Finsler Metrics of Scalar Flag Curvature with Special Non-Riemannian Curvature Properties

B. Najafi, Z. Shen* and A. Tayebi

July 27, 2006

Abstract

In this paper, we study Finsler metrics of scalar flag curvature. We find that a non-Riemannian quantity is closely related to the flag curvature. We show that this non-Riemannian quantity takes a special form if and only if the flag curvature takes a special form. This will lead to a better understanding on Finsler metrics of scalar flag curvature.

1 Introduction

In Finsler geometry, there are several geometric quantities: the Riemann curvature (the flag curvature), the distortion, the (mean) Cartan curvature, the S-curvature, the (mean) Landsberg curvature and the (mean) Berwald curvature, etc. (Cf. [7] [8]). Except for the Riemann curvature and its averaging quantity — the Ricci curvature, all the other mentioned quantities are non-Riemannian, namely, they vanish when the Finsler metric is Riemannian. We define these non-Riemannian quantities via covariant horizontal and vertical derivatives of two basic quantities defined on a Minkowskian tangent spaces: the distortion and the (mean) Cartan torsion. These two quantities are defined only for Minkwoski norms and any of the two vanishes if and only if the norm is Euclidean. The other non-Riemannian quantities simply tell us the rates of change of these two quantities along geodesics.

The study shows that the above mentioned non-Riemannian quantities are closely related to the Riemann curvature (the flag curvature). See [5] for some recent developments. Is there any other interesting non-Riemannian quantity to be singled out for further attention? In [1], H. Arkar-Zadeh considered a non-Riemannian quantity H which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. This very non-Riemannian quantity is also closely related to the flag curvature. More precisely, Akbar-Zadeh proved that for a Finsler metric of scalar flag curvature,

---

[*] Research is support in part by a National Natural Science Foundation of China (10371138) and a NSF grant on IR/D

the flag curvature is a scalar function on the manifold if and only if $H = 0$. Thus the quantity deserves further investigation.

Let us take a look at the example which is constructed in [9]. Let $a \in R^n$ be an arbitrary unit vector ($|a| = 1$). On the unit ball $B^n(1) \subset R^n$, define

$$\begin{aligned} F: &= \frac{\sqrt{(|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2 + |y|^2(1 - |x|^4)}}{1 - |x|^4} \\ &\quad - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1 - |x|^4}, \end{aligned} \tag{1}$$

where $y \in T_xR^n \equiv R^n$. This Finsler metric is of scalar flag curvature whose flag curvature is in a special form: $\mathbf{K} = 3\theta/F + \sigma$, where $\theta := \langle a, y\rangle$ is an exact 1-form and $\sigma := 3\langle a, x\rangle^2 - 2|x|^2$ is a scalar function on $B^n(1)$. This leads us to investigate Finsler metrics of scalar flag curvature with flag curvature in the above form.

In this paper, we are going to establish an equation between the the flag curvature $\mathbf{K}$ and H for Finsler metrics of scalar flag curvature and prove the following

Theorem 1.1 *Let F be a Finsler metric of scalar flag curvature on an n-dimensional manifold M ($n \geq 3$). Let θ be an arbitrary 1-form on M. Then*

$$H = \frac{(n^2 - 1)\theta}{2F}. \tag{2}$$

if and only if

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \tag{3}$$

where $\sigma = \sigma(x)$ is a scalar function on M.

If we set $\theta = 0$ in Theorem 1.1, we obtain Akbar-Zadeh's theorem [1]. By the Schur lemma in Finsler geometry ([2]), if $\mathbf{K} = \sigma(x)$ is a scalar function, then it must be constant in dimension $n > 2$. However, when $\theta \neq 0$, σ can not be a constant in general as shown by the above example. Nevertheless, θ and σ are still related. See Proposition 3.1 below.

If a Finsler metric has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function, then it has isotropic mean Berwald curvature, $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$. Taking the horizontal covariant derivative of $\mathbf{E}$ along a geodesic, one obtains (2) with $\theta = c_{x^i}y^i$. Thus Theorem 1.1 generalizes Theorem 1 in [4].

