

# A Characterization of Randers Metrics of Scalar Flag Curvature

Z. Shen\* and G.C. Yildirim

## Abstract

Randers metrics form a rich class of Finsler metrics. In the paper we find a sufficient and necessary condition that characterizes Randers metrics of scalar flag curvature. We also discuss Randers metrics of weakly isotropic flag curvature.

## 1 Introduction

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. The flag curvature on a Finsler manifold  $(M, F)$  is a function  $\mathbf{K} = \mathbf{K}(P, y)$ , where  $P \subset T_x M$  is a two-dimensional plane and  $y \in P \setminus \{0\}$ .  $F$  is said to be of *scalar flag curvature* if  $\mathbf{K} = \sigma(x, y)$  is independent of  $P$  containing  $y \in T_x M$ . It is said to be of *weakly isotropic flag curvature* if  $\mathbf{K} = 3\theta/F + \sigma(x)$ , where  $\theta$  is a 1-form on  $M$ , and it is said to be of *isotropic* if  $\mathbf{K} = \sigma(x)$  ( $=$  *constant* in dimension  $n \geq 3$  by the Schur lemma).

It is well-known that a Riemannian metric is of scalar flag curvature if and only if it is of isotropic sectional curvature ( $=$  *constant* in dimension  $n \geq 3$  by the Schur lemma). By Beltrami Theorem, a Riemannian metric is of constant sectional curvature if and only if it is locally projectively flat. Every locally projectively flat Finsler metric is of scalar flag curvature. There are Finsler metrics of scalar flag curvature, but not locally projectively flat, nor of weakly isotropic flag curvature [4]. There are Finsler metrics of weakly isotropic flag curvature, but not locally projectively flat, nor of isotropic flag curvature (see Example 1.1 below). Many known examples are of Randers type.

A Randers metric on a manifold  $M$  is a Finsler metric in the form  $F = \alpha + \beta$ , where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and  $\beta = b_i(x)y^i$  is a 1-form on  $M$  with  $b := \|\beta_x\|_\alpha < 1$  everywhere. Many known Finsler metrics of constant flag curvature are Randers metrics [17]. The problem to classify Randers metrics of constant flag curvature was first studied by Yasuda–Shimada [19] and Matsumoto [10]. In 2002, Bao–Robles [1] found a system of PDEs that characterize Randers metrics of constant flag curvature. The same result was simultaneously obtained by Matsumoto–Shimada [11] by a different method. Recently, using

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the results in [1], and motivated by Zermelo's navigation idea [17], Bao-Robles-Shen [3] have classified Randers metrics of constant flag curvature. However, one can not use the navigation idea to find a sufficient and necessary condition for a Randers metric to be of scalar curvature. In this paper, we take a different approach and give an affirmative answer to this problem.

Before we state our main theorems, let us introduce some notations. Let  $F = \alpha + \beta$  be a Randers metric on an  $n$ -dimensional manifold, where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and  $\beta = b_i(x)y^i$  is a 1-form on  $M$ . Let  $\nabla\beta = b_{i|j}y^i dx^j$  denote the covariant derivative of  $\beta$  with respect to  $\alpha$ . Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), & s_j &:= b^i s_{ij}, \\ q_{ij} &:= r_{im} s^m_j, & t_{ij} &:= s_{im} s^m_j, & t_j &:= b^i t_{ij}. \end{aligned}$$

Here and thereafter, we use  $a_{ij}$  to raise and lower the indices of tensors defined by  $b_i$  and  $b_{i|j}$ . We shall also denote  $y_i := a_{ij}y^j$ . The index "0" means the contraction with  $y^i$ . For example,  $t_{k0} := t_{kl}y^l$  and  $t_{00} := t_{kl}y^k y^l$ , and etc. We prove the following

**Theorem 1.1** *Let  $F = \alpha + \beta$  be a Randers metric on an  $n$ -dimensional manifold  $M$ .  $F$  is of scalar flag curvature,  $\mathbf{K} = \sigma(x, y)$ , if and only if the Riemann curvature  $\bar{R}^i_k$  of  $\alpha$  and the covariant derivatives of  $\beta$  satisfy the following equations*

$$\begin{aligned} \bar{R}^i_k &= \kappa \left\{ \alpha^2 \delta_k^i - y_k y^i \right\} + \alpha^2 t^i_k + t_{00} \delta_k^i - t_{k0} y^i - t^i_0 y_k - 3s^i_0 s_{k0}, & (1) \\ s_{ij|k} &= \frac{1}{n-1} \left\{ a_{ik} s^m_{j|m} - a_{jk} s^m_{i|m} \right\}. & (2) \end{aligned}$$

where  $\kappa = \kappa(x)$  is a scalar function on  $M$ .

