overline{M}$, the map $\psi_x : (0,\pi) \times S_x\overline{M} \to \overline{M} \setminus \{x, x^*\}$, $\psi_x(t,v) \mapsto \gamma_v(t)$ is a diffeomorphism and we have that

$$\begin{aligned} V(SS^n) &= V(S\overline{M}) \\ &= V\left(\bigcup_{\gamma_v(t) \in \overline{M} \setminus \{x,x^*\}} S_{\gamma_v(t)}\overline{M} \right) \\ &= \int_{\overline{M}\setminus\{x,x^*\}} \left\{ \int_{S_{\gamma_v(t)}\overline{M}} 1 \, d\dot{g}_{\gamma_v(t)} \right\} dg^{\dot{\gamma}_v(t)} \\ &= \int_{\overline{M}\setminus\{x,x^*\}} vol_{\dot{g}_{\gamma_v(t)}}\left(S_{\gamma_v(t)}\overline{M}\right) dg^{\dot{\gamma}_v(t)} \\ &\leq V(SS^n). \end{aligned}$$

The last line is obtained from the above argument and equality holds if and only if $(\overline{M}, F)$ is a Riemannian manifold, and hence F is a Riemannian metric.

Symmetry and parallel curvature

[Cartan '26] For Riemannian spaces, parallel curvature $\Longleftrightarrow$ geodesic reflections are isometries

In Finsler geometry, geodesic reflections are isometries $\Longrightarrow$ parallel curvature; cf. A Hilbert geometry

[Egloff '96] A Hilbert geometry is locally symmetric if and on if the metric is Riemannian.

[Foulon '97] Any compact negatively curved Finsler spaces with parallel curvature is a locally symmetric Riemannian space.

[Kim, '07] Any compact locally symmetric Finsler manifolds M with positive flag curvature must be Riemannian.

(*sketch of proof*) $\overline{M}$ diffeomorphic G/H and has a Berwald metric.

- G/H positively curved Riemannian symmetric spaces.
- $V(S(G/H)) \geq V(S\overline{M}) = V(S(G/H))$

Global rigidity properties, Kim '07

- If M simply connected and $\dim \mathrm{Iso}(M^n) \geq n(n-1)/2+1$, $n \neq 4$, then $\sec M = c > 0$.
- The two-point homogeneous Finsler spaces are Riemannian
  - if isometry group is transitive on equidistant pairs of points
  - isotropic at x if G_x is transitive on the S_xM at x
  - Banach-Mazur rotation problem
- Problem; Is it true that the vertical Laplacian is vanishing if and only if the mean tangent curvature is zero?
  - If $\mathrm{Ric}_{M^n} \geq (n-1)$, vanishing vertical Laplacian, and $\lambda_1(M) = \lambda_1(S_1^n)$, then $\sec M = 1$.
  - If $\mathrm{Ric}_{M^n} \geq (n-1)$, vanishing mean tangent curvature, and $\mathrm{diam}(M) = \mathrm{diam}(S_1^n)$, then $\sec M = 1$.

Group actions

Let M be an n-dimensional Finsler manifold of positive flag curvature and admit an effective isometric k-dimensional torus T^k-action.

- If n is even, then the fixed point set is not empty.
- If n is odd, then there is a circle orbit, i.e., not all isotropy subgroups can be finite.

[Symmetry rank, Kim '07] Assume that M is an n-dimensional Finsler manifold with positive flag curvature. If M admits an isometric T^k-action, then $k \leq [(n+1)/2]$ (the integer part).

Note that a sphere and a complex projective space with the quotient Riemannian metric from the Hopf fibration, $S^1 \to S^{2n+1} \to S^{2n+1}/S^1 = CP^n$, have the maximal symmetric rank.

Synge type properties

For a $(k+1)$-dimensional subspace $V \subset T_xM$, define the Ricci curvature Ric_V on V to be the trace of the Riemann curvature restricted to V. Ric_V is given by

$$\mathrm{Ric}_V(v) := \sum_{i=1}^{k+1} g^v\big(R^v(e_i), e_i\big), v \in V,$$

where $\{e_i\}_{i=1}^{k+1}$ is an orthonormal basis for (V, g^v).

[Synge type, Kim '07] Assume that M be an oriented Finsler manifold with k-th Ricci curvature $\mathrm{Ric}_k \geq k$. Let f be an isometry satisfied $\mathrm{dist}(x, f(x)) > \pi\sqrt{(k-1)/k}$ for all $x \in M$.

- If M is even dimensional, then f reverses the orientation.
- If M is odd dimensional, then f is orientation preserving.

Open problems

- Classify nonreversible Finsler metrics with positive constant flag curvature
- Find a Finsler metric with positive flag curvature
  - (Hopf-Finsler) Does there exist (non-)reversible Finsler metric with positive flag curvature on $S^2 \times S^2$?
  - Classify positively curved Finsler manifolds with large isometry groups; Grove-Wilking-Ziller
- Classify Finsler tori without conjugate points; Burago-Ivanov, Croke-Kleiner
- (Chern) Does every smooth manifold admit a Finsler Einstein metric? Hamilton-Perelman Ricci flow
- Brunn-Minkowski inequality and Ricci curvature bounded below on measure metric spaces; Sturm, Lott-Villani, Ohta

References

- Rigidity of noncompact Finsler manifolds, *Geometriae Dedicata* **81** (2000), 245–259 with Jin-Whan Yim
- Finsler manifolds with positive constant flag curvature, *Geometriae Dedicata* **98** (2003), 47–56 with Jin-Whan Yim
- Locally symmetric positively curved Finsler spaces, *Arch. Math. (Basel)* **88** (2007), 378–384
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- Some rigidity theorems for Finsler manifolds, *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis* **24** (2008)
- Finsler metrics with positive constant flag curvature, *preprint*

A summary of the main points

In this talk, I discuss Finsler metrics with positive flag curvature. We recall Weinstein's theorem in Finsler geometry that any isometry f of an n-dimensional oriented Finsler manifold of positive flag curvature has a fixed point if either n is even and f preserves orientation or n is odd and f reverses orientation. Since Synge's theorem for Finsler manifolds is an easy corollary of this, it should not be surprising that with a little more work was we can prove the following theorem. Assume that M be an oriented Finsler manifold with k-th Ricci curvature $\mathrm{Ric}_k \geq k$. Let f be an isometry satisfied $\mathrm{dist}(x, f(x)) > \pi\sqrt{(k-1)/k}$ for all $x \in M$.

(1) If M is even dimensional, then f reverses the orientation.

(2) If M is odd dimensional, then f is orientation preserving.