On Some Projectively Flat Finsler Metrics[*]

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Abstract

In this paper, we find a sufficient condition for certain class of Finsler metrics in the form $F = \alpha\phi(\beta/\alpha)$ to be locally projectively flat, where $\phi = \phi(s)$ is a positive C^∞ function satisfying certain conditions, α is a Riemannian metric and β is a 1-form. Using this sufficient condition, we construct a family of projectively flat metrics on an open subset in R^n.

1 Introduction

It is Hilbert's Fourth Problem in the regular case to study and characterize Finsler metrics on an open domain $\mathcal{U} \subset \mathrm{R}^n$ whose geodesics are straight lines. Finsler metrics with this property are called *projectively flat* metrics. It is well-known that every projectively flat metric is of scalar flag curvature, namely, the flag curvature $\mathbf{K}(P, y) = K(x, y)$ is independent of the section P containing $y \in T_x\mathcal{U}$ (see [2]). Thus projectively flat metrics become more interesting.

In 1903, G. Hamel found a system of partial differential equations that characterize projectively flat metrics $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$. That is,

$$F_{x^m y^i} y^m = F_{x^i}. \tag{1}$$

A natural problem is to find projectively flat metrics by solving (1). However, according to the Beltrami Theorem, a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ is projectively flat if and only if it is of constant sectional curvature. Thus this problem has been solved in Riemannian geometry.

In this paper, we are going to consider a class of Finsler metrics on a manifold M, which are expressed in the following form

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_o$ for $x \in M$, and $\phi = \phi(s)$ is a C^∞ function on $(-b_o, b_o)$ satisfying

$$\phi(0) = 1, \quad \phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \tag{2}$$

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where s and b are arbitrary numbers with $|s| \leq b < b_o$. The above condition is sufficient and necessary for $F = \alpha\phi(\beta/alpha)$ to be a Finsler metric. Finsler metrics in the above form are called (α, β)-metrics. The class of (α, β)-metric contain all Riemannian metrics ($\phi = 1$). It is a very rich class of metrics in Finsler geometry. In this paper, we shall restrict our attention to (α, β)-metrics. For some functions ϕ, the defined metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat if and only if α is projectively flat and β is parallel with respect to α. For example, the Matsumoto metric defined by $\phi = 1/(1-s)$ and the exponential metric defined by $\phi = \epsilon s + \exp(s)$ have this property. See [6] and [13]. For some functions ϕ, the 1-form of projectively flat metric $F = \alpha\phi(\beta/\alpha)$ is not necessarily parallel. The simplest one is the Randers metric $F = \alpha + \beta$ defined by $\phi = 1 + s$. It is known that $F = \alpha + \beta$ is projectively flat if and only if α is projectively flat and β is closed (see [3]). Another interesting metric is $F = (\alpha + \beta)^2/\alpha$ defined by $\phi = (1+s)^2$. In [11], it is shown that $F = (\alpha+\beta)^2/\alpha$ is projectively flat if and only if

$$b_{i|j} = \tau\Big\{(1 + 2b^2)a_{ij} - 3b_i b_j\Big\}, \tag{3}$$

and the spray coefficients G^i_α of α are in the form:

$$G^i_\alpha = \xi y^i - \tau\alpha^2 b^i, \tag{4}$$

where $b := \sqrt{a^{ij}b_i b_j}$, $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α, $\tau = \tau(x)$ is a scalar function and $\xi = a_i(x)y^i$ is a 1-form on M. The sufficient condition is first obtained in [8], where some non-trivial solutions of (3) and (4) are also given.

In this paper, we are going to consider a class of (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$, where $\phi = \phi(s)$ is a function satisfying (2) and

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s). \tag{5}$$

where p, r are constants. Note that $\phi := (1+s)^2$ satisfies (5) with $p = 1/2$ and $r = -1/2$. In general, the solution of (5) can not be expressed in terms of elementary functions. Nevertheless we find a sufficient condition for $F = \alpha\phi(\beta/\alpha)$ to be projectively flat.

Theorem 1.1 *Assume that $\phi = \phi(s)$ satisfies (2) and (5). Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on a manifold M. If*

$$b_{i|j} = 2\tau\Big\{(p + b^2)a_{ij} + (r-1)b_i b_j\Big\}, \tag{6}$$

and

$$G^i_\alpha = \xi y^i - \tau\alpha^2 b^i, \tag{7}$$

where $b := \sqrt{a^{ij}b_i bj}$ and $\xi = \xi_i(x)y^i$ is a 1-form on M, then F is locally projectively flat.