There are non-Riemannian quantities, the S-curvature  $\mathbf{S}$  ([14]) and the H-curvature,  $\mathbf{H}$  ([12]). For a Finsler metric  $F = F(x, y)$  on an  $n$ -dimensional manifold, the S-curvature is isotropic  $\mathbf{S} = (n+1)c(x)F$ , then  $\mathbf{H} = \frac{1}{2}(n^2-1)\theta/F$ , where  $\theta = c_{x^m}(x)y^m$ . It is shown that for a Finsler metric  $F$  of scalar flag curvature, it is of weakly isotropic flag curvature  $\mathbf{K} = 3\theta/F + \sigma(x)$  if and only if  $\mathbf{H} = \frac{1}{2}(n^2-1)\theta/F$  ([12]). In particular, if a Finsler metric  $F$  of scalar flag curvature has isotropic S-curvature, then it must be of weakly isotropic flag curvature [7]. Therefore it is necessary to study and characterize Finsler metrics of weakly isotropic flag curvature.

**Theorem 1.2** *Let  $F = \alpha + \beta$  be a Randers metric on an  $n$ -dimensional manifold  $M$ .  $F$  is of weakly isotropic flag curvature*

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \quad (3)$$

where  $\theta = \lambda_i(x)y^i$  is a 1-form and  $\sigma = \sigma(x)$  is a scalar function on  $M$  if and only if the Riemann curvature  $\bar{R}^i_k$  of  $\alpha$  and the covariant derivatives of  $\beta$  satisfy the following equations

$$\begin{aligned} \bar{R}^i_k &= \kappa \left\{ \alpha^2 \delta_k^i - y_k y^i \right\} \\ &\quad + \alpha^2 t^i_k + t_{00} \delta_k^i - t_{k0} y^i - t^i_0 y_k - 3s^i_0 s_{k0}, \end{aligned} \quad (4)$$

$$\begin{aligned} s_{ij|k} &= a_{ik} \left\{ (\sigma + c^2) b_j + 2cs_j + t_j + 2c_{x^j} \right\} \\ &\quad - a_{jk} \left\{ (\sigma + c^2) b_i + 2cs_i + t_i + 2c_{x^i} \right\} \end{aligned} \quad (5)$$

$$r_{ij} = -b_i s_j - b_j s_i + 2c(a_{ij} - b_i b_j), \quad (6)$$

where  $c = c(x)$  is a scalar function on  $M$  and

$$\kappa = \sigma - 2c^2 + (\sigma + c^2)b^2 + t_m b^m + 2c_{x^m} b^m. \quad (7)$$

In this case  $\theta = c_{x^m} y^m$ .

Theorem 1.2 in the case when  $\theta = 0$  and  $c = 0$  is obtained in [1].

Note that (4) is same as (1) and (5) implies (2).

The local metric structure of Randers metrics of scalar flag curvature with isotropic S-curvature has been determined using the navigation idea [6]. It is known that (6) is equivalent to that the S-curvature is isotropic ([5]). By Theorem 1.2, Randers metrics of weakly isotropic flag curvature must have isotropic S-curvature. Thus the local metric structure of Randers metrics of weakly isotropic flag curvature can be completely determined. Below is a special example.

**Example 1.1** ([16]) Let  $a \in \mathbb{R}^n$  be a non-zero constant vector. Define  $F = \alpha + \beta$  on an open ball  $B^n(1/\sqrt{|a|})$  in  $\mathbb{R}^n$  by

$$\begin{aligned} F : &= \frac{\sqrt{(1 - |a|^2|x|^4)|y|^2 + (|x|^2\langle a, y \rangle - 2\langle a, x \rangle\langle x, y \rangle)^2}}{1 - |a|^2|x|^4} \\ &\quad - \frac{|x|^2\langle a, y \rangle - 2\langle a, x \rangle\langle x, y \rangle}{1 - |a|^2|x|^4}. \end{aligned}$$

$F$  is of weakly isotropic flag curvature. More precisely, the flag curvature is given by  $\mathbf{K} = 3\theta/F + \sigma$ , where  $\theta = \langle a, y \rangle$  and  $\sigma = 3\langle a, x \rangle^2 - 2|a|^2|x|^2$ . See more examples in [6].

The remaining problem is to classify Randers metrics of scalar flag curvature.

