Two-Dimensional Finsler Metrics with Constant Flag Curvature

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Abstract

We construct infinitely many two-dimensional Finsler metrics on $\mathbb{S}^2$ and $\mathbb{D}^2$ with non-zero constant flag curvature. They are all not locally projectively flat.

1 Introduction

In Finsler geometry, the flag curvature is an analogue of sectional curvature in Riemannian geometry. A natural problem is to study and characterize Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature, due to the non-Riemannian features of general Finsler metrics. The first set of non-Riemannian Finsler metrics of constant flag curvature are the Hilbert-Klein metric and the Funk metric on a strongly convex domain. The Funk metric is positively complete and non-reversible with $\mathbf{K} = -1/4$ and the Hilbert-Klein metric is complete and reversible with $\mathbf{K} = -1$. Both metrics are locally projectively flat [Ok][Sh1]. P. Funk first completely determined the local structure of two-dimensional projectively flat Finsler metrics with constant flag curvature [Fk1][Fk2]. R. Bryant has shown that up to diffeomorphism, there is exactly a 2-parameter family of locally projectively flat Finsler metrics on $\mathbb{S}^2$ with $\mathbf{K} = 1$ and the only reversible one is the standard Riemannian metric [Br1][Br2]. He has also extended his construction to higher dimensional spheres $\mathbb{S}^n$ [Br3]. Recently, the author has completely determined the local structure of projectively flat analytic Finsler metrics of constant flag curvature in higher dimensions [Sh3]. Our method is different from Funk's.

The next problem is to classify non-projectively flat Finsler metrics of constant flag curvature. This problem turns out to be very difficult. The very first step might be to construct as many examples as possible. In 2000, D. Bao and the author first constructed a family of non-projectively flat Finsler metrics on $\mathbb{S}^3$ with $\mathbf{K} = 1$ using the Lie group structure of $\mathbb{S}^3$ [BaSh]. Our examples are in the form $F = \alpha + \beta$, where $\alpha(\mathbf{y}) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and

$\beta(\mathbf{y}) = b_i(x)y^i$ is a 1-form. Finsler metrics in this form are called Randers metrics [Ra]. Recently, the author has constructed an incomplete non-projectively flat Randers metric with $\mathbf{K} = 0$ in each dimension[Sh2].

The main technique in [Sh2] is described as follows. Given a Finsler metric Φ and a vector field $\mathbf{v}$ on a manifold M, define a function $F : TM \to [0, \infty)$ by

$$\Phi\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon \mathbf{v}_p\Big) = 1, \qquad \mathbf{y} \in T_pM. \tag{1}$$

where ϵ is a constant with $\Phi(-\epsilon\mathbf{v}_p) < 1$ at any point $p \in M$. It is easy to see that F is a Finsler metric when ϵ is small. An important relationship between Φ and F is that their (Busemann-Hausdorff) volume forms are equal, $dV_\Phi = dV_F$ [Sh2]. When Φ is a Riemannian metric, then $F = \alpha + \beta$ is a Randers metric.

The Finsler metric F defined in (1) can also be constructed in the following way. Let $\Phi^* : T^*M \to [0, \infty)$ denote the Finsler metric dual to Φ and $\mathbf{v}^* : T^*M \to \mathrm{R}$ denote the 1-form dual to $\mathbf{v}$. They are defined by

$$\Phi^*(\xi) := \sup_{\mathbf{y}\in T_pM} \frac{\xi(\mathbf{y})}{\Phi(\mathbf{y})}, \qquad \mathbf{v}^*(\xi) = \xi(\mathbf{v}), \qquad \xi \in T_p^*M.$$

Let

$$F^* := \Phi^* + \epsilon\mathbf{v}^*. \tag{2}$$

Since Φ^* is a co-Finsler metric, the function $F^* : T^*M \to [0, \infty)$ is a Finsler metric when ϵ is small. In this case when F^* is a co-Finsler metric, we let $F : TM \to [0, \infty)$ denote the Finsler metric dual to F^*, i.e.,

$$F(\mathbf{y}) := \sup_{\xi\in T_p^*M} \frac{\xi(\mathbf{y})}{F^*(\xi)}, \qquad \mathbf{y} \in T_pM.$$

It is easy to verify that the Finsler metric F defined in (2) is just the one defined in (1). In this sense, the formula (1) is dual to the formula (2).

The formula (2) is used by A.B. Katok [Ka] to construct many interesting Randers metrics on $\mathbb{S}^n$ with special properties of closed geodesics. See [Zi] for further discussion. However, the curvature properties are not discussed in their papers.

By choosing an appropriate Finsler metric Φ and an appropriate vector field $\mathbf{v}$, one obtains a Finsler metric F with the same curvature properties as Φ. For example, if Φ has constant flag curvature and constant S-curvature, then for an appropriate vector field $\mathbf{v}$, the Finsler metric F defined in (1) or (2) has the same curvature properties. This fact was discovered by the author accidently using Maple in [Sh2] when he was studying the shortest time problem.

In this paper, using (1) we are going to construct a family of Randers metrics on $\mathbb{S}^2$ with $\mathbf{K} = 1$ and a family of Randers metrics on a disk $\mathbb{D}^2(\rho)$ with $\mathbf{K} = -1$ or $\mathbf{K} = -1/4$. They are all not locally projectively flat. These examples show that the classification problem of non-projectively flat Finsler metrics of constant flag curvature is very difficult.