Unfortunately, we are not able to prove that the conditions (6) and (7) are necessary. A key step is to show that β is closed. Some progresses have been made for certain types of functions ϕ. After the first draft of this paper was released, Y.B. Shen and L. Zhao have shown that when $r = -1/4$, (6) and (7) are necessary conditions (see [12]). For some functions ϕ satisfying (5), the conditions (6) and (7) might be necessary.

For a function ϕ satisfying (5), we can also find a family of particular solutions (α, β) to (6) and (7). Then we obtain some special solutions $F = \alpha\phi(\beta/\alpha)$ of (1) as given below.

Theorem 1.2 *Let $\phi = \phi(s)$ be a function satisfying (2) and (5). Let*

$$h := \frac{1}{\sqrt{1+\mu|x|^2}}\Big\{C_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}}\Big\}, \tag{8}$$

and let $\rho = \rho(t)$ be given by

$$\rho(t) = \begin{cases} \frac{(C_2)^2}{p}\Big(C_3 + \eta t - \frac{1}{2}\mu t^2\Big) & \text{if } r = 0 \\ \ln\Big[-\frac{2r(C_2)^2}{p}\Big(C_3 + \eta t - \frac{1}{2}\mu t^2\Big)\Big]^{-\frac{1}{2r}} & \text{if } r \neq 0 \end{cases} \tag{9}$$

where η and C_i are constants ($C_2 > 0$) and $a \in \mathrm{R}^n$ is a constant vector. Define

$$\alpha := e^{\rho(h)}\alpha_\mu, \qquad \beta := C_2 e^{(r+1)\rho(h)} dh,$$

where

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+\mu|x|^2}.$$

Then α and β satisfy (6) and (7) with

$$\tau = \frac{\rho'(h)}{2C_2 e^{(r+1)\rho(h)}}.$$

Thus the Finsler metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat.

It is not easy to verify that $F = \alpha\phi(\beta/\alpha)$ defined in Theorem 1.2 satisfies (1).

2 Preliminaries

In this section, for a function $\phi = \phi(s)$ satisfying (2) and (5), we shall find a sufficient condition for an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ to be projectively flat.

Let $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Let $b_{i|j}dx^i \otimes dx^j$ denote covariant derivative of β with respect to α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s_j := b^j s_{i|j}.$$

Clearly, β is closed if and only if $s_{ij} = 0$.

The geodesic coefficients G^i are given by

$$G^i = G^i_\alpha + Py^i + Q^i. \tag{10}$$

where

$$\begin{aligned} P &= \alpha^{-1}\Theta\Big(-2\alpha Q s_0 + r_{00}\Big) \\ Q^i &= \alpha Q s^i{}_0 + H\Big(-2\alpha Q s_0 + r_{00}\Big) b^i \end{aligned}$$

where

$$\begin{aligned} \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \\ Q &= \frac{\phi'}{\phi - s\phi'} \\ H &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

When $\phi'' \neq 0$, we can also express G^i as follows:

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + H\Big(-2\alpha Q s_0 + r_{00}\Big)\Big\{ \chi \frac{y^i}{\alpha} + b^i \Big\},$$

where

$$\chi := \frac{(\phi - s\phi')\phi'}{\phi\phi''} - s.$$

Note that if α is projectively flat and β is parallel, then $F = \alpha\phi(\beta/\alpha)$ is projectively flat too.

Below is a list of some solutions of (5) which can be expressed in terms of elementary functions.

(a) If $r = -1$,

$$\phi = \begin{cases} \sqrt{1 - \tilde{s}^2} + \tilde{s}\arctan\left(\frac{\tilde{s}}{\sqrt{1-\tilde{s}^2}}\right) + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{p}}s \ \text{ if } \ p > 0 \\ \sqrt{1 + \tilde{s}^2} - \tilde{s}\ln(\tilde{s} + \sqrt{1 + \tilde{s}^2}) + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{-p}}s \ \text{ if } \ p < 0. \end{cases}$$

(b) If $r = 1$, then

$$\phi = \begin{cases} \sqrt{1 + \tilde{s}^2} + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{p}}s \ \text{ if } p > 0 \\ \sqrt{1 - \tilde{s}^2} + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{-p}}s \ \text{ if } p < 0. \end{cases}$$