## 2 Preliminaries

A spray on a manifold  $M$  is a vector field on  $TM$  which is locally expressed in the following form

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where  $G^i = G^i(x, y)$  are local functions with the positive homogeneity:  $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ ,  $\forall \lambda > 0$ . Every Finsler metric  $F$  induces a spray by

$$G^i(x, y) = \frac{g^{il}}{4} \left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}.$$

Every spray  $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$  on a manifold  $M$  determines a local basis  $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}$ , for  $T(TM \setminus \{0\})$ , where  $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - \frac{G^j}{\partial y^j} \frac{\partial}{\partial y^i}$ . The Riemann curvature  $R = R^i_k \frac{\delta}{\delta x^i} \otimes dx^k$  of  $G$  is defined by

$$R^i_k := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature is the trace of the Riemann curvature,

$$\text{Ric} := R^m_m.$$

Let

$$A^i_k := R^i_k - R \delta^i_k, \quad R := \frac{R^m_m}{n-1}.$$

Then the Weyl curvature  $W = W^i_k \frac{\delta}{\delta x^i} \otimes dx^k$  of  $G$  is defined by

$$W^i_k := A^i_k - \frac{1}{n+1} \frac{\partial A^m_k}{\partial y^m} y^i.$$

A spray  $\tilde{G} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i \frac{\partial}{\partial y^i}$  is projectively equivalent to  $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$  if their spray coefficients are related by

$$G^i = \tilde{G}^i + P y^i.$$

An important fact is that the Weyl curvature  $W = W^i_k \frac{\delta}{\delta x^i} \otimes dx^k$  of  $G$  is equal to the Weyl curvature  $\tilde{W} = \tilde{W}^i_k \frac{\delta}{\delta x^i} \otimes dx^k$  of  $\tilde{G}$ . Thus the Weyl curvature is a projective invariant. See [15] for more details.

Let  $F = F(x, y)$  be a Finsler metric on an  $n$ -dimensional manifold  $M$  and  $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$  the induced spray. For a tangent plane  $\Pi \subset T_x M$  and a vector  $y \in \Pi$ , the *flag curvature*  $\mathbf{K} = \mathbf{K}(\Pi, y)$  is defined by

$$\mathbf{K}(\Pi, y) := \frac{g_{ij} R^i_k y^k u^j}{[g_{ij} y^i y^j][g_{kl} u^k u^l] - [g_{ik} y^i u^k]^2},$$

where  $u \in \Pi$  such that  $\Pi = \text{span}\{y, u\}$ . It is easy to verify that  $\mathbf{K} = \sigma(x, y)$  if

$$R^i_k = \sigma(x, y) \left\{ F^2 \delta^i_k - F F_{y^k} y^i \right\}. \quad (8)$$

Moreover, (8) is equivalent to that  $W^i_k = 0$  [9] [18].

Denote the Hausdorff-Busemann volume form of  $F$  by

$$dV_F = \sigma_F(x) dx^1 \cdots dx^n,$$

where

$$\sigma_F(x) := \frac{\text{Vol}(B^n(1))}{\text{Vol}\{(y^i) | F(x, y) < 1\}}.$$

The S-curvature of  $F$  is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} \ln \sigma_F.$$

A Finsler metric is of isotropic S-curvature if

$$\mathbf{S} = (n+1)cF,$$

where  $c = c(x)$  is a scalar function. We have the following

**Lemma 2.1** ([5]) *For a Randers metric  $F = \alpha + \beta$  on an  $n$ -dimensional manifold, the S-curvature is isotropic,  $\mathbf{S} = (n+1)cF$  if and only if  $r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2)$ .*

For Finsler metrics of scalar flag curvature and isotropic S-curvature, we have following

**Theorem 2.2** ([7]) *Let  $F$  be a Finsler metric on an  $n$ -dimensional manifold. Suppose that  $F$  is of scalar flag curvature with isotropic S-curvature  $\mathbf{S} = (n+1)cF$  for some scalar function  $c = c(x)$ . Then the flag curvature takes the following form:*

$$\mathbf{K} = \frac{3c_x^m y^m}{F} + \sigma,$$

where  $\sigma = \sigma(x)$  is a scalar function.

### 3 A Formula for the Weyl Curvature

For a Randers metric  $F = \alpha + \beta$ , in order to find equations on  $\alpha$  and  $\beta$  that characterize Randers metrics of scalar flag curvature, we shall first compute the Weyl curvature.

**Lemma 3.1** *For a Randers metric  $F = \alpha + \beta$ ,*

$$\begin{aligned} W^i_k &= \bar{W}^i_k + 2\alpha s^i_{0|k} - \alpha s^i_{k|0} - \alpha^{-1} s^i_{0|0} y_k + 3s^i_0 s_{k0} + t^i_0 y_k - \alpha^2 t^i_k \\ &\quad - \frac{1}{n-1} \left\{ 2\alpha s^m_{0|m} - 2t_{00} - \alpha^2 t^m_m \right\} \delta^i_k \\ &\quad + \frac{1}{n-1} \left\{ \alpha^{-1} s^m_{0|m} y_k + \alpha s^m_{k|m} - 2t_{k0} - t^m_m y_k \right\} y^i, \end{aligned}$$

where  $\bar{W}^i_k$  denote the Weyl curvature of  $\alpha$ .

