On Douglas Metrics*†

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1 Introduction

In Finsler geometry, there are several important classes of Finsler metrics. The Berwald metrics were first investigated by L. Berwald. Every Finsler metric F on a manifold M induced a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ which determines the geodesics. By definition, a Finsler metric F is a Berwald metric if the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_x M$ at every point x, i.e.,

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k.$$

Riemannian metrics are special Berwald metrics. In fact, Berwald metrics are "almost Riemannian" in the sense that every Berwald metric is affinely equivalent to a Riemannian metric, i.e., the geodesics of any Berwald metric are the geodesics of some Riemannian metric [12]. The Douglas metrics are more generalized ones than Berwald metrics. A Finsler metric is called a Douglas metric if the spray coefficients $G^i = G^i(x, y)$ are in the following form:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i. \tag{1}$$

Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. The study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities.

There are two imoportant non-Riemannian quantities: the mean Berwald curvature (E-curvature) and the Landsberg curvature (L-curvature). If $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is the spray of a Finsler metric F, then the mean Berwald tensor $\mathcal{E} = E_{ij} dx^i \otimes dx^j$ and the Landsberg tensor $\mathcal{L} = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ are respectively defined by

$$E_{ij} := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\Big(\frac{\partial G^m}{\partial y^m}\Big), \qquad L_{ijk} := -\frac{1}{2} y^s g_{ms} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}.$$

Clearly, for Berwad metrics, $\mathcal{E} = 0$ and $\mathcal{L} = 0$. Finsler metrics with $\mathcal{E} = 0$ are called weakly Berwald metrics and those with $\mathcal{L} = 0$ are called Landsberg

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metrics. There are many weakly Berwald metrics which are non-Berwaldian, but so far, we do not know if every Landsberg metric is Berwaldian. This is a long existing open problem in Finsler geometry. The class of Douglas metrics is much larger than that of Berwald metrics. In this paper, we are prove the following theorem for a general Douglas metrics.

Theorem 1.1 *Let (M, F) be a non-Riemannian Douglas manifold of dimension $n \geq 3$. Then the following are equivalent:*

(a) F has isotropic mean Berwald curvature;

(b) F has relatively isotropic Landsberg curvature.

2 Preliminaries

Let M be a manifold of dimension n and let π^*TM denote the pull-back tangent bundle over the slit tangent bundle $TM_o := TM \setminus \{0\}$ by the natural projection $\pi : TM_o \to M$. Denote by $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ be the local natural frame and $\{dx^i, dy^i\}$ the local natural coframe for $T(TM_o)$ in a standard local coordinate system (x^i, y^i) in TM_o. Denote by $\{\partial_i\}$ the local natural frame for π^*TM corresponding to the local natural frame $\{\frac{\partial}{\partial x^i}\}$ for the tangent bundle TM.

Let F be a Finsler metric on manifold M of dimension n. We have two tensors

$$g := g_{ij} dx^i \otimes dx^j, \qquad h := h_{ij} dx^i \otimes dx^j,$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and $h_{ij} := F F_{y^i y^j} = g_{ij} - F_{y^i} F_{y^j}$. The first impoprtant quantity is the Cartan tensor $\mathcal{C} = C_{ijk} dx^i \otimes dx^j \otimes dx^l$, where $C_{ijk} = \frac{1}{4}[F^2]_{y^i y^k y^l}$. Let G be the induced spray. $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a special vector field on TM_o, which is defined by

$$G^i = \frac{1}{4} g^{il} \Big\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \Big\},$$

where $(g^{ij}) := (g_{ij})^{-1}$. The spray determines the geodesics of F by

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0.$$

More precisely, the geodesics of F are the projections of the integral curves of G.