Now let us describe our examples. Let $\Phi(\mathbf{y}) = \sqrt{h(\mathbf{y},\mathbf{y})}$ denote the standard Riemannian metric on the unit sphere $\mathbb{S}^2$ and $\mathbf{v}$ denote the vector field on $\mathbb{S}^2$ defined by

$$\mathbf{v}_p = \Big(-y,\ x,\ 0 \Big) \quad \text{at } p = \Big(x,\ y,\ z \Big) \in \mathbb{S}^2, \tag{3}$$

Define $F : T\mathbb{S}^2 \to [0,\infty)$ by (1). Then $F = \alpha + \beta$ is a Randers metric, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha := \frac{\sqrt{\epsilon^2 h(\mathbf{v},\mathbf{y})^2 + h(\mathbf{y},\mathbf{y})\Big(1 - \epsilon^2 h(\mathbf{v},\mathbf{v})\Big)}}{1 - \epsilon^2 h(\mathbf{v},\mathbf{v})}, \qquad \beta := -\frac{\epsilon h(\mathbf{v},\mathbf{y})}{1 - \epsilon^2 h(\mathbf{v},\mathbf{v})}. \tag{4}$$

F is defined on the whole sphere for $|\epsilon| < 1$ and it is defined only on the open disks around the north pole and south pole with radius $\rho = \sin^{-1}(1/|\epsilon|)$ for $|\epsilon| \geq 1$. Note that when $\epsilon = 0$, $F = \Phi$ is the standard Riemannian metric on $\mathbb{S}^2$.

Theorem 1.1 *Let $F = \alpha + \beta$ be any Finsler metric on $\mathbb{S}^2$ given in (4). It has the following properties*

(a) $\mathbf{K} = 1$;

(b) $\mathbf{S} = 0$;

(c) F is not locally projectively flat unless $\epsilon = 0$;

(d) the Gauss curvature $\bar{\mathbf{K}}$ of α is not a constant unless $\epsilon = 0, \pm 1$. When $\epsilon = 0$, $\bar{\mathbf{K}} = 1$; When $\epsilon = \pm 1$, $\bar{\mathbf{K}} = -4$.

According to Yasuda-Shimada[YaSh], if a Randers metric $F = \alpha + \beta$ is of positive constant flag curvature, then β must be a Killing form of constant length with respect to α. However, β in (4) is not a Killing form when $\epsilon \neq 0$. This shows that the Yasuda-Shimada theorem in the positive constant flag curvature case is wrong.

Similarly, let $\Phi(\mathbf{y}) = \sqrt{h(\mathbf{y},\mathbf{y})}$ denote the standard Klein metric on the unit disk $\mathbb{D}^2$ and $\mathbf{v}$ denote the vector field on $\mathbb{D}^2$ defined by

$$\mathbf{v}_p = (-y,\ x) \quad \text{at } p = (x,y) \in \mathbb{D}^2. \tag{5}$$

Define $F : T\mathbb{D}^2 \to [0,\infty)$ by (1). Then $F = \alpha + \beta$ is a Randers metric, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha := \frac{\sqrt{\epsilon^2 h(\mathbf{v},\mathbf{y})^2 + h(\mathbf{y},\mathbf{y})\Big(1 - \epsilon^2 h(\mathbf{v},\mathbf{v})\Big)}}{1 - \epsilon^2 h(\mathbf{v},\mathbf{v})}, \qquad \beta := -\frac{\epsilon h(\mathbf{v},\mathbf{y})}{1 - \epsilon^2 h(\mathbf{v},\mathbf{v})}. \tag{6}$$

F is a Finsler metric defined on the disk $\mathbb{D}^2(\rho)$ with radius $\rho = 1/\sqrt{1+\epsilon^2}$. Note that when $\epsilon = 0$, F is the Klein metric on the unit disk.

Theorem 1.2 *Let $F = \alpha + \beta$ be the Finsler metric on the disk $\mathbb{D}^2(\rho)$ given in (6). It has the following properties*

(a) $\mathbf{K} = -1$;

(b) $\mathbf{S} = 0$;

(c) *F is not locally projectively if $\epsilon \neq 0$;*

(d) *the Gauss curvature $\bar{\mathbf{K}}$ of α is not constant unless $\epsilon = 0$. When $\epsilon = 0$,* $\bar{\mathbf{K}} = -1$.

According to Yasuda-Shimada [YaSh] if a Randers metric $F = \alpha + \beta$ is of negative constant flag curvature, then the Riemannian metric α is of negative constant flag curvature. However, the Randers metric defined in (6) do not have this property when $\epsilon \neq 0$. This shows that the Yasuda-Shimada theorem in the negative constant flag curvature case is wrong.

Besides the Klein metric, the hyperbolic metric can be expressed in many other forms, such as the Poincare metric and the one arising from the proof of Theorem 1.1. We can use them to construct many non-projectively flat Finsler metrics with negative constant flag curvature. See Remark 4.1 below.

Finally, let $\Phi(\mathbf{y}) = \sqrt{h(\mathbf{y},\mathbf{y})} + h(\mathbf{u},\mathbf{y})$ denote the Funk metric on the unit disk $\mathbb{D}^2$, where h is the Klein metric on $\mathbb{D}^2$ and $\mathbf{u} = (1-x^2-y^2)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) \in T_{(x,y)}\mathbb{D}^2$ is a vector field. $\Phi(\mathbf{y})$, $\mathbf{y} \in T_p\mathbb{D}^2$, is defined by

$$\frac{\mathbf{y}}{\Phi(\mathbf{y})} + p \in \partial\mathbb{D}^2. \tag{7}$$

Let $\mathbf{v}$ denote the vector field on $\mathbb{D}^2$ defined by (5). Define $F : T\mathbb{D}^2 \to [0,\infty)$ by (1), i.e.,

$$\sqrt{h\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v},\ \frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}\Big)} + h\Big(\mathbf{u},\ \frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}\Big) = 1. \tag{8}$$

Then $F = \alpha + \beta$ is a Randers metric on $\mathbb{D}^2(\rho)$ where $\rho = 1/\sqrt{1+\epsilon^2}$.

Theorem 1.3 *Let $F = \alpha + \beta$ be the Randers metric on $\mathbb{D}^2(\rho)$ defined in (8). It has the following properties:*

(a) $\mathbf{K} = -1/4$;

(b) $\mathbf{S} = \frac{3}{2}F$;

(c) *F is not locally projectively flat unless $\epsilon = 0$;*

(d) *the Gauss curvature $\bar{\mathbf{K}}$ of α is not a constant unless $\epsilon = 0$. When $\epsilon = 0$,* $\bar{\mathbf{K}} = -1$.