*Proof:* The spray coefficients  $G^i$  of  $F$  are given by

$$G^i = \bar{G}^i + Py^i + Q^i,$$

where  $\bar{G}^i$  denote the spray coefficients of  $\alpha$  and

$$P := \frac{r_{00} - 2\alpha s_0}{2F}, \quad Q^i := \alpha s^i_0.$$

Let

$$\tilde{G} := y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i \frac{\partial}{\partial y^i},$$

where

$$\tilde{G}^i := \bar{G}^i + Q^i.$$

Clearly,  $G$  and  $\tilde{G}$  are projectively equivalent. Thus they have the same Weyl curvature.

$$W^i_k = \tilde{W}^i_k.$$

We can compute  $W^i_k$  using  $\tilde{G}$ . First we have

$$\tilde{R}^i_k = \bar{R}^i_k + H^i_k, \quad \tilde{R} = \bar{R} + H, \quad (9)$$

where  $\tilde{R} := \frac{1}{n-1}\tilde{R}^m_m$ ,  $\bar{R} := \frac{1}{n-1}\bar{R}^m_m$ , and

$$\begin{aligned} H^i_k &:= 2Q^i_{|k} - Q^i_{|j.k}y^j + 2Q^jQ^i_{.j.k} - Q^i_{.j}Q^j_{.k}, \\ H &:= \frac{H^m_m}{n-1}. \end{aligned}$$

By (9), we get

$$\tilde{A}^i_k = \tilde{R}^i_k - \tilde{R}\delta^i_k = \bar{A}^i_k + H^i_k - H\delta^i_k.$$

Then

$$W^i_k = \tilde{W}^i_k = \tilde{A}^i_k - \frac{1}{n+1} \frac{\partial \tilde{A}^m_k}{\partial y^m} y^i = \bar{W}^i_k + \Theta^i_k,$$

where  $\bar{W}^i_k$  denote the Weyl curvature of  $\alpha$  and

$$\Theta^i_k := \left( H^i_k - H\delta^i_k \right) - \frac{1}{n+1} \frac{\partial}{\partial y^m} \left( H^m_k - H\delta^m_k \right) y^i.$$

By a direct computation we get

$$\begin{aligned} Q^i_{|k} &= \alpha s^i_{0|k}, \\ Q^i_{|j.k}y^j &= \alpha^{-1}s^i_{0|0}y_k + \alpha s^i_{k|0}, \\ Q^jQ^i_{.j.k} &= s^i_0s_{k0} + t^i_0y_k, \\ Q^i_{.j}Q^j_{.k} &= -s^i_0s_{k0} + t^i_0y_k + \alpha^2t^i_k. \end{aligned}$$

Then we obtain

$$H^i_k = 2\alpha s^i_{0|k} - \alpha s^i_{k|0} - \alpha^{-1} s^i_{0|0} y_k + 3s^i_0 s_{k0} + t^i_0 y_k - \alpha^2 t^i_k.$$

$$H = \frac{1}{n-1} \left\{ 2\alpha s^m_{0|m} - 2t_{00} - \alpha^2 t^m_m \right\}.$$

Finally, we get

$$\begin{aligned} \Theta^i_k &= 2\alpha s^i_{0|k} - \alpha s^i_{k|0} - \alpha^{-1} s^i_{0|0} y_k + 3s^i_0 s_{k0} + t^i_0 y_k - \alpha^2 t^i_k \\ &\quad - \frac{1}{n-1} \left\{ 2\alpha s^m_{0|m} - 2t_{00} - \alpha^2 t^m_m \right\} \delta^i_k \\ &\quad + \frac{1}{n-1} \left\{ \alpha^{-1} s^m_{0|m} y_k + \alpha s^m_{k|m} - 2t_{k0} - t^m_m y_k \right\} y^i. \end{aligned}$$

This proves the lemma.

Q.E.D.

## 4 Proof of Theorem 1.1

We know that a Finsler metric is of scalar flag curvature if and only if the Weyl curvature vanishes. Using the Weyl curvature, we can characterize Randers metrics of scalar flag curvature.