By definition, F is a Berwald metric if the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_x M$ at every point $x \in M$. For a general Finsler metrics it is natural to consider the following term

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

It is easy to verify that $\mathcal{B} := B_j{}^i{}_{kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_o. We call $\mathcal{B}$ the *Berwald tensor*. The mean Berwald tensor $\mathcal{E} = E_{ij} dx^i \otimes dx^j$ is defined by

$$E_{ij} := \frac{1}{2} B_m{}^m{}_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \Big(\frac{\partial G^m}{\partial y^m} \Big). \tag{2}$$

The Landsberg tensor $\mathcal{L} = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk} := -\frac{1}{2} y^s g_{sm} B_i{}^m{}_{jk} = -\frac{1}{2} y^s g_{sm} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}. \tag{3}$$

We have the following important Bianchi identity:

$$\frac{\partial L_{jkl}}{\partial y^m} - \frac{\partial L_{jkm}}{\partial y^l} = \frac{1}{2} g_{il} B_m{}^i{}_{kj} - \frac{1}{2} g_{im} B_l{}^i{}_{kj}. \tag{4}$$

See (10.12) in [11] for a proof.

Definition 2.1 Let F be a Finsler metric on an n-dimensional manifold M.

(a) F has isotropic mean Berwald curvature if

$$E_{ij} = \frac{n+1}{2} c F_{y^i y^j}. \tag{5}$$

(b) F has relatively isotropic Landsberg curvature if

$$L_{ijk} + cFC_{ijk} = 0. \tag{6}$$

(c) F has isotropic Berwald curvature if

$$B_j{}^i{}_{kl} = c\Big\{F_{y^j y^k} \delta^i_l + F_{y^j y^l} \delta^i_k + F_{y^k y^l} \delta^i_j + F_{y^j y^k y^l} y^i\Big\}. \tag{7}$$

Here $c = c(x)$ in (5)-(7) is a scalar function on M.

Since $h_{ij} = F F_{y^i y^j}$ and $h^i_j := g^{ik} h_{jk} = \delta^i_j - F^{-2} g_{is} y^s y^m$, (7) can expressed as

$$B_j{}^i{}_{kl} = cF^{-1}\Big\{h_{jk} h^i_l + h_{jl} h^i_k + h_{kl} h^i_j + 2C_{jkl} y^i\Big\}. \tag{8}$$

Example 2.2 The Funk metric $\Theta = \Theta(x, y)$ on a strongly convex domain in R^n has isotropic Berwald curvature with $c = \frac{1}{2}$.

Let

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i\Big),$$

It is easy to verify that $\mathcal{D} := D_j{}^i{}_{kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_o. We call $\mathcal{D}$ the *Douglas tensor*. By a direct comptation, one can express $D_j{}^i{}_{kl}$ as follows.

$$D_j{}^i{}_{kl} := B_j{}^i{}_{kl} - \frac{2}{n+1}\Big\{E_{jk} \delta^i_l + E_{jl} \delta^i_k + E_{kl} \delta^i_j + \frac{\partial E_{jk}}{\partial y^l}\, y^i\Big\}. \tag{9}$$

It is easy to verify that (1) holds if and only if

$$G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i = \gamma^i_{jk}(x) y^j y^k, \tag{10}$$

for some set of local functions $\gamma^i_{jk}(x)$; and (10) holds if and only if $D_j{}^i{}_{kl} = 0$. Thus Douglas metrics are also characterized by the curvature equation $\mathcal{D} = 0$. By (9), one can easily see that every Berwald metric is a Douglas metric. There are many non-Berwaldian Douglas metrics. For example, a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed [1].

In [6], we prove that a Randers metric $F = \alpha+\beta$ with β closed has isotropic mean Berwald curvature if and only if it has relatively isotropic Landsberg curvature [6]. In this paper, we generalize this result.

Proposition 2.3 *Let F be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 3$. The following are equivalent.*

(a) F is of isotropic Berwald curvature;

(b) F is a Douglas metric with isotropic mean Berwald curvature;

(c) F is a Douglas metric with relatively isotropic Landsberg curvature.

From Proposition 2.3, we see that every Finsler metric of isoropic Berwald curvature is a Douglas metric.

Besides Randers metrics, there are many interesting Douglas metrics.