If the Finsler metric F is *reversible*, $F(x, -y) = F(x, y)$, then (2) holds if and only if $H = 0$; and (3) holds if and only if $\mathbf{K} = \sigma$.

2 Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold. We always require that F is positive definite, i,e., the Hessian $g_{ij} := \frac{1}{2}\frac{\partial^2[F^2]}{\partial y^i \partial y^j}(x, y)$ of $F^2/2$

is positive definite for any $y \in T_xM \setminus \{0\}$. The geodesics of F are characterized by the following system of second order ordinary differential equations in local coordinates

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0, \tag{4}$$

where

$$G^i(x,y) := \frac{1}{4} g^{il}\Big\{ \frac{\partial^2 [F^2]}{\partial x^m \partial y^l} y^m - \frac{\partial [F^2]}{\partial x^l} \Big\}.$$

The local function $G^i = G^i(x,y)$ denote a vector field on TM,

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

G is called the *spray* of F.

The Riemann curvature $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := \; 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

Let $R_j{}^i{}_{kl}$ denote the hh-curvature of the Berwald connection. Let $R^i{}_{kl} := y^j R_j{}^i{}_{kl}$. Then

$$\begin{aligned} R_j{}^i{}_{kl} &= \frac{1}{3}\Big\{ \frac{\partial^2 R^i{}_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i{}_l}{\partial y^j \partial y^k} \Big\}, && (5) \\ R^i{}_{kl} &= \frac{1}{3} y^j \Big\{ \frac{\partial^2 R^i{}_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i{}_l}{\partial y^j \partial y^k} \Big\}. && (6) \end{aligned}$$

Conversely, we have

$$R^i{}_k = R^i{}_{kl} y^l = y^j R_j{}^i{}_{kl} y^l.$$

Thus the essential part of the Riemann curvature $R^i{}_k$.

Let $R_{jk} := g_{ij} R^i{}_k$. The key important property of R_{jk} is given as follows:

$$R_{jk} y^j = 0, \qquad R_{jk} = R_{kj}.$$

Thus one can compare R_{jk} with the angular form $\mathbf{h}_y = h_{ij} dx^i \otimes dx^j$, where

$$h_{jk} := g_{jk} - F^{-2} g_{jp} y^p g_{kq} y^q.$$

The flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ of a flag (P, y), where $P = \mathrm{span}\{y, u\} \subset T_xM$, is defined by

$$\mathbf{K} = \frac{R_{jk}(x,y) u^j u^k}{F(x,y)^2 h_{jk}(x,y) u^j u^k}.$$

One of the fundamental problems in Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature $\mathbf{K} = \mathbf{K}(x,y)$, or isotropic flag curvature $\mathbf{K} = \sigma(x)$ or constant flag curvature $\mathbf{K} = \sigma$.

It is an important fact that for a Riemannian metric, the flag curvature $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$ for any section $P \subset T_xM$. Thus if F is of scalar flag curvature, then it must be isotropic. Moreover, it must be constant, when dimension $n > 2$ by the Schur Lemma.

However, general Finsler metrics do not have the above properties. As shown by the example in (1) there are Finsler metrics with $\mathbf{K} = \mathbf{K}(x, y)$ depending on y. The Schur lemma in Finsler geometry tells us that if $\mathbf{K} = \sigma(x)$, then $\mathbf{K} = \textit{comstant}$ in dimension $n > 2$.

Definition 2.1 Let $F = F(x, y)$ be a Finsler metric on a manifold M.