Again Theorem 1.3 is inconsistent with Yasuda-Shimada's result in dimension two, since α does not have constant sectional curvature when $\epsilon \neq 0$.

In a recent paper by Bao-Robles [BaRo], they characterize Randers metrics with constant flag curvature by three equations, giving a correct version of the theorem in [YaSh]. Bao-Robles show that the Yasuda-Shimada theorem is still true for Randers metrics under an additional condition. Moreover, they use the technique in [Sh2] to construct two-dimensional Randers metrics with the Gauss curvature $\mathbf{K} = \mathbf{K}(x)$ independent of the directions, a three-dimensional Randers metric on S^3 with $\mathbf{K} = 1$ and a three-dimensional Randers metric on B^3 with $\mathbf{K} = -1$. We should point out that their example on S^3 is not equivalent to that in [BaSh]. With the examples in [BaSh][BaRo][Sh2], now we have non-projectively flat Randers metrics with constant flag curvature of any sign in higher dimensions. Recently, the author learns that Matsumoto-Shimada have written a joint paper [MaSh], giving a correct version of the result in citeYaSh.

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2 Preliminaries

Let F be a Finsler metric on a manifold M. In a standard local coordinate system (x^i, y^i) in TM, $F = F(x, y)$ is a function of (x^i, y^i). Let

$$g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$$

and $(g^{ij}) := (g_{ij})^{-1}$. The geodesics of F are characterized locally by

$$\frac{d^2 x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where

$$G^i = \frac{1}{4} g^{ik}\Big\{2\frac{\partial g_{pk}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^k}\Big\} y^p y^q.$$

The coefficients of the Riemann curvature $\mathbf{R_y} = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}$ are given by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{9}$$

F is said to be of constant flag curvature $\mathbf{K} = \lambda$, if

$$R^i{}_k = \lambda\Big\{F^2 \delta^i_k - F F_{y^k} y^i\Big\}.$$

When $F = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $R^i{}_k = R_j{}^i{}_{kl}(x) y^j y^l$, where $R_j{}^i{}_{kl}(x)$ denote the coefficients of the usual Riemannian curvature tensor. Thus the quantity $\mathbf{R_y}$ in Finsler geometry is still called the Riemann curvature.

There are many interesting non-Riemannian quantities in Finsler geometry. In this paper, we will only discuss the S-curvature [Sh1]. Express the (Busemann-Hausdorff) volume form of F by

$$dV_F = \sigma(x) dx^1 \cdots dx^n.$$

The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - \frac{y^i}{\sigma(x)} \frac{\partial \sigma}{\partial x^i}(x). \tag{10}$$

See [Sh1] for a related discussion on the S-curvature.

Randers metrics are among the simplest non-Riemannian Finsler metrics, so that many well-known geometric quantities are computable.

Let $F = \alpha + \beta$ be a Randers metric on a manifold M, where

$$\alpha(y) = \sqrt{a_{ij}(x) y^i y^j}, \qquad \beta(y) = b_i(x) y^i$$

with $\|\beta\|_x := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$. Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j \theta_i^{\ j},$$

where $\theta^i := dx^i$ and $\theta_i^{\ j} := \tilde{\Gamma}^j_{ik} dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big),$$

$$s^i_{\ j} := a^{ih} s_{hj}, \qquad s_j := b_i s^i_{\ j}, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then G^i are given by

$$G^i = \bar{G}^i + \frac{e_{00}}{2F} y^i - s_0 y^i + \alpha s^i_{\ 0}, \tag{11}$$

where $e_{00} := e_{ij} y^i y^j$, $s_0 := s_i y^i$, $s^i_{\ 0} := s^i_{\ j} y^j$ and $\bar{G}^i$ denote the geodesic coefficients of α. See [AIM].

According to Lemma 3.1 in [ChSh],

$$\mathbf{S} = c(n+1)F \quad \Longleftrightarrow \quad e_{00} = 2c(\alpha^2 - \beta^2). \tag{12}$$

where $c = c(x)$ is a scalar function. See also Proposition 5.1 in [Sh2] in the case when $c = 0$.

Assume that $\mathbf{S} = c(n+1)F$ for some constant c. Then

$$G^i = \bar{G}^i + c(\alpha - \beta) y^i - s_0 y^i + \alpha s^i_{\ 0}. \tag{13}$$

By a direct computation, one obtains a formula for the Riemann curvature is given by

$$
\begin{aligned}
R^i{}_k &= \bar{R}^i{}_k + 3c^2\Big(\alpha^2\delta^i_k - y^i y_k\Big) - c^2\beta\Big(\beta\delta^i_k - b_k y^i\Big) \\
&\quad + \Big(s_{0|0}\delta^i_k - s_{0|k}y^i\Big) + s_0\Big(s_0\delta^i_k - s_k y^i\Big) + \Big(s_{k|0} - s_{0|k}\Big)y^i \\
&\quad - \Big(\alpha^2 s^i{}_j s^j{}_k - y_k s^i{}_j s^j{}_0\Big) + 6c s_{k0} y^i + 3 s_{k0} s^i{}_0 \\
&\quad - \Big\{(c^2\beta + 2cs_0 + s_j s^j{}_0)\Big(\alpha^2\delta^i_k - y_k y^i\Big) + c^2\alpha^2\Big(\beta\delta^i_k - b_k y^i\Big) \\
&\quad + 2c\alpha^2\Big(s_0\delta^i_k - s_k y^i\Big) - \Big(\alpha^2 s^i{}_{0|k} - y_k s^i{}_{0|0}\Big) \\
&\quad + \alpha^2\Big(s_j s^j{}_0\delta^i_k - s_j s^j{}_k y^i\Big) + \alpha^2\Big(s^i{}_{k|0} - s^i{}_{0|k}\Big)\Big\}\alpha^{-1}. \qquad (14)
\end{aligned}
$$