Example 2.4 Let $F = (\alpha^2 + \beta^2)/\alpha$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta = b_iy^i$ be a 1-form on a manifold M with $b := \sqrt{a_{ij}b^ib^j} < 1$ at any point $x \in M$. Then F is a Finsler metric. M. Matsumto [10] has proved that when $n = \dim M \geq 3$, F is a Douglas metric if and only if

$$b_{i|j} = c\Big\{(1 + 2b^2)a_{ij} - 3b_ib_j\Big\}.$$

where $c = c(x)$ is a scalar function on M. In this case, we can show that

$$G^i = \bar{G}^i + c\Big\{Py^i + \alpha^2 b^i\Big\},$$

where $P = -2\beta^3/(\alpha^2 + \beta^2)$ and $\bar{G}^i = \bar{G}^i(x, y)$ are the spray coefficients of α. One has

$$\frac{\partial G^m}{\partial y^m} = \frac{\partial \bar{G}^m}{\partial y^m} + c\{(n + 1)P + 2\beta\}.$$

Thus

$$E_{ij} = \frac{n + 1}{2} cP_{y^iy^j}.$$

Clearly, F does not have isotropic mean Berwald curvature.

The Douglas tensor is a projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent,

$$G^i = \bar{G}^i + Py^i,$$

where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$. Thus if a Finsler metric is projectively equivalent to a Berwald metric, then it is a Douglas metric. However, it is still an open problem whether or not every Douglas metric is (locally) projectively equivalent to a Berwald metric.

3 Proof of Theorem 1.1

Lemma 3.1 *Let $F = F(x, y)$ be a Douglas metric on an n-dimensional manifold M. If $\mathcal{E} = \frac{1}{2}(n+1)cF^{-1}h$, then $\mathcal{L} + cF\mathcal{C} = 0$.*

Proof: By (9),$\mathcal{D} = 0$ if and only if

$$B_j{}^i{}_{kl} = \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l}\, y^i\Big\}. \tag{11}$$

Plugging $E_{jk} = \frac{1}{2}(n+1)cF_{y^iy^j}$ into (11), one obtains

$$\begin{aligned} B_j{}^i{}_{kl} &= c\Big\{F_{y^jy^k}\delta^i_l + F_{y^jy^l}\delta^i_k + F_{y^ky^l}\delta^i_j + F_{y^jy^ky^l}y^i\Big\} \\ &= cF^{-1}\Big\{h_{jk}h^i_l + h_{jl}h^i_k + h_{kl}h^i_j + 2C_{jkl}y^i\Big\}. \end{aligned} \tag{12}$$

Contracting with (12) with $-\frac{1}{2}y^s g_{si}$, one immediately obtains

$$L_{jkl} = -\frac{1}{2}y^s g_{si}B_j{}^i{}_{kl} = -cFC_{jkl}.$$

Q.E.D.

The converse is almost true.

Lemma 3.2 *Let $F = F(x, y)$ be a non-Riemannian Douglas metric on an n-dimensional manifold M $(n \geq 3)$. If $\mathcal{L} + cF\mathcal{C} = 0$, then $\mathcal{E} = \frac{1}{2}(n+1)\lambda F^{-1}h$ for some scalar function $\lambda = \lambda(x)$. Moreover, $\lambda(x) = c(x)$ at any point x where F_x is not Euclidean.*

Proof: By assumption, (11) holds and

$$L_{jkl} = -cFC_{jkl}. \tag{13}$$

Contracting $B_j{}^i{}_{kl}$ with $h^m_i := \delta^m_i - F^{-2}g_{is}y^sy^m$ and using (3) and (13), one obtains

$$h^m_iB_j{}^i{}_{kl} = B_j{}^m{}_{kl} + 2F^{-2}L_{jkl}y^m = B_j{}^m{}_{kl} - 2cF^{-1}C_{jkl}y^m. \tag{14}$$

Contracting (11) with h^m_i and using (14), one obtains

$$B_j{}^m{}_{kl} = \frac{2}{n+1}\Big\{E_{jk}h^m_l + E_{jl}h^m_k + E_{kl}h^m_j\Big\} + 2cF^{-1}C_{jkl}y^m. \tag{15}$$