(c) If $r = -1/2$, then

$$\phi = \begin{cases} 1 + \tilde{s}^2 + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{2p}}s \ \text{ if } p > 0 \\ 1 - \tilde{s}^2 + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{-2p}}s \ \text{ if } p < 0. \end{cases}$$

(d) If $r = 1/2$, then

$$\phi = \begin{cases} 1 + \tilde{s}\arctan(\tilde{s}) + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{2p}}s \text{ if } \ p > 0 \\ 1 + \tilde{s}\ln\sqrt{\frac{1-\tilde{s}}{1+\tilde{s}}} + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{-2p}}s \text{ if } \ p < 0. \end{cases}$$

(e) If $r = -1/3$, then

$$\phi = \begin{cases} \left(1 + \frac{1}{2}\tilde{s}^2\right)\sqrt{1-\tilde{s}^2} + \frac{3}{2}\tilde{s}\arctan\left(\frac{\tilde{s}}{\sqrt{1-\tilde{s}^2}}\right) + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{3p}}s \text{ if } \ p > 0 \\ \left(1 - \frac{1}{2}\tilde{s}^2\right)\sqrt{1+\tilde{s}^2} - \frac{3}{2}\tilde{s}\ln\left(\tilde{s} + \sqrt{1+\tilde{s}^2}\right) + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{-3p}}s \text{ if } \ p < 0. \end{cases}$$

(f) If $r = 1/3$, then

$$\phi = \begin{cases} \sqrt{1+\tilde{s}^2} + \frac{\tilde{s}^2}{\sqrt{1+\tilde{s}^2}} + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{3p}}s \text{ if } \ p > 0 \\ \sqrt{1-\tilde{s}^2} - \frac{\tilde{s}^2}{\sqrt{1-\tilde{s}^2}} + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{\sqrt{-3p}}s \text{ if } \ p < 0. \end{cases}$$

(g) If $r = -1/4$, then

$$\phi = \begin{cases} 1 + 2\tilde{s}^2 - \frac{1}{3}\tilde{s}^4 + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{2\sqrt{p}}s \text{ if } \ p > 0 \\ 1 - 2\tilde{s}^2 - \frac{1}{3}\tilde{s}^4 + \epsilon\tilde{s}, & \tilde{s} = \frac{1}{2\sqrt{-p}}s \text{ if } \ p < 0. \end{cases}$$

(h) If $r = 1/4$, then

$$\phi = \begin{cases} \frac{2+3\tilde{s}^2}{2(1+\tilde{s}^2)} + \frac{3}{2}\tilde{s}\arctan(\tilde{s}) + \epsilon\tilde{s}, & \tilde{s} = \frac{s}{2\sqrt{p}} \text{ if } \ p > 0 \\ \frac{2-3\tilde{s}^2}{2(1-\tilde{s}^2)} + \frac{3}{2}\tilde{s}\ln\sqrt{\frac{1-\tilde{s}}{1+\tilde{s}}} + \epsilon\tilde{s}, & \tilde{s} = \frac{s}{2\sqrt{-p}} \text{ if } \ p < 0. \end{cases}$$

We can easily write down a formula for the (α, β)-metric defined by any of the above function ϕ. These special (α, β)-metrics have not been systematically studied yet.

3 Proof of Theorem 1.1

Proof of Theorem 1.1: Assume that $\phi = \phi(s)$ satisfies (5). Then

$$H = \frac{1}{2[p + b^2 + (r-1)s^2]}.$$

Since β is closed, $s_{ij} = 0$, we get from

$$G^i = G^i_\alpha + \frac{r_{00}}{2\Big[(p+b^2)\alpha^2 + (r-1)\beta^2\Big]}\Big\{\chi\frac{y^i}{\alpha} + b^i\Big\}\alpha^2.$$

By assumption, β satisfies (6), then

$$G^i = G^i_\alpha + \tau\Big\{\chi\frac{y^i}{\alpha} + b^i\Big\}\alpha^2.$$

The spray G of F defined by G^i is projectively equivalent to an affine spray defined by $\bar{G}^i = G^i_\alpha + \tau\alpha^2 b^i$. Thus F is a Douglas metric.