(a) *F is of almost isotropic flag curvature if* $\mathbf{K} = 3c_{x^m}y^m/F + \sigma$, *where* $c = c(x)$ *and* $\sigma = \sigma(x)$ *are scalar functions on* M;

(b) *F is of weakly isotropic flag curvature if* $\mathbf{K} = 3\theta/F + \sigma$, *where* θ *is a 1-form and* $\sigma = \sigma(x)$ *is a scalar function on* M;

For a Riemannian metric, the local functions $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in y, where Γ^i_{jk} are the Christoffel symbols. However for general Finsler metrics, $G^i = G^i(x, y)$ are not quadratic in y. Thus it is natural to consider the following quantity

$$B_j{}^i{}_{kl}(x, y) := \frac{\partial^3 G^i}{\partial y^i \partial y^j \partial y^k}(x, y).$$

We obtain a quantity $\mathbf{B}_y = B_j{}^i{}_{kl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ which can be expressed as a family of multi-linear form on each tangent space. We call $\mathbf{B}$ the *Berwald curvature* because L. Berwald first used this quantity to characterized Finsler metrics with quadratic G^i, including Riemannian metrics. These metrics are therefore called *Berwald metrics*.

The mean Berwald curvature $\mathbf{E}_y = E_{ij}dx^i \otimes dx^j$ is defined by

$$E_{ij} := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\Big(\frac{\partial G^m}{\partial y^m}\Big).$$

Finsler metrics with $E_{ij} = 0$ are not necessarily Berwald metrics. There are plenty of examples.

The well-known S-curvature is defined by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial(\ln \sigma_F)}{\partial x^m},$$

where $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is the Busemann-Hausdorff volume form (Cf.[7], [8]). Then the mean Berwald curvature $\mathbf{E} = E_{ij}dx^i \otimes dx^j$ can be defined using the S-curvature $\mathbf{S}$.

$$E_{ij} = \frac{1}{2}\frac{\partial^2 \mathbf{S}}{\partial y^i \partial y^j}.$$

The quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics. More precisely,

$$H_{ij} := E_{ij|m}y^m,$$

where "|" denotes the horizontal covariant differentiation with respect to the Berwald connection. In local coordinates,

$$\begin{aligned} 2H_{ij} &= y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} \\ &\quad - \frac{\partial G^m}{\partial y^i} \frac{\partial^3 G^k}{\partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^j} \frac{\partial^3 G^k}{\partial y^i \partial y^k \partial y^m}. \end{aligned}$$

Clearly, E_{ij} and H_{ij} have the following property

$$E_{ij} y^i = 0, \qquad H_{ij} y^i = 0.$$

Thus one can compare them with the angular tensor $\mathbf{h} = h_{ij} dx^i \otimes dx^j$, where $h_{ij} = g_{ij} - F^{-2} g_{ip} y^p g_{jq} y^q$.

We can take further averaging on $\mathbf{E}$ and $\mathbf{H}$ as follows:

$$E := g^{ij} E_{ij}, \qquad H = g^{ij} H_{ij}.$$

E and H are scalar functions on the slit tangent bundle $TM \setminus \{0\}$.

Let us take a look at the example in (1). By a direct computation, we get

$$\mathbf{S} = (n+1)\langle a, x\rangle F.$$

Thus we have

$$E_{ij} = \frac{(n+1)\langle a, x\rangle}{2F} h_{ij}, \qquad H_{ij} = \frac{(n+1)\langle a, y\rangle}{2F} h_{ij}.$$

This example leads us to investigate Finsler metrics on a manifold M satisfying

$$E_{ij} = \frac{(n+1)c}{2F} h_{ij} \qquad \Big(\text{resp. } E = \frac{(n^2-1)c}{2F}\Big)$$

or

$$H_{ij} = \frac{(n+1)\theta}{2F} h_{ij} \qquad \Big(\text{resp. } H = \frac{(n^2-1)\theta}{2F}\Big),$$

where $c = c(x)$ is a scalar function and $\theta = a_i(x) dx^i$ is a 1-form on M.