*Proof of Theorem 1.1:* First we need the following fact:

$$A + \alpha B + \alpha^{-1} C = 0,$$

where  $A, B$  and  $C$  are polynomials in  $y$ , if and only if

$$A = 0, \quad \alpha^2 B + C = 0.$$

Assume that  $F$  is of scalar flag curvature, then  $W^i_k = 0$ . By Lemma 4.1, we obtain the following two equations:

$$\begin{aligned} \bar{W}^i_k + 3s^i_0 s_{k0} + t^i_0 y_k - \alpha^2 t^i_k + \frac{1}{n-1} \left\{ 2t_{00} + \alpha^2 t^m_m \right\} \delta^i_k \\ - \frac{1}{n-1} \left\{ 2t_{k0} + t^m_m y_k \right\} y^i = 0. \end{aligned} \quad (10)$$

$$\begin{aligned} 2\alpha^2 s^i_{0|k} - \alpha^2 s^i_{k|0} - s^i_{0|0} y_k - \frac{1}{n-1} \left\{ 2\alpha^2 s^m_{0|m} \right\} \delta^i_k \\ + \frac{1}{n-1} \left\{ s^m_{0|m} y_k + \alpha^2 s^m_{k|m} \right\} y^i = 0, \end{aligned} \quad (11)$$

Let  $\bar{W}_{ik} := \bar{W}^m_k a_{im}$ . Rewrite (10) as follows

$$\begin{aligned} \bar{W}_{ik} &= -3s_{i0} s_{k0} - t_{i0} y_k + \alpha^2 t_{ik} \\ &\quad - \frac{2}{n-1} \left\{ t_{00} a_{ik} - t_{k0} y_i \right\} - \frac{t^m_m}{n-1} \left\{ \alpha^2 a_{ik} - y_k y_i \right\}. \end{aligned} \quad (12)$$

Then

$$\overline{W}_{ik} - \overline{W}_{ki} = \frac{n+1}{n-1} \{t_{k0}y_i - t_{i0}y_k\}. \quad (13)$$

On the other hand, letting  $\overline{\text{Ric}} = \overline{\text{Ric}}_{ij}y^i y^j$ , we have

$$\overline{W}^i_k = \overline{R}^i_k - \frac{1}{n-1} \overline{\text{Ric}}_{pq}y^p y^q \delta_k^i + \frac{1}{n-1} \overline{\text{Ric}}_{kl}y^l y^i. \quad (14)$$

Thus

$$\overline{W}_{ik} = \overline{R}_{ik} - \frac{1}{n-1} \overline{\text{Ric}}_{pq}y^p y^q a_{ik} + \frac{1}{n-1} \overline{\text{Ric}}_{kl}y^l y_i. \quad (15)$$

Using the fact that  $\overline{R}_{ik} = \overline{R}_{ki}$ , we get from (15) that

$$\overline{W}_{ik} - \overline{W}_{ki} = \frac{1}{n-1} \{ \overline{\text{Ric}}_{kl}y^l y_i - \overline{\text{Ric}}_{il}y^l y_k \}. \quad (16)$$

Comparing (16) with (13), we obtain

$$\{ \overline{\text{Ric}}_{kl}y^l - (n+1)t_{k0} \} y_i = \{ \overline{\text{Ric}}_{il}y^l - (n+1)t_{i0} \} y_k.$$

There is a scalar function  $\lambda = \lambda(x)$  such that

$$\overline{\text{Ric}}_{ij} = (n+1)t_{ij} + (n-1)\lambda a_{ij}. \quad (17)$$

Plugging (17) into (15) we obtain

$$\overline{W}_{ik} = \overline{R}_{ik} - \frac{n+1}{n-1} \{t_{00}a_{ik} - t_{k0}y_i\} - \lambda \{ \alpha^2 a_{ik} - y_k y_i \}. \quad (18)$$

By (12) and (18), we get

$$\begin{aligned} \overline{R}_{ik} &= -3s_{i0}s_{k0} + (\alpha^2 t_{ik} - t_{i0}y_k) + (t_{00}a_{ik} - t_{k0}y_i) \\ &\quad + \left( \lambda - \frac{1}{n-1} t^m_m \right) \{ \alpha^2 a_{ik} - y_i y_k \}. \end{aligned} \quad (19)$$

Letting  $\kappa := \lambda - t^m_m/(n-1)$ , we obtain (1).