Plugging (13) and (15) into (4), one obtains

$$E_{km}h_{jl} + E_{jm}h_{kl} - E_{kl}h_{jm} - E_{jl}h_{km} = 0. \tag{16}$$

Contracting (16) with g^{jm} yields

$$E_{kl} = \frac{1}{2}(n+1)\lambda F^{-1}h_{kl}, \tag{17}$$

where

$$\lambda := \frac{2}{n^2-1} F g^{jm} E_{jm}.$$

Next we are going to show that $\lambda = \lambda(x, y)$ is independent of $y \in T_x M$ at any point $x \in M$. Plugging (17) into (11) and (15) respectively, one obtains

$$\begin{aligned} B_j{}^i{}_{kl} &= \frac{\lambda}{F}\Big\{h_{jk}\delta^i_l + h_{jl}\delta^i_k + h_{kl}\delta^i_j\Big\} + \Big[\lambda F^{-1} h_{jk}\Big]_{y^l} y^i \\ &= \frac{\lambda}{F}\Big\{h_{jk}h^i_l + h_{jl}h^i_k + h_{kl}h^i_j\Big\} + 2cF^{-1}C_{jkl}y^i. \end{aligned}$$

Comparing the above two identities yields

$$\lambda_{y^l} h_{jk} = 2(c-\lambda)C_{jkl}. \tag{18}$$

Contracting (18) with g^{jk} yields

$$\lambda_{y^l} = \frac{2}{n-1}(c-\lambda)I_l. \tag{19}$$

Plugging (19) into (18), one obtains

$$(c-\lambda)\Big\{(n-1)C_{jkl} - I_l h_{jk}\Big\} = 0. \tag{20}$$

Contracting the above identity with g^{jk} yields

$$(n-2)(c-\lambda)I_l = 0.$$

Since $n \geq 3$, the above equation becomes

$$(c-\lambda)I_l = 0.$$

Then it follows from (19) that $\lambda_{y^l} = 0$. Thus $\lambda = \lambda(x)$ is independent of $y \in T_x M$.

Now it follows from (18) that

$$(c-\lambda)C_{jkl} = 0.$$

At any point $x \in M$ where $F_x = F|_{T_x M}$ is not Euclidean, $C_{jkl}(x, y) \neq 0$ for some $y \in T_x M \setminus \{0\}$. Then $c(x) = \lambda(x)$. This completes the proof. Q.E.D.

Lemma 3.2 might be true too in dimension two. But we do not find a short proof for this conjecture.

4 Isotropic Berwald Curvature

In this section, we are going to prove Proposition 2.3.

Lemma 4.1 *Let F be a Finsler metric on an n-dimensional manifold M. The following are equivalent.*

(a) F is of isotropic Berwald curvature satisfying (8) for a scalar function $c = c(x)$ on M;

(b) $\mathcal{D} = 0$ and $\mathcal{E} = \frac{1}{2}(n+1)cF^{-1}h$ for a scalar function $c = c(x)$ on M.

Proof: Assume that (8) holds. Then by (2), one obtains

$$E_{jk} = \frac{1}{2}B_j{}^m{}_{km} = \frac{1}{2}(n+1)cF^{-1}h_{jk} = \frac{1}{2}(n+1)cF_{y^iy^j}.$$

Plugging it into (18) yields (11). Thus $D_j{}^i{}_{kl} = 0$. Conversely, suppose that $F = F(x,y)$ is a Douglas metric with $\mathcal{E} = \frac{1}{2}(n+1)cF^{-1}h$. Plugging $E_{jk} = \frac{1}{2}(n+1)cF_{y^jy^k}$ into (11) yields (12). Thus F has isotropic Berwald curvature. This is already proved in Lemma 3.1. Q.E.D.

By Lemmas 3.2 and 4.1, one can easily show the following

Lemma 4.2 *Let F be a Finsler metric on an n-dimensional manifold M. The following hold*

(a) If F is of isotropic Berwald curvature satisfying (8) for a scalar function $c = c(x)$ on M, then $\mathcal{D} = 0$ and $\mathcal{L} + cF\mathcal{C} = 0$;

(b) ($n \geq 3$) If $\mathcal{D} = 0$ and $\mathcal{L} + cF\mathcal{C} = 0$ for a scalar function $c = c(x)$ on M, then F is of isometric Berwald curvature satisfying (8) for a scalar function $\lambda = \lambda(x)$. In this case, $\lambda(x) = c(x)$ at point x where F_x is not Euclidean.

One immediately obtains the following corollary.

Corollary 4.3 ([2]) *Let F be a Douglas metric on a manifold M of dimension $n \geq 3$. Suppose that $\mathcal{L} = 0$, then F is a Berwald metric.*

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