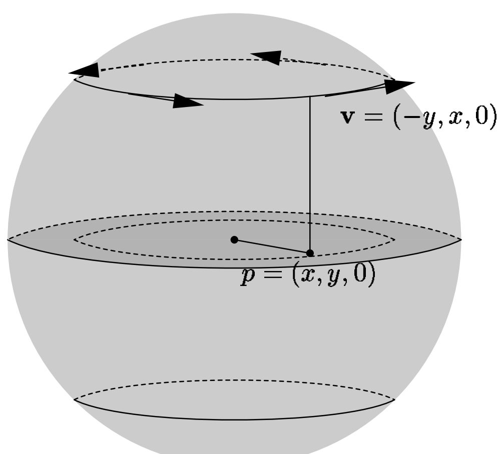
Taking the trace of $R^i{}_k$, we obtain a formula for the Ricci curvature **Ric** of F which is expressed in terms of the Ricci curvature $\overline{\mathbf{Ric}}$ of α and the covariant derivatives of β with respect to α.

$$
\begin{aligned}
\mathbf{Ric} &= \overline{\mathbf{Ric}} + (n-1)\Big\{c^2(\alpha^2+\beta^2) + 2c^2(\alpha^2-\beta^2) + s_{0|0} + s_0 s_0\Big\} \\
&\quad + 2s_{k0}s^k{}_0 - \alpha^2 s^k{}_j s^j{}_k \\
&\quad + \Big\{2s^k{}_{0|k} - (n-1)\Big(4cs_0 + 2s_j s^j{}_0 + 2c^2\beta\Big)\Big\}\alpha. \qquad (15)
\end{aligned}
$$

3 Proof of Theorem 1.1

The Finsler metric in Theorem 1.1 is constructed by solving the equation (1), i.e.,

$$
\Phi\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}\Big) = \sqrt{h\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v},\ \frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}\Big)} = 1. \qquad (16)
$$



Let $\psi : \mathrm{R}^2 \to \mathbb{S}^2_+$ by

$$\psi(x, y) := \Big(\frac{x}{\sqrt{1+x^2+y^2}}, \ \frac{y}{\sqrt{1+x^2+y^2}}, \ \frac{1}{\sqrt{1+x^2+y^2}}\Big).$$

With this map, the standard Riemannian metric Φ on $\mathbb{S}^2$ can be expressed on R^2 by

$$\Phi(\mathbf{y}) = \frac{\sqrt{(u^2+v^2)+(xv-yu)^2}}{1+x^2+y^2},$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \in T_{(x,y)}\mathrm{R}^2$. The Finsler metric defined by (16) is a Randers metric $F = \alpha + \beta$, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha : = \frac{\sqrt{\Big(1+(1-\epsilon^2)(x^2+y^2)\Big)(u^2+v^2) + \Big(1+\epsilon^2+x^2+y^2\Big)(xv-yu)^2}}{\Big(1+(1-\epsilon^2)(x^2+y^2)\Big)\sqrt{1+x^2+y^2}}$$

$$\beta : = -\frac{\epsilon(xv-yu)}{1+(1-\epsilon^2)(x^2+y^2)}.$$

Note that when $|\epsilon| > 1$, F is defined only on the open disk $\mathbb{D}^2(r)$ of radius $r = 1/\sqrt{\epsilon^2-1}$. The corresponding domain on $\mathbb{S}^2$ is a metric disk $B(\rho)$ around the north pole with radius $\rho = \sin^{-1}(1/|\epsilon|)$.

To compute the curvatures of F, we express it in a polar coordinate system, $x = r\cos(\theta)$, $y = r\sin(\theta)$. For $\mathbf{y} = \mu\frac{\partial}{\partial r} + \nu\frac{\partial}{\partial \theta}$, $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha = \frac{\sqrt{\Big(1+(1-\epsilon^2)r^2\Big)\mu^2 + r^2\Big(1+r^2\Big)^2\nu^2}}{\Big(1+(1-\epsilon^2)r^2\Big)\sqrt{1+r^2}}$$

$$\beta = -\frac{\epsilon r^2 \nu}{1+(1-\epsilon^2)r^2}$$

Express $\alpha = \sqrt{a_{11}\mu^2 + a_{12}\mu\nu + a_{21}\nu\mu + a_{22}\nu^2}$ and $\beta = b_1\mu + b_2\nu$, where

$$a_{11} = \frac{1}{(1+r^2)(1+(1-\epsilon^2)r^2)}, \qquad a_{12} = 0 = a_{21}, \qquad a_{22} = \frac{r^2(1+r^2)}{(1+(1-\epsilon^2)r^2)^2},$$

$$b_1 = 0, \qquad b_2 = -\frac{\epsilon r^2}{1+(1-\epsilon^2)r^2}.$$

The geodesic coefficients $\bar{G}^1$ and $\bar{G}^2$ of α are given by

$$\bar{G}^1 = \frac{\Big(1+(1-\epsilon^2)(1+2r^2)\Big)r}{2(1+r^2)\Big(1+(1-\epsilon^2)r^2\Big)}\,\mu^2 - \frac{(1+r^2)\Big(1+2r^2-(1-\epsilon^2)r^2\Big)r}{2\Big(1+(1-\epsilon^2)r^2\Big)^2}\,\nu^2$$

$$\bar{G}^2 = \frac{1+2r^2-(1-\epsilon^2)r^2}{(1+r^2)\Big(1+(1-\epsilon^2)r^2\Big)r}\,\mu\nu$$

We immediately obtain the Gauss curvature $\bar{\mathbf{K}}$ of α,

$$\bar{\mathbf{K}} = \frac{1-5\epsilon^2+(1-\epsilon^4)r^2}{1+(1-\epsilon^2)r^2}.$$