Contracting (11) with  $y_i := a_{ij}y^j$  yields

$$s_{k0|0} - \frac{1}{n-1} \{ s^m_{0|m} y_k - s^m_{k|m} \alpha^2 \} = 0. \quad (20)$$

Rewrite (20) as

$$s^i_{0|0} = \frac{1}{n-1} \{ s^m_{0|m} y^i - s^m_{|m} \alpha^2 \}. \quad (21)$$

Plugging (21) into (11) we get

$$2s^i_{0|k} - s^i_{k|0} + \frac{1}{n-1} \{ s^m_{|m} y_k - 2s^m_{0|m} \delta_k^i + s^m_{k|m} y^i \} = 0. \quad (22)$$

Differentiating (22) with respect to  $y^j$  yields

$$2s^i_{j|k} - s^i_{k|j} + \frac{1}{n-1} \left\{ s^{mi}_{|m} a_{kj} - 2s^m_{j|m} \delta^i_k + s^m_{k|m} \delta^i_j \right\} = 0. \quad (23)$$

Rewriting (23) as

$$2s_{ij|k} - s_{ik|j} = -\frac{1}{n-1} \left\{ s^m_{i|m} a_{jk} - 2s^m_{j|m} a_{ik} + s^m_{k|m} a_{ij} \right\}.$$

Then

$$2s_{ik|j} - s_{ij|k} = -\frac{1}{n-1} \left\{ s^m_{i|m} a_{jk} - 2s^m_{k|m} a_{ij} + s^m_{j|m} a_{ik} \right\}.$$

From the above two equations we get (2).

Conversely, we assume that (1) and (2) hold. It suffices to prove (10) and (11) hold.

By (1), we get (17) with  $\lambda := \kappa + t^m_m/(n-1)$ . Plugging (1) and (17) into (14), we get (10).

By (2) we have

$$\begin{aligned} s^i_{0|0} &= \frac{1}{n-1} \left( s^m_{0|m} y^i - s^{mi}_{|m} \alpha^2 \right), \\ s^i_{0|k} &= \frac{1}{n-1} \left( s^m_{0|m} \delta^i_k - s^{mi}_{|m} y_k \right) \\ s^i_{k|0} &= \frac{1}{n-1} \left( s^m_{k|m} y^i - s^{mi}_{|m} y_k \right) \end{aligned}$$

Then (11) follows from the above three identities.

Q.E.D.

To find a formula for the flag curvature, we need the following

**Lemma 4.1** ([2], p 220) *Let  $F = \alpha + \beta$  be a Randers metric on a manifold  $M$ . Then the Ricci curvature of  $F$  is given by*

$$\begin{aligned} \text{Ric} &= \overline{\text{Ric}} + \left\{ 2\alpha s^m_{0|m} - 2t_{00} - \alpha^2 t^m_m \right\} \\ &+ (n-1) \left\{ \frac{3(r_{00} - 2s_0\alpha)^2}{4(\alpha + \beta)^2} + \frac{4\alpha[q_{00} - t_0\alpha] - [r_{00|0} - 2s_{0|0}\alpha]}{2(\alpha + \beta)} \right\}. \end{aligned} \quad (24)$$

where  $\overline{\text{Ric}}$  denotes the Ricci curvature of  $\alpha$ .

## 5 Weak Einstein Metrics

A Finsler metric on an  $n$ -dimensional manifold is called a *weak Einstein metric* if

$$\text{Ric} = (n-1) \left\{ \frac{3\theta}{F} + \sigma \right\} F^2, \quad (25)$$

where  $\theta$  is a 1-form and  $\sigma = \sigma(x)$  are scalar functions. Clearly, if a Finsler metric is of weakly isotropic flag curvature, then it is a weak Einstein metric.

Using (24), we can prove the following

**Proposition 5.1** *Let  $F = \alpha + \beta$  be a Randers metric on a manifold  $M$ . Then  $F$  is a weak Einstein metric satisfying (25) if and only if*

$$\begin{aligned} \overline{\text{Ric}} &= (n-1)\left\{(\sigma - 3c^2)\alpha^2 + (\sigma + c^2)\beta^2 + (3\theta - c_0)\beta - s_{0|0} - s_0^2\right\} \\ &\quad + 2t_{00} + \alpha^2 t_m^m, \end{aligned} \quad (26)$$

$$s_{0|m}^m = (n-1)\left\{(\sigma + c^2)\beta + 2cs_0 + t_0 + \frac{3\theta + c_0}{2}\right\}, \quad (27)$$

$$r_{00} = -2s_0\beta + 2c(\alpha^2 - \beta^2), \quad (28)$$

where  $c = c(x)$  is a scalar function on  $M$  and  $c_0 := c_{x^i}(x)y^i$ .