It seems that there are more Finsler metrics with $\mathbf{H} = 0$ (resp.$H = 0$) than those with $\mathbf{E} = 0$ (resp. $E = 0$). But no example with $\mathbf{H} = 0$ (resp. $H = 0$) and $\mathbf{E} \neq 0$ (resp. $E \neq 0$) has been found yet. The important of the quantity $H = 0$ lies in the following

Theorem 2.2 ([1], [6]) *Let F be a Finsler metric of scalar flag curvature on an n-dimensional manifold M $(n \geq 3)$. Then the flag curvature $\mathbf{K} =$ constant is a scalar function on M if and only if $H = 0$.*

3 Generalized Schur Lemma

First we recall some important structure equations using the Berwald connection. One can refer to [7] for details. Using the Berwald connection, we have the following Bianchi identities for the hh-curvature $R_j{}^i{}_{kl}$ and the Berwald curvature $B_j{}^i{}_{kl}$:

$$R_j{}^i{}_{kl|m} + R_j{}^i{}_{lm|k} + R_j{}^i{}_{mk|l} = 0; \tag{7}$$

$$R_j{}^i{}_{kl\cdot m} = B_j{}^i{}_{ml|k} - B_j{}^i{}_{km|l}; \tag{8}$$

$$B_j{}^i{}_{kl\cdot m} = B_j{}^i{}_{km\cdot l}. \tag{9}$$

Contracting (7) with y^j, we obtain a Bianchi identity for $R^i{}_{kl} = R_j{}^i{}_{kl} y^j$.

$$R^i{}_{kl|m} + R^i{}_{lm|k} + R^i{}_{mk|l} = 0. \tag{10}$$

Contracting (10) with y^l yields

$$R^i{}_{k|m} - R^i{}_{m|k} + R^i{}_{mk|l} y^l = 0. \tag{11}$$

Assume that a Finsler metric F is of scalar curvature $\mathbf{K} = \mathbf{K}(x, y)$. In local coordinates,

$$R^i{}_k = \mathbf{K} F^2 h^i_k, \tag{12}$$

where

$$h^i_k = \delta^i_j - \frac{1}{F} F_{\cdot j} y^i.$$

For simplicity, let

$$\mathbf{K}_{\cdot i} := \mathbf{K}_{y^i}, \qquad \mathbf{K}_{\cdot i \cdot j} := \mathbf{K}_{y^i y^j}, \qquad \cdots.$$

Plugging it into (6), we get

$$R^i{}_{kl} = \frac{1}{3} \mathbf{K}_{\cdot l} F^2 \, h^i_k - \frac{1}{3} \mathbf{K}_{\cdot k} F^2 \, h^i_l + \mathbf{K} \Big\{ F F_{\cdot l} h^i_k - F F_{\cdot k} h^i_l \Big\}. \tag{13}$$

Differentiating (13) along the direction $y^m \frac{\delta}{\delta x^m}$ yields

$$R^i{}_{kl|m} y^m = \frac{1}{3} \mathbf{K}_{\cdot l|m} y^m F^2 \, h^i_k - \frac{1}{3} \mathbf{K}_{k|m} y^m F^2 \, h^i_l + \mathbf{K}_{|m} y^m \Big\{ F F_{\cdot l} h^i_k - F F_{\cdot k} h^i_l \Big\}. \tag{14}$$

Differentiating (12) along the direction $\frac{\delta}{\delta x^l}$ yields

$$R^i{}_{k|l} = \mathbf{K}_{|l} F^2 \, h^i_k. \tag{15}$$

Plugging (14) and (15) into the Bianchi identity (11), we obtain

$$\begin{aligned} \mathbf{K}_{|l} F^2 \, h^i_k - \mathbf{K}_{|k} F^2 \, h^i_l - \mathbf{K}_{|m} y^m \Big\{ F F_{\cdot l} h^i_k - F F_{\cdot k} h^i_l \Big\} \\ = \frac{\mathbf{K}_{\cdot l|m} y^m}{3} F^2 \, h^i_k - \frac{\mathbf{K}_{k|m} y^m}{3} F^2 \, h^i_l. \end{aligned} \tag{16}$$

Proposition 3.1 *Let F be a Finsler metric of scalar flag curvature on a manifold M of dimension $n > 2$. Assume that*

$$\mathbf{K} = \frac{3\theta}{F} + \sigma,$$

where $\theta = t_i(x)y^i$ is a 1-form and $\sigma = \sigma(x)$ is a scalar function. Then

$$\theta_{|l} - \theta_{\cdot l|m}y^m + (F\sigma_{|s} + 2\theta_{|s})h^s_l = 0. \tag{17}$$