Note that for $\epsilon = \pm 1$, α has negative constant Gauss curvature,

$$\bar{\mathbf{K}} = -4.$$

Now we are going to find the geodesic coefficients G^1 and G^2 of F. By (11), we first compute r_{ij}, ${s^i}_j$ and s_i, etc. A direct computation yields that

$$\begin{aligned}
r_{11} &= 0 = r_{22} \\
r_{12} &= \frac{\epsilon^3 r^3}{(1+r^2)\Big(1+(1-\epsilon^2)r^2\Big)^2} = r_{21} \\
s_{11} &= 0 = s_{22} \\
s_{12} &= \frac{\epsilon r}{\Big(1+(1-\epsilon^2)r^2\Big)^2} = -s_{21} \\
{s^1}_1 &= 0 = {s^2}_2 \\
{s^1}_2 &= \frac{\epsilon r(1+r^2)}{1+(1-\epsilon^2)r^2} \\
{s^2}_1 &= -\frac{\epsilon}{r(1+r^2)} \\
s_1 &= \frac{\epsilon^2 r}{(1+r^2)\Big(1+(1-\epsilon^2)r^2\Big)} \\
s_2 &= 0.
\end{aligned}$$

We obtain that

$$e_{ij} := r_{ij} + b_i s_j + b_j s_i = 0$$

This is equivalent to that $\mathbf{S} = 0$. By (13) and the above identities, we obtain

$$\begin{aligned}
G^1 &= \bar{G}^1 - \frac{\epsilon^2 r}{(1+r^2)\Big(1+(1-\epsilon^2)r^2\Big)}\,\mu^2 + \frac{\epsilon r(1+r^2)}{1+(1-\epsilon^2)r^2}\,\alpha\,\nu \\
G^2 &= \bar{G}^2 - \frac{\epsilon^2 r}{(1+r^2)(1+(1-\epsilon^2)r^2)}\,\mu\nu - \frac{\epsilon}{r(1+r^2)}\,\alpha\,\mu
\end{aligned}$$

Plugging them into (9), we obtain

$$R^i{}_k = F^2\Big\{\delta^i_k - \frac{F_{y^k}}{F}y^i\Big\}. \tag{17}$$

We conclude that the Gauss curvature $\mathbf{K} = 1$.

We can also use (15) and the above identities to verify that $\mathbf{K} = 1$. To do so, it suffices to compute $s_{0|0}$ and $s^k{}_{0|k}$. They are given by

$$
\begin{aligned}
s_{0|0} &= \frac{\epsilon^2\Big(1-(1-\epsilon^2)r^4\Big)}{\Big(1+r^2\Big)^2\Big(1+(1-\epsilon^2)r^2\Big)^2}\mu^2 + \frac{\epsilon^2 r^2\Big(1+(1+\epsilon^2)r^2\Big)}{\Big(1+(1-\epsilon^2)r^2\Big)^3}\nu^2 \\
s^k{}_{0|k} &= -\frac{\epsilon(1-\epsilon^2)r^2(1+r^2)\nu}{\Big(1+(1-\epsilon^2)r^2\Big)^2}.
\end{aligned}
$$

Plugging them into (15) gives

$$\mathbf{Ric} = F^2.$$

We conclude that $\mathbf{K} = \mathbf{Ric}/F^2 = 1$.

Remark 3.1 Express the spherical metric in a radial form

$$\Phi(\mathbf{y}) = \sqrt{u^2 + \sin^2(r)v^2},$$

where $\mathbf{y} = u\frac{\partial}{\partial r} + v\frac{\partial}{\partial \theta} \in T_{(r,\theta)}((0,\infty)\times \mathrm{S}^1)$. Take $\mathbf{v} = \frac{\partial}{\partial\theta} \in T_{(r,\theta)}((0,\infty)\times \mathrm{S}^1)$ and define F by (1). We obtain

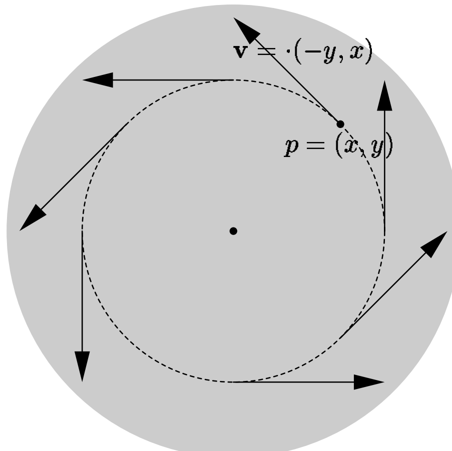
$$F = \frac{\sqrt{\Big(1-\epsilon^2\sin^2(r)\Big)u^2 + \sin^2(r)v^2} - \epsilon\sin^2(r)v}{1-\epsilon^2\sin^2(r)}. \tag{18}$$

F satisfies that $\mathbf{K} = 1$ and $\mathbf{S} = 0$, but it is not locally projectively flat.

4 Proof of Theorem 1.2

The Finsler metric in Theorem 1.2 is also constructed by solving the equation (1), i.e.,

$$\Phi\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}\Big) = \sqrt{h\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v},\ \frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}\Big)} = 1. \tag{19}$$



The Klein metric Φ on $\mathbb{D}^2$ is given by

$$\Phi(\mathbf{y}) = \frac{\sqrt{(u^2+v^2) - (xv - yu)^2}}{1 - (x^2+y^2)},$$

where $\mathbf{y} = (u, v) \in T_{(x,y)}\mathrm{R}^2$. The Finsler metric defined by (19) is a Randers metric $F = \alpha + \beta$, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha := \frac{\sqrt{\Big(1 - (1+\epsilon^2)(x^2+y^2)\Big)(u^2+v^2) - \Big(1 - \epsilon^2 - (x^2+y^2)\Big)(xv - yu)^2}}{\Big(1 - (1+\epsilon^2)(x^2+y^2)\Big)\sqrt{1 - x^2 - y^2}}$$

$$\beta := -\frac{\epsilon(xv - yu)}{1 - (1+\epsilon^2)(x^2+y^2)}.$$

To compute the curvatures of F, we take a polar coordinate system, $x = r\cos(\theta), y = r\sin(\theta)$. For a vector $\mathbf{y} = \mu\frac{\partial}{\partial r} + \nu\frac{\partial}{\partial \theta}$, $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha = \frac{\sqrt{\Big(1 - (1+\epsilon^2)r^2\Big)\mu^2 + r^2\Big(1 - r^2\Big)^2\nu^2}}{\Big(1 - (1+\epsilon^2)r^2\Big)\sqrt{1 - r^2}}$$