*Proof:* First we assume that the Ricci curvature of  $F$  satisfies (25). Multiplying both sides of (24) by  $4(\alpha + \beta)^2$  and using (25), we get

$$\begin{aligned} &4(\alpha + \beta)^2 \left\{ \overline{\text{Ric}} + 2\alpha s_{0|m}^m - 2t_{00} - \alpha^2 t_m^m \right\} \\ &+ (n-1) \left\{ 3(r_{00} - 2s_0\alpha)^2 + 8\alpha(q_{00} - t_0\alpha)(\alpha + \beta) \right. \\ &\left. - 2(r_{00|0} - 2s_{0|0}\alpha)(\alpha + \beta) - 4[3\theta(\alpha + \beta)^3 + \sigma(\alpha + \beta)^4] \right\} = 0. \end{aligned} \quad (29)$$

Rewriting the above equation as

$$A\alpha^4 + B\alpha^2 + C + \alpha(D\alpha^2 + E) = 0,$$

where  $A, B, C, D$ , and  $E$  are polynomials in  $y^i$ . We have

$$A\alpha^4 + B\alpha^2 + C = 0, \quad D\alpha^2 + E = 0.$$

By this, we see that (29) is equivalent to the following equations:

$$\begin{aligned} &4(\alpha^2 + \beta^2) \left\{ \overline{\text{Ric}} - 2t_{00} - \alpha^2 t_m^m \right\} + 16\alpha^2 \beta s_{0|m}^m \\ &+ (n-1) \left\{ 3(r_{00}^2 + 4s_0^2\alpha^2) + 4s_{0|0}\alpha^2 - 2r_{00|0}\beta + 8\alpha^2(q_{00} - t_0\beta) \right. \\ &\left. - 12\theta(3\alpha^2\beta + \beta^3) - 4\sigma(\alpha^4 + 6\alpha^2\beta^2 + \beta^4) \right\} = 0 \end{aligned} \quad (30)$$

and

$$\begin{aligned} &8\beta \left\{ \overline{\text{Ric}} - 2t_{00} - \alpha^2 t_m^m \right\} + 8(\alpha^2 + \beta^2)s_{0|m}^m \\ &+ (n-1) \left\{ -12r_{00}s_0 - 2r_{00|0} + 4s_{0|0}\beta + 8q_{00}\beta - 8t_0\alpha^2 \right. \\ &\left. - 12\theta(\alpha^2 + 3\beta^2) - 16\sigma(\alpha^2 + \beta^2)\beta \right\} = 0 \end{aligned} \quad (31)$$

Multiplying (31) by  $\beta$  and subtracting the resulting equation from (30) we obtain

$$\begin{aligned} &4(\alpha^2 - \beta^2) \left\{ \overline{\text{Ric}} - 2t_{00} - \alpha^2 t_m^m + 2\beta s_{0|m}^m \right\} \\ &+ 4(n-1)(\alpha^2 - \beta^2) \left\{ 3s_0^2 + s_{0|0} + 2q_{00} - 6\theta\beta - \sigma(\alpha^2 + 3\beta^2) \right\} \\ &+ 3(n-1)(r_{00} + 2s_0\beta)^2 = 0. \end{aligned} \quad (32)$$

Note that the polynomial  $\alpha^2 - \beta^2$  is irreducible under the strong convexity condition  $\|\beta\|_\alpha < 1$ . Thus there is a scalar function  $c = c(x)$  such that

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2). \quad (33)$$

Then (32) becomes

$$\begin{aligned} & \overline{\text{Ric}} - 2t_{00} - \alpha^2 t_m^m + 2\beta s_{0|m}^m \\ & + (n-1) \left\{ 3s_0^2 + s_{0|0} + 2q_{00} - 6\theta\beta - \sigma(\alpha^2 + 3\beta^2) \right\} \\ & + 3(n-1)c^2(\alpha^2 - \beta^2) = 0. \end{aligned} \quad (34)$$

Multiplying (34) by  $8\beta$  and subtracting the resulting equation from (31) yield

$$\begin{aligned} 8(\alpha^2 - \beta^2)s_{0|m}^m &= 2(n-1) \left\{ 6(r_{00} + 2s_0\beta)s_0 + r_{00|0} + 2s_{0|0}\beta \right. \\ & \quad \left. + 4q_{00}\beta + 4t_0\alpha^2 \right. \\ & \quad \left. + [6\theta + 4\sigma\beta + 12c^2\beta](\alpha^2 - \beta^2) \right\}. \end{aligned} \quad (35)$$

Differentiating (33), we get

$$r_{00|0} + 2s_{0|0}\beta = 4s_0^2\beta + 8cs_0\beta^2 + \left\{ 2c_0 - 4cs_0 - 8c^2\beta \right\}(\alpha^2 - \beta^2). \quad (36)$$

By (33), we get

$$q_{00} = -s_0^2 - 2cs_0\beta - t_0\beta. \quad (37)$$

By (33), (36) and (37), we can simplify (35) to (27). Plugging (27) and (37) back into (34) gives (26).

Conversely, if (26), (27) and (28) are satisfied, then (36) and (37) hold. Plugging them into (24), we get (25). Q.E.D.