We have

$$\mathbf{K}_{|l} = \frac{3\theta_{|l}}{F} + \sigma_{|l}, \qquad \mathbf{K}_{\cdot l|m}y^m = 3\frac{\theta_{\cdot l|m}y^m F - \theta_{|m}y^m F_{\cdot l}}{F^2}.$$

Here the covariant derivatives are taken with respect to the Berwald connection of F. For the scalar function $c = c(x)$, we view it as a scalar function on $TM \setminus \{0\}$. Thus

$$c_{|k} = c_{x^k}.$$

For the 1-form θ on M, we view it as a scalar function on $TM \setminus \{0\}$. Thus

$$\theta_{|l} = \theta_{x^l} - \theta_{y^m}N^m_l,$$

where $N^m_l = \frac{\partial G^m}{\partial y^l}$.

By (16) we get

$$\begin{aligned}
&\Big(\frac{3\theta_{|l}}{F} + \sigma_{|l}\Big)F^2h^i_k - \Big(\frac{3\theta_{|k}}{F} + \sigma_{|k}\Big)F^2h^i_l \\
&\Big(\frac{3\theta_{|m}y^m}{F} + \sigma_{|m}y^m\Big)(FF_{\cdot l}h^i_k - FF_{\cdot k}h^i_l) \\
&= (\theta_{\cdot l|m}y^m F - \theta_{|m}y^m F_{\cdot l})h^i_k - (\theta_{\cdot k|m}y^m F - \theta_{|m}y^m F_{\cdot k})h^i_l.
\end{aligned}$$

Rewrite the above identity as follows

$$Q_l h^i_k = Q_k h^i_l, \tag{18}$$

where

$$Q_l := \theta_{|l} - \theta_{\cdot l|m}y^m + (F\sigma_{|m} + 2\theta_{|m})h^m_l.$$

Taking a trace of (18) over i and k, we get

$$(n-2)Q_l = 0.$$

Now we assume that $n > 2$. The above identity becomes (17). Q.E.D.

4 An Important Identity

In this section, we are going to show an important relationship between the flag curvature $\mathbf{K}$ and the quantity $\mathbf{H}$ for Finsler metrics of scalar flag curvature. Throughout this paper, we shall use the notations in [7].

Proposition 4.1 *Let F be a Finsler metric of scalar flag curvature on an n-dimensional manifold M. For any 1-form θ on M, the flag curvature $\mathbf{K}$ and the quantity $\mathbf{H}$ satisfy*

$$\frac{\partial^2(F\tilde{\mathbf{K}})}{\partial y^i\partial y^j}-\tilde{\mathbf{K}}\frac{\partial^2 F}{\partial y^i\partial y^j}=-\frac{6}{n+1}F^{-1}\Big\{H_{ij}-\frac{(n+1)\theta}{2}\frac{\partial^2 F}{\partial y^i\partial y^j}\Big\}. \tag{19}$$

where $\tilde{\mathbf{K}}:=\mathbf{K}-3\theta/F$.

Proof: Plugging (12) into (5) gives

$$\begin{aligned} R_j{}^i{}_{kl} &= \frac{\mathbf{K}_{\cdot j\cdot l}}{3}F^2h^i_k-\frac{\mathbf{K}_{\cdot j\cdot k}}{3}F^2h^i_l+\mathbf{K}_{\cdot j}\Big\{FF_{\cdot l}h^i_k-FF_{\cdot k}h^i_l\Big\}\\ &\quad+\frac{1}{3}\mathbf{K}_{\cdot l}\Big\{2FF_{\cdot j}\delta^i_k-g_{jk}y^i-FF_{\cdot k}\delta^i_j\Big\}\\ &\quad-\frac{1}{3}\mathbf{K}_{\cdot k}\Big\{2FF_{\cdot j}\delta^i_l-g_{jl}y^i-FF_{\cdot l}\delta^i_j\Big\}+\mathbf{K}\Big\{g_{jl}\delta^i_k-g_{jk}\delta^i_l\Big\} \end{aligned} \tag{20}$$