$$\beta = -\frac{\epsilon r^2 \nu}{1 - (1+\epsilon^2)r^2}.$$

Express $\alpha = \sqrt{a_{11}\mu^2 + a_{12}\mu\nu + a_{21}\nu\mu + a_{22}\nu^2}$ and $\beta = b_1\mu + b_2\nu$, where

$$a_{11} = \frac{1}{\Big(1 - (1+\epsilon^2)r^2\Big)(1 - r^2)} \qquad a_{12} = 0 = a_{21}, \qquad a_{22} = \frac{r^2(1-r^2)}{\Big(1 - (1+\epsilon^2)r^2\Big)^2},$$

$$b_1 = 0, \qquad b_2 = -\frac{\epsilon r^2}{1-(1+\epsilon^2)r^2}.$$

The geodesic coefficients $\bar{G}^1$ and $\bar{G}^2$ of α are given by

$$\begin{aligned} \bar{G}^1 &= \frac{\Big(1+(1+\epsilon^2)(1-2r^2)\Big)r}{2(1-r^2)\Big(1-(1+\epsilon^2)r^2\Big)}\,\mu^2 - \frac{\Big(1-2r^2+(1+\epsilon^2)r^2\Big)(1-r^2)r}{2\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\nu^2 \\ \bar{G}^2 &= \frac{1-2r^2+(1+\epsilon^2)r^2}{(1-r^2)\Big(1-(1+\epsilon^2)r^2\Big)r}\,\mu\nu \end{aligned}$$

The Gauss curvature $\bar{\mathbf{K}}$ of α is given by

$$\bar{\mathbf{K}} = \frac{-1-5\epsilon^2+(1-\epsilon^4)r^2}{1-(1+\epsilon^2)r^2}. \tag{20}$$

We see that $\bar{\mathbf{K}}$ is not a constant unless $\epsilon = 0$.

Now we are going to find the geodesic coefficients G^1 and G^2 of $F = \alpha + \beta$. Let $r_{ij}, s_{ij}, s^i{}_j, s_j$ and e_{ij} as above. A direct computation yields that

$$\begin{aligned} r_{11} &= 0 = r_{22} \\ r_{12} &= \frac{\epsilon^3 r^3}{(1-r^2)\Big(1-(1+\epsilon^2)r^2\Big)^2} = r_{21} \\ s_{11} &= 0 = s_{22} \\ s_{12} &= \frac{\epsilon r}{\Big(1-(1+\epsilon^2)r^2\Big)^2} = -s_{21} \\ s^1{}_1 &= 0 = s^2{}_2 \\ s^1{}_2 &= \frac{\epsilon r(1-r^2)}{1-(1+\epsilon^2)r^2} \\ s^2{}_1 &= -\frac{\epsilon}{(1-r^2)r} \\ s_1 &= \frac{\epsilon^2 r}{(1-r^2)\Big(1-(1+\epsilon^2)r^2\Big)} \\ s_2 &= 0. \end{aligned}$$

We immediately see that

$$e_{ij} := r_{ij} + b_i s_j + b_j s_i = 0.$$

Thus the S-curvature vanishes, $\mathbf{S} = 0$. By (13) and the above identities, we obtain

$$\begin{aligned} G^1 &= \bar{G}^1 - \frac{\epsilon^2 r}{(1-r^2)(1-(1+\epsilon^2)r^2)}\,\mu^2 + \frac{\epsilon r(1-r^2)}{1-(1+\epsilon^2)r^2}\,\alpha\nu \\ G^2 &= \bar{G}^2 - \frac{\epsilon^2 r}{(1-r^2)(1-(1+\epsilon^2)r^2)}\,\mu\nu - \frac{\epsilon}{(1-r^2)r}\,\alpha\mu. \end{aligned}$$

Plugging them into (9), we immediately obtain

$$R^i{}_k = -\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}. \tag{21}$$

Thus the Gauss curvature $\mathbf{K} = -1$.

We can also use (15) and the above identities to verify that $\mathbf{K} = -1$. To do so, it suffices to compute $s_{0|0}$ and $s^k{}_{0|k}$. They are given by

$$\begin{aligned} s_{0|0} &= \frac{\epsilon^2\Big(1-(1+\epsilon^2)r^4\Big)}{\Big(1-r^2\Big)^2\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\mu^2 + \frac{\epsilon^2 r^2\Big(1-(1-\epsilon^2)r^2\Big)}{\Big(1-(1+\epsilon^2)r^2\Big)^3}\,\nu^2, \\ s^k{}_{0|k} &= \frac{\epsilon(1+\epsilon^2)r^2(1-r^2)}{\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\nu. \end{aligned}$$

Plugging them into (15), we obtain

$$\mathbf{Ric} = -F^2.$$

Again, we conclude that $\mathbf{K} = \mathbf{Ric}/F^2 = -1$.

Remark 4.1 Express the Klein metric in the radial form,

$$\Phi(\mathbf{y}) = \sqrt{u^2 + \sinh^2(r)v^2},$$

where $\mathbf{y} = u\frac{\partial}{\partial r} + v\frac{\partial}{\partial \theta} \in T_{(r,\theta)}((0,\infty)\times \mathrm{S}^1)$. Take $\mathbf{v} = \frac{\partial}{\partial\theta} \in T_{(r,\theta)}(\mathrm{R}\times\mathrm{S}^1)$ and define F by (1). We obtain

$$F = \frac{\sqrt{\Big(1-\epsilon^2\sinh^2(r)\Big)u^2 + \sinh^2(r)v^2} - \epsilon\sinh^2(r)v}{1-\epsilon^2\sinh^2(r)}, \tag{22}$$

where $\mathbf{y} = u\frac{\partial}{\partial r} + v\frac{\partial}{\partial \theta} \in T_{(r,\theta)}((0,\infty)\times \mathrm{S}^1)$. F satisfies $\mathbf{K} = -1$ and $\mathbf{S} = 0$, but it is not locally projectively flat.