## 6 Proof of Theorem 1.2

*Proof of Theorem 1.2:* Suppose that  $F$  has weakly isotropic flag curvature satisfying (3),  $\mathbf{K} = 3\theta/F + \sigma$ . We are going to prove that (4)-(6) hold.

By Theorem 1.1, (1) and (2) hold. (4) is just (1).  $F$  is a weak Einstein metric satisfying (25). By Proposition 5.1, (26)-(28) hold. (6) is just (28).

Now we are going to prove (5). Plugging (27) into (2), we obtain

$$\begin{aligned} s_{ij|k} &= a_{ik} \left\{ (\sigma + c^2)b_j + 2cs_j + t_j + \frac{3\lambda_j + c_j}{2} \right\} \\ & \quad - a_{jk} \left\{ (\sigma + c^2)b_i + 2cs_i + t_i + \frac{3\lambda_i + c_i}{2} \right\} \end{aligned} \quad (38)$$

Observe that

$$\begin{aligned} s_{ij} &= b_{|j}^m s_{mi} + b^m s_{mi|j} \\ &= q_{ji} - t_{ji} + b_j \left\{ (\sigma + c^2)b_i + 2cs_i + t_i + \frac{3\lambda_i + c_i}{2} \right\} - a_{ij}\delta, \end{aligned}$$

where

$$\delta := (\sigma + c^2)b^2 + t_m b^m + \frac{3\lambda_m b^m + c_m b^m}{2}.$$

Thus

$$s_{0|0} = q_{00} - t_{00} + \left\{ (\sigma + c^2)\beta + 2cs_0 + t_0 + \frac{3\theta + c_0}{2} \right\} \beta - \delta\alpha^2.$$

By (37), we obtain

$$s_{0|0} + s_0^2 = -t_{00} + (\sigma + c^2)\beta^2 + \frac{3\theta + c_0}{2}\beta - \delta\alpha^2. \quad (39)$$

Plugging it into (26), we get

$$\text{Ric} = (n-1) \left\{ (\sigma - 3c^2 + \delta)\alpha^2 + \frac{3}{2}(\theta - c_0) \right\} + (n+1)t_{00} + t_m^m \alpha^2. \quad (40)$$

On the other hand, by (1),

$$\text{Ric} = (n-1)\kappa\alpha^2 + (n+1)t_{00} + t_m^m \alpha^2. \quad (41)$$

It follows from (40) and (41) that

$$(\kappa - \sigma + 3c^2 - \delta)\alpha^2 = \frac{3}{2}(\theta - c_0)\beta.$$

We must have

$$\theta = c_0 \quad (42)$$

and

$$\kappa = \sigma - 3c^2 + \delta. \quad (43)$$

By (42), we obtain (5) from (38) and we obtain (7) from (43).

Conversely, assume that (4)-(6) hold. By Theorem 1.1, (4) and (5) imply that  $F$  is of scalar flag curvature. By (4),

$$\overline{\text{Ric}} = (n+1)\kappa\alpha^2 + (n+1)t_{00} + t_m^m \alpha^2. \quad (44)$$

By (5), we have

$$s_{0|m}^m = (n-1) \left\{ (\sigma + c^2)\beta + 2cs_0 + t_0 + 2c_0 \right\}. \quad (45)$$

By the same argument as above we have

$$s_{0|0} + s_0^2 = -t_{00} + (\sigma + c^2)\beta^2 + 2c_0\beta - \delta\alpha^2,$$

where  $\delta$  is given by (7). The only difference is that  $(3\theta + c_0)/2$  in (39) is replaced by  $2c_0$  since  $\theta = c_0$ . We can express  $\overline{\text{Ric}}$  in (44) as

$$\overline{\text{Ric}} = (n-1) \left\{ (\sigma - 3c^2)\alpha^2 + (\sigma + c^2)\beta^2 + 2c_0\beta - s_{0|0} - s_0^2 \right\} + 2t_{00} + t_m^m \alpha^2.$$

By Proposition 5.1,  $F$  is a weak Einstein metric satisfying (25) with  $\theta = c_0$ . Thus

$$\mathbf{K} = \frac{1}{n-1} \frac{\text{Ric}}{F^2} = \frac{3c_0}{F} + \sigma.$$

Q.E.D.

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Zhongmin Shen  
Center of Mathematical Sciences  
Zhejiang University  
Hangzhou, Zhejiang Province 310027  
P.R. China

and

Department of Mathematical Sciences  
Indiana University Purdue University Indianapolis (IUPUI)  
402 N. Blackford Street  
Indianapolis, IN 46202-3216  
USA  
zshen@math.iupui.edu

Gulcin Civi Yildirim  
Istanbul Technical University  
Faculty of Science and Letters  
Department of Mathematics  
Maslak, Istanbul  
TURKEY  
civi@itu.edu.tr