By assumption G^i_α are in the form (7), then we can eliminate the term b^i, i.e.,

$$G^i = (\xi + \alpha^{-1}\tau\chi)y^i.$$

Therefore, F is projectively flat. Q.E.D.

We are not be able to prove that (6) and (7) are necessary for $F = \alpha\phi(\beta/\alpha)$ to be projectively flat when ϕ satisfies (5). To find a necessary condition, one can apply the following

Lemma 3.1 ([11]) *$F = \alpha\phi(\beta/\alpha)$ is projectively flat on an open subset $\mathcal{U} \subset \mathrm{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H(-2\alpha Q s_0 + r_{00})(b_l\alpha^2 - \beta y_l) = 0. \qquad (11)$$

If we assume that β is closed,i.e., $s_{ij} = 0$, then (11) is reduced to

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \frac{\alpha^2 r_{00}}{2[(p+b^2)\alpha^2 + (r-1)\beta^2]}(b_l\alpha^2 - \beta y_l) = 0.$$

By the above equation, one can easily show that there is a scalar function $\tau = \tau(x)$ such that

$$r_{00} = 2\tau\Big\{(p+b^2)\alpha^2 + (r-1)\beta^2\Big\}. \qquad (12)$$

Note that (12) is same as (6) since $s_{ij} = 0$. Thus in order to prove (6) and (7) are necessary conditions for projectively flat (α, β)-metrics defined by certain functions ϕ satisfying (5), a key step is to prove that β is closed.

4 Special solutions

In this section we are going to find special solutions of (6) and (7) on an open ball $\mathrm{B}^n \subset \mathrm{R}^n$ in the case when $\tau\beta$ is closed. It is not clear whether or not $\tau\beta$ is always closed. Throughout this section, for a scalar function $f = f(x)$ on B^n, we always denote

$$f_0 := f_{x^i}y^i, \qquad f_{00} := f_{x^i x^j}y^i y^j.$$

Since β and $\tau\beta$ are closed, we may let

$$\beta = \frac{1}{2}\sigma_0, \qquad \tau\beta = \frac{1}{2}\rho_0.$$

where $\sigma = \sigma(x)$ and $\rho = \rho(x)$ are scalar functions on B^n. We have

$$2\tau\beta = \tau\sigma_0 = \rho_0. \tag{13}$$

Let $\alpha := e^{\rho}\alpha_\mu$, where α_μ is given by

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}.$$

α_μ is a projectively flat Riemannian metric of constant curvature μ. Its spray coefficients $G^i_{\alpha_\mu}$ are given by

$$G^i_{\alpha_\mu} = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2}\, y^i.$$

By a simple computation, we get the spray coefficients of α:

$$\begin{aligned} G^i_\alpha &= G^i_{\alpha_\mu} + \rho_0 y^i - \frac{1}{2}a^{ij}\rho_{x^j}\alpha^2 \\ &= \xi y^i - \tau b^i \alpha^2, \end{aligned} \tag{14}$$

where

$$\xi = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2} + \tau\sigma_0.$$

Thus (7) is satisfied.

Now we are going to solve (6). Since β is closed, $r_{00} = b_{0|0}$. By (14), we get

$$\begin{aligned} r_{00} &= \frac{\partial b_i}{\partial x^j} y^i y^j - 2b_m G^m_\alpha \\ &= \frac{1}{2}\sigma_{00} - \sigma_0\Big\{ -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2} + \tau\sigma_0 \Big\} + 2\tau b^2\alpha^2. \end{aligned} \tag{15}$$

Then (6) is equivalent to the following equation:

$$\sigma_{00} + \frac{2\mu\langle x, y\rangle}{1 + \mu|x|^2}\sigma_0 = 4p\tau\alpha^2 + \tau(r+1)\sigma_0^2. \tag{16}$$

We assume that $\rho = \rho(x)$ and $\sigma = \sigma(x)$ are defined by a common function $h = h(x)$ in the following form

$$\rho = \rho(h), \qquad \sigma = \sigma(h).$$

Then (13) is equivalent to

$$\tau\sigma' = \rho'. \tag{17}$$

(16) is reduced to

$$h_{00} + \frac{2\mu\langle x, y\rangle}{1 + \mu|x|^2} h_0 = \frac{4p\rho'}{(\sigma')^2}\alpha^2 + \Big\{(r+1)\rho' - \frac{\sigma''}{\sigma'}\Big\} h_0^2. \tag{18}$$

We need the following

Lemma 4.1 *Let*

$$h = \frac{1}{\sqrt{1+\mu|x|^2}}\Big\{C_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}}\Big\},$$

where η is a constant and $a \in \mathrm{R}^n$ is a constant vector. Then h satisfies

$$h_{00} + \frac{2\mu\langle x, y\rangle}{1+\mu|x|^2}h_0 = (\eta - \mu h)\alpha_\mu^2. \tag{19}$$

The proof of the lemma is straight forward, so is omitted.