Differentiating (20) with respect to y^m gives a formula for $R_j{}^i{}_{kl\cdot m}$ expressed in terms of $\mathbf{K}$ and its derivatives. Contracting $R_j{}^i{}_{kl\cdot m}=B_j{}^i{}_{ml|k}-B_j{}^i{}_{km|l}$ with y^k, we obtain

$$\begin{aligned} B_j{}^i{}_{ml|k}y^k &= 2\mathbf{K}C_{jlm}y^i\\ &\quad-\frac{\mathbf{K}_{\cdot j}}{3}\Big\{FF_{\cdot l}\delta^i_m+FF_{\cdot m}\delta^i_l-2g_{lm}y^i\Big\}\\ &\quad-\frac{\mathbf{K}_{\cdot l}}{3}\Big\{FF_{\cdot j}\delta^i_m+FF_{\cdot m}\delta^i_j-2g_{jm}y^i\Big\}\\ &\quad-\frac{\mathbf{K}_{\cdot m}}{3}\Big\{FF_{\cdot j}\delta^i_l+FF_{\cdot l}\delta^i_j-2g_{jl}y^i\Big\}\\ &\quad-\frac{\mathbf{K}_{\cdot j\cdot m}}{3}F^2\,h^i_l-\frac{\mathbf{K}_{\cdot j\cdot l}}{3}F^2\,h^i_m-\frac{\mathbf{K}_{\cdot l\cdot m}}{3}F^2\,h^i_j, \end{aligned} \tag{21}$$

Note that B^i_{jkl} is symmetric in j,k,l and $B^m_{jkm}=2E_{jk}$. It follows from (21) that

$$H_{ij}:=E_{jl|k}y^k=-\frac{(n+1)F}{6}\Big\{\mathbf{K}_{\cdot j}F_{\cdot l}+\mathbf{K}_{\cdot l}F_{\cdot j}+\mathbf{K}_{\cdot j\cdot l}F\Big\}. \tag{22}$$

Let θ be a 1-form and

$$\tilde{\mathbf{K}}:=\mathbf{K}-\frac{3\theta}{F}.$$

Then

$$(F\tilde{\mathbf{K}})_{\cdot i\cdot j}=(F\mathbf{K})_{\cdot i\cdot j}=F_{\cdot i\cdot j}\mathbf{K}+F_{\cdot j}\mathbf{K}_{\cdot i}+F_{\cdot i}\mathbf{K}_{\cdot j}+F\mathbf{K}_{\cdot i\cdot j}. \tag{23}$$

$$\tilde{\mathbf{K}}F_{\cdot i\cdot j} = \mathbf{K}F_{\cdot i\cdot j} - 3\theta F^{-1}F_{\cdot i\cdot j}. \tag{24}$$

By (22), (23) and (24) we arrive at the following identity:

$$(F\tilde{\mathbf{K}})_{\cdot i\cdot j} - \tilde{\mathbf{K}}F_{\cdot i\cdot j} = -\frac{6}{n+1}F^{-1}\Big\{H_{ij} - \frac{n+1}{2}\theta F_{\cdot i\cdot j}\Big\}.$$

We obtain (19).

5 Proof of Theorem 1.1

To prove the main theorem, we need the following

Lemma 5.1 *Let (M, F) be an n-dimensional Finsler manifold ($n \geq 3$). Let $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(x, y)$ is a positively homogeneous function of degree zero on the tangent bundle TM and $\sigma = \sigma(x)$ be a scalar function on M. Then the following equations are equivalent*

$$\frac{\partial^2(F\tilde{\mathbf{K}})}{\partial y^i \partial y^j} = \tilde{\mathbf{K}}\frac{\partial^2 F}{\partial y^i \partial y^j} \tag{25}$$

$$g^{ij}\frac{\partial^2(F\tilde{\mathbf{K}})}{\partial y^i \partial y^j} = (n-1)F^{-1}\tilde{\mathbf{K}}, \tag{26}$$