The Poincare metric on the disk $\mathbb{D}^2$ is given by

$$\Phi(\mathbf{y}) = \frac{2\sqrt{u^2+v^2}}{1-x^2-y^2}, \tag{23}$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \in T_{(x,y)}\mathbb{D}^2$. The Poincare metric has negative constant sectional curvature $\mathbf{K} = -1$. Take $\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in T_{(x,y)}\mathbb{D}^2$ and define F by (1). We obtain

$$F = \frac{\sqrt{\epsilon^2(xv-yu)^2 + (u^2+v^2)\Big(\frac{1}{4}(1-x^2-y^2)^2 - \epsilon^2(x^2+y^2)\Big)} - \epsilon(xv-yu)}{\frac{1}{4}(1-x^2-y^2)^2 - \epsilon^2(x^2+y^2)}. \tag{24}$$

F satisfies $\mathbf{K} = -1$ and $\mathbf{S} = 0$, but it is not locally projectively flat.

The Riemannian metric α from Theorem 1.1 is given by

$$\Phi(\mathbf{y}) := \sqrt{\frac{u^2 + v^2 + (xv - yu)^2}{1 + x^2 + y^2} + (xv - yu)^2},$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \in T_{(x,y)}\mathrm{R}^2$. Φ has constant sectional curvature $\mathbf{K} = -4$. Take $\mathbf{v}_p = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ at $p = (x, y)$ and define F by (1). We obtain a Randers metric $F = \alpha + \beta$, where $\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\begin{aligned} \alpha : &= \frac{\sqrt{u^2 + v^2 + (2 + x^2 + y^2)(xv - yu)^2 - \epsilon^2(xu + yv)^2(1 + x^2 + y^2)}}{\sqrt{1 + x^2 + y^2}\Big(1 - \epsilon^2(x^2 + y^2)(1 + x^2 + y^2)\Big)} \\ \beta : &= -\frac{\epsilon(1 + x^2 + y^2)(xv - yu)}{1 - \epsilon^2(x^2 + y^2)(1 + x^2 + y^2)}. \end{aligned}$$

F satisfies $\mathbf{K} = -4$ and $\mathbf{S} = 0$, but it is not locally projectively flat when $\epsilon \neq 0$.

5 Proof of Theorem 1.3

Let Φ denote the Funk metric on $\mathbb{D}^2$. It is given by

$$\Phi(\mathbf{y}) = \frac{\sqrt{(u^2 + v^2) - (xv - yu)^2} + xu + yv}{1 - x^2 - y^2},$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \in T_{(x,y)}\mathbb{D}^2$. The Finsler metric in Theorem 1.3 is defined by (8). Solving the equation (8), we obtain

$$F := \frac{\sqrt{u^2 + v^2 - \Big(\epsilon(xu + yv) + (xv - yu)\Big)^2} + (xu + yv) - \epsilon(xv - yu)}{1 - (1 + \epsilon^2)(x^2 + y^2)}. \tag{25}$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \in T_{(x,y)}\mathrm{R}^2$. $F = \alpha + \beta$ is a Randers metric on the disk $\mathbb{D}^2(\rho)$ with $\rho = 1/\sqrt{1 + \epsilon^2}$, where α and β are given by

$$\begin{aligned} \alpha &= \frac{\sqrt{u^2 + v^2 - \Big(\epsilon(xu + yv) + (xv - yu)\Big)^2}}{1 - (1 + \epsilon^2)(x^2 + y^2)} \\ \beta &= \frac{(xu + yv) - \epsilon(xv - yu)}{1 - (1 + \epsilon^2)(x^2 + y^2)} \end{aligned}$$

To compute the curvatures of $F = \alpha + \beta$, we express the Randers metric in a polar coordinate system $x = r\cos\theta, y = r\sin\theta$. For a vector $\mathbf{y} = \mu\frac{\partial}{\partial r} + \nu\frac{\partial}{\partial \theta}$,

$\alpha = \alpha(\mathbf{y})$ and $\beta = \beta(\mathbf{y})$ are given by

$$\alpha = \frac{\sqrt{\mu^2 + r^2\nu^2 - r^2\Big(r\nu + \epsilon\mu\Big)^2}}{1-(1+\epsilon^2)r^2}$$
$$\beta = \frac{r\mu - \epsilon r^2\nu}{1-(1+\epsilon^2)r^2}.$$

Express $\alpha = \sqrt{a_{11}\mu^2 + a_{12}\mu\nu + a_{21}\nu\mu + a_{22}\nu^2}$ and $\beta = b_1\mu + b_2\nu$, where

$$a_{11} = \frac{1-\epsilon^2 r^2}{\Big(1-(1+\epsilon^2)r^2\Big)^2},$$
$$a_{12} = -\frac{\epsilon r^3}{\Big(1-(1+\epsilon^2)r^2\Big)^2} = a_{21},$$
$$a_{22} = \frac{r^2(1-r^2)}{\Big(1-(1+\epsilon^2)r^2\Big)^2},$$

$$b_1 = \frac{r}{1-(1+\epsilon^2)r^2}, \qquad b_2 = -\frac{\epsilon r^2}{1-(1+\epsilon^2)r^2}.$$

The geodesic coefficients $\bar{G}^1$ and $\bar{G}^2$ of α are given by

$$\bar{G}^1 = \frac{\Big(\epsilon^2 - 5\epsilon^2 r^2 - \epsilon^4 r^2 + 2 - 2r^2\Big)r}{2\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\mu^2 + \frac{\epsilon\Big(1-r^2+\epsilon^2 r^2\Big)r^2}{\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\mu\nu - \frac{\Big(1-r^2+\epsilon^2 r^2\Big)(1-r^2)r}{2\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\nu^2$$

$$\bar{G}^2 = \frac{\epsilon\Big(-3+r^2+3\epsilon^2 r^2\Big)}{2\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\mu^2 + \frac{(1-\epsilon^2 r^2)\Big(1-r^2+\epsilon^2 r^2\Big)}{\Big(1-(1+\epsilon^2)r^2\Big)^2 r}\,\mu\nu - \frac{\epsilon\Big(1-r^2+\epsilon^2 r^2\Big)r^2}{2\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\nu^2$$

The Gauss curvature $\bar{\mathbf{K}}$ of α is given by

$$\bar{\mathbf{K}} = \frac{-1-5\epsilon^2+(1-\epsilon^4)r^2}{1-(1+\epsilon^2)r^2}. \tag{26}$$

We see that $\bar{\mathbf{K}}$ is not a constant unless $\epsilon = 0$.