Comparing (18) and (19),we see that if σ satisfies

$$(r+1)\rho' - \frac{\sigma''}{\sigma'} = 0, \tag{20}$$

$$4p\rho'\Big(\frac{e^\rho}{\sigma'}\Big)^2 = \eta - \mu h, \tag{21}$$

then (18) holds.

It follows from (20) that

$$\sigma' = 2C_2 e^{(r+1)\rho}, \tag{22}$$

where C_2 is a positive constant. It follows from (21) and (22) that

$$\rho' e^{-2r\rho} = \frac{(C_2)^2}{p}(\eta - \mu h), \tag{23}$$

If $r = 0$, then

$$\rho = \frac{(C_2)^2}{p}\Big(C_3 + \eta h - \frac{1}{2}\mu h^2\Big).$$

If $r \neq 0$, then

$$\rho = \ln\Big[-\frac{2r(C_2)^2}{p}\Big(C_3 + \eta h - \frac{1}{2}\mu h^2\Big)\Big]^{-\frac{1}{2r}},$$

where C_3 is a constant.

By (17), the scalar function τ is given by

$$\tau = \frac{\rho'(h)}{2C_2 e^{(r+1)\rho(h)}}.$$

Q.E.D.

5 Examples

In this section, we are going to discuss some exmples coming out of Theorem 1.2. Throughout of this section, we always let

$$h := \frac{1}{\sqrt{1+\mu|x|^2}}\Big\{C_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}}\Big\}.$$

Example 5.1 ($r = 1, p = 1$) Let

$$\alpha := \lambda\alpha_\mu, \qquad \beta := C_2\lambda^2 h_0,$$

where

$$\lambda := \Big[-2(C_2)^2(C_3 + \eta h - \frac{1}{2}\mu h^2)\Big]^{-1/2}.$$

Then

$$F := \sqrt{\alpha^2+\beta^2} + \epsilon\beta$$

is projectively flat. F is a Randers metric. Thus

$$\tilde{\alpha} := \sqrt{\alpha^2+\beta^2}$$

is projectively flat. By Beltrami theorem, $\tilde{\alpha}$ is of constant sectional curvature.

Example 5.2 ($r = -1/2, p = 1/2$) Let

$$\alpha := \lambda\alpha_\mu, \qquad \beta := C_2\sqrt{\lambda}h_0,$$

where

$$\lambda := 2(C_2)^2\Big[C_3 + \eta h - \frac{1}{2}\mu h^2\Big].$$

Then

$$F := \alpha + \frac{\beta^2}{\alpha} + \epsilon\beta$$

is projectively flat. This is Theorem 1.1 in [8].

Example 5.3 ($r = 1/3, p = 1/3$) Let

$$\alpha := \lambda\alpha_\mu, \qquad \beta := C_2\lambda^{4/3}h_0,$$

where

$$\lambda := \Big[-2(C_2)^2(C_3 + \eta h - \frac{1}{2}\mu h^2)\Big]^{-3/2}.$$

Then

$$F := \sqrt{\alpha^2+\beta^2} + \frac{\beta^2}{\sqrt{\alpha^2+\beta^2}} + \epsilon\beta$$

is projectively flat.

Let

$$\tilde{\alpha} := \sqrt{\alpha^2 + \beta^2}.$$

Then

$$F = \tilde{\alpha} + \frac{\beta^2}{\tilde{\alpha}} + \epsilon\beta.$$

This metric has the same type as the one in Example 5.2.

Example 5.4 ($r = -1/4, p = 1/4$) Let

$$\alpha := \lambda\alpha_\mu, \qquad \beta := C_2\lambda^{3/4}h_0,$$

where

$$\lambda := \Big[2(C_2)^2(C_3 + \eta h - \frac{1}{2}\mu h^2)\Big]^2.$$

Then

$$F := \alpha + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3} + \epsilon\beta$$

is projectively flat.

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