$$\tilde{\mathbf{K}} = \sigma. \tag{27}$$

Proof: It is easy to see that (25) implies (26) and (27) implies (25). Thus it surfaces to prove (26) implies (27). We assume that (26) holds. Fix an arbitrary point $x \in M$. Let $S_xM := \{y \in T_xM | F(x, y) = 1\}$ denote the unit sphere at x. There is a naturally induced Riemannian metric $\hat{g}_x$ on S_xM. Let (u^a) be a local coordinate system in S_xM. Then the position function $y = y(u)$ of S_xM satisfies the following Varga equation

$$y^i_{;a;b} + \mathbf{C}^c_{ab}y^i_{;c} + \hat{g}_{ab}y^i = 0, \tag{28}$$

where the covariant derivatives are taken with respect to $\hat{g}_x = \hat{g}_{ab}du^a \otimes du^b$ where

$$\hat{g}_{ab} = F_{y^iy^j}y^i_{;a}y^j_{;b}.$$

For a positively homogeneous function f of degree k on T_xM, its restriction $\bar{f}$ on S_xM satisfies

$$\bar{f}_{;a;b} = f_{y^iy^j}y^i_{;a}y^j_{;b} - \mathbf{C}^c_{ab}\bar{f}_{;c} - \hat{g}_{ab}k\bar{f}, \tag{29}$$

where the indices $a, b, c = 1, \cdots, n-1$. Using the above identities (25), one can easily show that (25) is equivalent to the following equation

$$\tilde{\mathbf{K}}_{;a;b} + \mathbf{C}^c_{ab}\tilde{\mathbf{K}}_{;c} = 0, \tag{30}$$

and (26) is equivalent to the following equation

$$\Delta\tilde{\mathbf{K}} + \mathbf{I}^c\tilde{\mathbf{K}}_{;c} = 0. \tag{31}$$

where $\mathbf{I}^c := \hat{g}^{ab}\mathbf{C}^c_{ab}$.

By the Hopf maximum principle in the case when $\dim S_xM = n-1 \geq 2$, we conclude from (31) that $\tilde{\mathbf{K}} = \mathit{constant}$ on S_xM at each point $x \in M$. Thus there is a scalar function $\sigma = \sigma(x)$ such that

$$\tilde{\mathbf{K}} = \sigma.$$

In the dimensional two case $n = 2$, S_xM has dimension one. Parametrizing S_xM by $y = y(s)$ where s is the arc-length of $\hat{g}_x$, we obtain two functions $\mathbf{I} := \mathbf{I}^1(x, y(s))$ and $\tilde{\mathbf{K}}(s) = \tilde{\mathbf{K}}(x, y(s))$. $\mathbf{I}$ is the so-called *main scalar*. Then (31) can be written as

$$\tilde{\mathbf{K}}''(s) + \mathbf{I}(s)\tilde{\mathbf{K}}'(s) = 0.$$

We obtain

$$\tilde{\mathbf{K}}'(s) = \exp\Big[-\int_{s_o}^{s}\mathbf{I}(t)dt\Big]\tilde{\mathbf{K}}'(s_o),$$

where s_o is any fixed number. Choose s_o such that $\tilde{\mathbf{K}}(s_o) = \max\tilde{\mathbf{K}}$. Then $\tilde{\mathbf{K}}'(s_o) = 0$. By the above identity we conclude that $\tilde{\mathbf{K}}' = 0$. We conclude that

$$\tilde{\mathbf{K}}(s) = \sigma(x).$$

Q.E.D.

Proof of Theorem 1.1: By (19),

$$g^{ij}\frac{\partial^2(F\tilde{\mathbf{K}})}{\partial y^i \partial y^j} - (n-1)F^{-1}\tilde{\mathbf{K}} = -\frac{6}{n+1}F^{-1}\Big\{H - \frac{(n^2-1)\theta}{2F}\Big\}. \tag{32}$$

By (32), one can easily see that (2) is equivalent to (26). By Lemma 5.1, this is equivalent to (3). Q.E.D.