Now we are going to find the geodesic coefficients G^1 and G^2 of $F = \alpha + \beta$. Let $r_{ij}, s_{ij}, s^i{}_j, s_j$ and e_{ij} as above. A direct computation yields that

$$\begin{aligned}
r_{11} &= \frac{1 - r^2 - 3\epsilon^2 r^2}{\Big(1 - (1+\epsilon^2)r^2\Big)^2} \\
r_{12} &= -\frac{\epsilon(1-\epsilon^2)r^3}{\Big(1 - (1+\epsilon^2)r^2\Big)^2} \;=\; r_{21} \\
r_{22} &= \frac{\Big(1 - r^2 + \epsilon^2 r^2\Big)r^2}{\Big(1 - (1+\epsilon^2)r^2\Big)^2} \\
s_{11} &= 0 \;=\; s_{22} \\
s_{12} &= \frac{\epsilon r}{\Big(1 - (1+\epsilon^2)r^2\Big)^2} \;=\; -s_{21} \\
s^1{}_1 &= -\frac{\epsilon^2 r^2}{1 - (1+\epsilon^2)r^2} \;=\; -s^2{}_2 \\
s^1{}_2 &= \frac{\epsilon r(1-r^2)}{1 - (1+\epsilon^2)r^2} \\
s^2{}_1 &= -\frac{\epsilon(1-\epsilon^2 r^2)}{(1 - (1+\epsilon^2)r^2)r} \\
s_1 &= \frac{\epsilon^2 r}{1 - (1+\epsilon^2)r^2} \\
s_2 &= \frac{\epsilon r^2}{1 - (1+\epsilon^2)r^2}.
\end{aligned}$$

We immediately see that

$$e_{ij} := r_{ij} + b_i s_j + b_j s_i = a_{ij} - b_i b_j. \tag{27}$$

By Lemma 3.1 in [ChSh], (27) is equivalent to that

$$\mathbf{S} = \frac{3}{2}F.$$

By (13) and the above identities, we obtain

$$\begin{aligned}
G^1 &= \bar{G}^1 + \frac{1}{2}(\alpha - \beta)\mu - \frac{\epsilon r\Big(\epsilon\mu + r\nu\Big)}{1 - (1+\epsilon^2)r^2}\,\mu - \frac{\epsilon r\Big(\epsilon r\mu - \nu + r^2\nu\Big)}{1 - (1+\epsilon^2)r^2}\,\alpha \\
G^2 &= \bar{G}^2 + \frac{1}{2}(\alpha - \beta)\nu - \frac{\epsilon r\Big(\epsilon\mu + r\nu\Big)}{1 - (1+\epsilon^2)r^2}\,\nu - \frac{\epsilon\Big(\mu - \epsilon^2 r^2\mu - \epsilon r^3\nu\Big)}{r\Big(1 - (1+\epsilon^2)r^2\Big)}\,\alpha
\end{aligned}$$

Plugging them into (9), we immediately obtain

$$R^i{}_k = -\frac{1}{4}\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}.$$

Thus the Gauss curvature $\mathbf{K} = -1/4$.

We can also use (15) to verify that $\mathbf{K} = -1/4$. To do so, it suffices to compute $s_{0|0}$ and $s^k{}_{0|k}$. They are given by

$$\begin{aligned} s_{0|0} &= \frac{\epsilon^2(1+r^2-\epsilon^2 r^2)}{\Big(1-(1+\epsilon^2)r^2\Big)^3}\,\mu^2 - \frac{4\epsilon^3 r^3}{\Big(1-(1+\epsilon^2)r^2\Big)^3}\,\mu\nu + \frac{\epsilon^2 r^2(1-r^2+\epsilon^2 r^2)}{\Big(1-(1+\epsilon^2)r^2\Big)^3}\,\nu^2 \\ s^k{}_{0|k} &= -\frac{\epsilon^2(1+\epsilon^2)r^3}{\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\mu + \frac{\epsilon(1+\epsilon^2)r^2(1-r^2)}{\Big(1-(1+\epsilon^2)r^2\Big)^2}\,\nu. \end{aligned}$$

Plugging $c = 1/2$ and the above identities into (15) gives

$$\mathbf{Ric} = -\frac{1}{4}F^2.$$

We conclude that $\mathbf{K} = \mathbf{Ric}/F^2 = -1/4$.

Remark 5.1 Below is a byproduct. Let

$$\begin{aligned} \alpha : &= \frac{\sqrt{u^2+v^2-\Big(\epsilon(xu+yv)+(xv-yu)\Big)^2}}{1-(1+\epsilon^2)(x^2+y^2)} \\ \tilde{\alpha} : &= \frac{\sqrt{\Big(1-(1+\epsilon^2)(x^2+y^2)\Big)(u^2+v^2)-\Big(1-\epsilon^2-(x^2+y^2)\Big)(xv-yu)^2}}{\Big(1-(1+\epsilon^2)(x^2+y^2)\Big)\sqrt{1-x^2-y^2}}. \end{aligned}$$

α and $\tilde{\alpha}$ are two Riemannian metrics on $\mathbb{D}^2(\rho)$ with radius $\rho = 1/\sqrt{1+\epsilon^2}$. According to (20) and (26), The Gauss curvatures of α and $\tilde{\alpha}$ are equal and given by

$$\bar{\mathbf{K}} = \frac{-1-5\epsilon^2+(1-\epsilon^4)(x^2+y^2)}{1-(1+\epsilon^2)(x^2+y^2)}.$$

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