6 Some Remarks

In this section, we will link our main theorem to the results in [4] and discuss some related problems.

It is easy to verify the following fact:

$$\mathbf{S} = (n+1)cF \quad\Longrightarrow\quad \mathbf{E} = \frac{(n+1)c}{2F}\mathbf{h} \quad\Longrightarrow\quad \mathbf{H} = \frac{(n+1)c_{x^m}y^m}{2F}\mathbf{h}.$$

$$E = \frac{(n^2-1)c}{2F} \quad\Longrightarrow\quad H = \frac{(n^2-1)c_{x^m}y^m}{2F}.$$

Suppose that a scalar function $\mathbf{K} = \mathbf{K}(x, y)$ on TM satisfies

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma,$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M. Then the scalar function $\tilde{\mathbf{K}} := \mathbf{K} - 3c_{x^i}y^i/F$ satisfies

$$\frac{\partial^2(F\tilde{\mathbf{K}})}{\partial y^i \partial y^j} - \tilde{\mathbf{K}}\frac{\partial^2 F}{\partial y^i \partial y^j} = 0.$$

By Lemma 5.1, there is a scalar function $\sigma = \sigma(x)$ such that

$$\tilde{\mathbf{K}} = \sigma.$$

Now we have the following diagram for Finsler metrics of scalar flag curvature. The first row is the relationship between the S-curvature and the flag curvature which is established in [4].

$$\begin{tikzcd}
\mathbf{S} = (n+1)cF \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathbf{K} = \frac{3c_{x^i}y^i}{F} + \sigma \arrow[dd, Rightarrow] \\
E = \frac{(n^2-1)c}{2F} \arrow[d, Rightarrow] & \\
H = \frac{(n^2-1)\theta}{2F} \arrow[r, Leftrightarrow] & \mathbf{K} = \frac{3\theta}{F} + \sigma
\end{tikzcd}$$

A natural question arises: for a Finsler metric of scalar flag curvature, is the equation $\mathbf{S} = (n+1)cF$ equivalent to the equation $\mathbf{K} = 3c_{x^i}y^i/F + \sigma$? This is verified for Randers metrics [10]. But it is still unknown for general Finsler metrics.

References

[1] H. Akbar-Zadeh, *Sur les espaces de Finsler A courbures sectionnelles constantes*, Acad. Roy. Belg. Bull. Cl. Sci. (5), **74** (1988),271-322.

[2] D. Bao, S. S. Chern and Z. Shen, *An introduction to Finsler geometry*, Graduate Texts in Mathematics 200, Springer-Verlag, New York, 2000.

[3] S.S. Chern and Z. Shen, *Riemann-Finsler geometry*, World Scientific Publishing Co., Singapore, 2005.

[4] X. Chen, X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, J. of London Math. Soc., (2) **68** (2003), 762-780.

[5] X. Cheng and Z. Shen, *Randers metrics of scalar flag curvature*, preprint, 2005.

[6] T. Sakaguchi, *On Finsler spaces of scalar curvature*, Tensor N.S. **38** (1982), 211-219.

[7] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[8] Z. Shen, *Lectures on Finsler Geometry*, World Scientific Publishing Co., Singapore, 2001.

[9] Z.Shen, *Landsberg curvature, S-curvature, and Riemann curvature, In "A Sampler of Finsler Geometry" MSRI series*, Cambridge University Press, 2004.

[10] Z. Shen and G.C. Yildirim, *Randers metrics with special flag curvature*, preprint, 2005.

B. Najafi
No. 25, Ordibehesht 2 alley
13 Aban, Shahid Rajaei St.
1883748773, Tehran-Iran
b_najafi@cic.aut.ac.ir

Z. Shen
Department of Mathematical Sciences
Indiana University Purdue University Indianapolis (IUPUI)
402 N. Blackford Street
Indianapolis, IN 46202-3216, USA
zshen@math.iupui.edu

A. Tayebi
Department of Mathematics and Computer Science
Amirkabir University
Tehran, Iran
akbar_tayebi@cic.aut.ac.ir